Dynamic test for buckling of bar subjected to tensile load

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ABSTRACT: The phenomenon of buckling of bar under tensile load is revealed in many cases of loading and found consideration in some structures and mechanisms. The tensile buckling load depends on dimensions and material of the bar as well as on flexibility of connections at its ends.

This paper describes the dynamic method for determining the tensile buckling load of bar on example of bar subjected to tensile load by a force directed towards the pole. Connection at the lateral non-movable end may vary from the hinge to the restrained including semi-rigid. The theoretical basis of the method is the dependence of the bar lateral stiffness on tensile load. The lateral stiffness is determined for various values of the tensile load based on free vibrations frequencies of the bar with mass at its tip as well as its own generalized mass. The tension critical load is obtained by extrapolation of the dependency between the stiffness and tensile load up to its zero value. The accuracy of the method is verified by comparing the experimental results and corresponding analytical values. Experimental measurement of critical tensile load of the stretched bar loaded by force directed to a pole is performed for the first time.

KEY WORDS: Dynamic method; Tensile buckling; End-fixity factor; Force directed towards a pole.

1 INTRODUCTION

Structures in which the loss of stability occurs due to tensile loads have consideration in scientific literature [1][2][3][4]. The buckling mode shapes have been investigated and calculation methods for determining the critical loads as well as optimization methods have been proposed. Scientific and technical significance of such structures requires further research in the direction of experimental verification of theoretical statements and particularly determining of critical tensile load. The methodological interest is also represented by statement and an experiment technique for non-destructive determining the critical load value, since in the literature such results are not presented.

The aim of the present research is non-destructive experimental determining of tensile critical load of bar with semi-rigid connection on the one end and with hinge slider moving along a circle trajectory on the other. Necessity of generalization of the fixed connection scheme on the one end by inclusion rotation spring is stipulated by the fact that due to compliance of connection elements an perfectly rigid clamping can not be produced. Since the radial reaction on the slider corresponds to the applied load, resistance to its movement by friction significantly affect the process and results of the experiment.

For decreasing of the friction influence, the original design solution is proposed by us and the dynamic method is adopted for determining the critical tensile load [5][6][7], as sliding friction has a character of Coulomb damping and doesn't influence the frequency of free vibrations [8]. The critical load is determined by linear extrapolation of dependence of a square of the natural frequency on axial load to its zero value [5][9]. The possibility of such linear approximation for beams under tensile load presented in [10], however it not include unconventional boundary conditions, such an end mass. Since the tested system has a slider with mass that significantly higher than mass of the tested bar, for justification and an assessment of linear approximation error the dynamic model with one degree of freedom is accepted. The model consists of tested bar as elastic element and generalized mass [11] on its laterally movable end that includes the masses of the bar and the slider.

2 THE LATERAL STIFFNESS OF BAR SUBJECTED TO TENSILE LOAD

The straight homogeneous bar subjected to tension by tensile load P with semi-rigid fixity condition at the one end and possibility of free rotation and lateral moving in sway mode at the other is considered in this paper (Figure 1). The end conditions that supplied the sway mode is considered in two possible configurations: (a) - with constant direction of tensile force and (b) - with changeable direction of force directed towards the pole.

For quantity definition of semi-rigid end condition we use the end-fixity factor [12][13]

\[ r = \left(1 + \frac{CH}{SEI} \right)^{-1} \]  

(1)

where C- the rotational stiffness at the support; EI- the bending stiffness of the bar cross-section; H- the bar length. The fixity condition at the end varies in the diapason from 0 in case of perfectly rigid to 1 in case of pinned support.
Figure 1. Elastic bar subjected to tensile load with semi-rigid connection at the lower end and with hinge at the upper end with two cases of sliding: (a) – along a straight line normal to initial bar’s axis; (b) – along a circular guiding line.

For the case presented in figure Figure 1a, as result of integration of the bar elastic deformations equation taking into account the end fixity factor [14], the following expression for lateral stiffness is received:

\[
K_t = \frac{P}{h_u} \left( \frac{3(1-r) + ru \tanh u}{3(1-r)u - [3(1-r) - ru^2] \tanh u} \right)
\]

where

\[
u = \frac{P}{\sqrt{EI}}
\]

The value of \(K_t\) is minimal for \(r=1\) (pinned-pinned rod) and equal to \(P/H\). The lateral stiffness \(K_{t,0}\) of the bar at \(P=0\) is determined as

\[
K_{t,0} = \lim_{P \to 0} K_t = \frac{3(1-r)}{H^3}
\]

The inclination of the dependence \(K_t(P)\) at \(P=0\)

\[
\alpha = \frac{dK_t}{dP} \bigg|_{P \to 0} = \frac{1}{H} \left[ 1 + 0.2(1-r)^2 \right]
\]

For convenience of the analysis a dimensionless parameters of the bar’s lateral stiffness are introduced as follow:

\[
K'_t = K_t \frac{H^3}{EI}
\]

Then, based on (2) and (4)

\[
K'_t = u^3 \left( \frac{3(1-r) + ru \tanh u}{3(1-r)u - [3(1-r) - ru^2] \tanh u} \right)
\]

and

\[
K'_{t,0} = 3(1-r)
\]

For the case presented in figure Figure 1b, when the traction force directed towards the pole we get

\[
R = P - \frac{K_t}{H}
\]

where \(R\) is the radius of sliding curvature that leads to a variation in direction of the tensile reaction force. At \(P=0\) \(K_{t,0}=K_{t,0}\) and inclination of the dependency \(K_t(P)\) derived as

\[
\beta = \frac{dK_t}{dP} \bigg|_{P \to 0} = \frac{1}{H} \left[ 1 + 0.2(1-r)^2 - \frac{H}{R} \right]
\]

Using (3) and (9) and (7) we have dimensionless parameters for stiffness

\[
K'_t = K'_t - \frac{H}{R}
\]

and for stiffness variation tendency

\[
\beta' = 1 + 0.2(1-r)^2 - \frac{H}{R}
\]

The graphs of dependency \(K'_t(P^2)\) for various values of \(H/R\) ratio and end-fixity factor \(r\) are presented in figure Figure 3a,b. The graphs show existence of critical load corresponding to stability losing under tension [2].

The equation for determining the critical value of load parameter \(u\) is obtained from (11) and (7) satisfying condition of null stiffness [14] \(K'_t=0\) as follows:

\[
\frac{3(1-r) + ru \tanh u}{3(1-r)u - [3(1-r) - ru^2] \tanh u} \frac{H}{R} = 0
\]

The values of roots of Eq. (13) are presented in table Table 1.
stable for \( H< R \) and non-stable for \( H> R \) without dependency on tensile load value.

The graphs presented in figure Figure 3 illustrates a character of the dependency \( K' \) on parameter \( \omega^2 \) and consequently dependence of square of bar's natural frequency on tensile load. For ratios \( H/R \) closer to 1 and end-fixity factors 0–0.25 the dependency \( K'_r(\omega^2) \) has a strong non-linearity. For \( H/R = 1.5 \) for all values of end-fixity factor the dependency \( K'_r(\omega^2) \) is close to linear. Thus, methodology of critical load determining based on experimental results should take into account the \( K'_r \) variation character depending on \( H/R \).

3 THE METHODOLOGY FOR CRITICAL LOAD DETERMINATION BASED ON NON-DESTRUCTIVE TESTING RESULTS

The experiment consists in determining the natural frequency of the bar's vibrations under tensile load variable in allowable range of its values and constant generalized mass. The upper bound of allowable range of tensile loads defined by ability of sufficiently accurate measuring of low frequencies in vicinity of critical load value due to influence of friction and the system imperfection. Experimentally determined that, depending on the imperfection and friction, the tensile force up to (0.8–0.85)\( P_{cr} \) can be achieved. The problem of critical load determination based on results of non-destructive testing consist in extrapolation of the results up to value corresponds to \( K'_r = 0 \) (or \( \omega = 0 \)).

A linear model in form

\[
K_r = K_{r,0} - bP
\]

is adopted for approximation of \( K_r(\omega^2) \) dependency in the loads range \( P \in [0, P_{cr}] \). This model is determined by the least squares method [16] based on results of experimental measurements. This simple model is less sensitive to measurement errors due to fewer unknown parameters. When the \( K_r(\omega^2) \) dependency has a strong non-linear character, the extrapolation results depend on selection of experimental values used for it. Independence from the selection can be achieved by definition of \( H/R \) ratios' range for which the dependence \( K_r(\omega^2) \) is close to linear. Since for larger ratio \( H/R \) the dependence \( K_r(\omega^2) \) became more close to linear, as it was shown in figure Figure 3, such definition of the range bounds is possible.

To assess the proximity of \( K_r(\omega^2) \) dependence character to linear, we have calculate a maximal deviation \( D_{\omega,\omega} \% \) of theoretical values \( K'_r \) calculated according to (11) in the loads range \( \omega \in [0, \omega_c] \) from linear line passing through the points \( [0, K_{r,0}] \) and \( [\omega_c^2, 0] \) and described by following expression:

\[
K'_r = K'_{r,0} \left( 1 - \frac{\omega^2}{\omega_c^2} \right)
\]

The maximal deviations for various values of sliding curvature and end-fixity parameters are presented in table Table 2. Based on the results shown in the table, the minimal allowable ratio \( H/R \), for which the linear approximation can be used, depends on accepted accuracy. For example, for accepted accuracy in 1.7% and for rigid end-fixity condition the linear model can be used for ratios \( H/R \geq 2 \).
Table 2. The maximal deviations \( D_{\text{max}} \) of linear approximation line from theoretical values \( K_r' \) in the loads range \([0, u_{cr}]\) for several values of end-fixity factor \( r \) and dimensionless curvature \( H/R \).

<table>
<thead>
<tr>
<th>( r )</th>
<th>0.00</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H/R )</td>
<td>1.3</td>
<td>12.6</td>
<td>8.8</td>
<td>1.9</td>
</tr>
<tr>
<td>1.4</td>
<td>9.5</td>
<td>4.8</td>
<td>1.9</td>
<td>0.7</td>
</tr>
<tr>
<td>1.5</td>
<td>6.8</td>
<td>3.8</td>
<td>1.5</td>
<td>0.1</td>
</tr>
<tr>
<td>2.0</td>
<td>1.7</td>
<td>1.3</td>
<td>0.5</td>
<td>0.1</td>
</tr>
<tr>
<td>2.5</td>
<td>0.4</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Since
\[
K_r = m \omega^2
\]

where \( m \)-generalized mass at the laterally movable bar's end, the linear model (15) can be presented in the following form:
\[
\omega^2 - \omega_0^2 = b_\omega P
\]

when \( \omega_0 \) denotes the free vibration frequency at \( P=0 \). If the frequency \( \omega_0 \) cannot be measured experimentally due to experimental conditions, then it defined as unknown parameter of the linear model and determined together with inclination \( b_\omega \). Based on determined parameters of linear model the critical force is calculated as follows:
\[
P_{cr} = \frac{\omega_0}{b_\omega}
\]

The advantage of using this model is its independence from generalized mass, i.e. during experiment performing its determination isn't required.

4 EXPERIMENTAL

The objectives of experimental investigation are:
- quality analysis of lateral stiffness' dependency on tensile load and curvature parameter \( H/R \);
- the critical load determination by methodology described in Chapter 3 and comparison with calculated values.

The experimental setup presented in Figure 4. To reduce the influence of friction, the curvilinear slider is constructed as rigid wheel with pinned connector holding the upper end of the tested bar. The wheel is statically equilibrated and its rotation axis is rigidly connected to guiding rail on which a carriage moves along the wheel radius. A clamp is mounted on the carriage to produce rigid or semi-rigid connection of lower end of the tested bar. The tensile force applied on the bar is produced by weights mounted on the carriage. The carriage's own weight is equilibrated by counterbalancing weight. The radius of curvilinear translation is \( R=327 \) mm. The tested bar made of spring steel strip with \( 20 \) mm width.

The graphs of dependency of lateral stiffness \( K_r \) on tensile load for two bars with lengths \( 360 \) and \( 460 \) mm and thickness \( 0.97 \) mm are presented on figure Figure 5. The graphs illustrates the significant dependency \( K_r \) on \( H/R \) ratio and approach to linearity at its increase. It confirms the thesis about possibility of using of linear approximation of stiffness \( K_r \) function subject to the limitation of its application region depending on \( H/R \) ratio.
To determine the critical load based on non-destructive experimental measurements, the end-fixity factor at the bottom end has been found out experimentally. Necessity of appropriate accounting of end-fixity factors follows from the analysis of roots given in table Table 1. As it follows from (3) and (13) a critical load is defined as

\[ P_{cr} = u_{cr}^2 \frac{EI}{H^2} \]  

(20)

and has a significant dependency on end-fixity conditions. For example, for \( H=450 \text{ mm} \) \( R=327 \text{ mm} \) the \( u_{cr} \) values from (13) are 3.166 for \( r=0.16 \) and 3.810 for \( r=0 \) (perfectly clamped end) and according to (20) it corresponds to critical loads ratio 0.69.

At conducting the experiment, rotation of bottom clamped end is caused by elastic deformation of its connection with movable carriage. Hence the theoretical values of critical loads were obtained accounting the end fixity factors. The natural frequency \( f_0 \) of unloaded bar \((P=0)\) and end-fixity factor \( r \) were obtained using additional masses according to setup scheme presented in figure Figure 6.

The elastic bar of the tested system is made of steel strip with length 435 mm and thickness 1.48 mm and without any axial forces. Two equal weights with general mass \( M_b \) hang on a flexible belt at the distance \( R_b=315 \text{ mm} \). From the expression for natural frequency \( f_0 \) of the system [17] follows

\[ M_b(R_b / R)^2(2\pi f)^2 = K_{r,0} - K_{r,0}(f / f_0)^2 \]  

(21)

For improving the measurements accuracy by statistical processing of results, the measurements are performed at three values of additional mass \( M_b \). As a result of linear approximation of the dependency by least squares [16]

\[ y = c + dx \]  

(22)

where

\[ y = M_b(R_b / R)^2(2\pi f)^2 \]  

(23)

\[ x = f^2 \]  

(24)

the parameters \( K_{r,0} = K_{r,0}/c \) and \( f_0^2 = -c/d \) can be obtained. With known material elasticity module of and cross-section dimensions, end factor is obtained from (4) as follows:

\[ r = 1 - \frac{K_{r,0} H}{3 \frac{E}{I}} \]  

(25)

For elasticity steel \( E=210 \text{ GPa} \) [17] and cross-section 19.9x1.48 mm the initial stiffness obtained based on experimental results presented in figure Figure 7 is \( K_{r,0} = K_{r,0}/34.4 \text{ N/m} \), and therefore end-fixity factor \( r=0.165 \).

Experiment on determination of critical tensile load has been performed for constant sliding radius and five different bar's lengths as presented in table Table 3. The minimal bar's length used in the experiment is limited by accuracy requirement \((H/R>1.4)\), whereas the maximal length is limited by range of tensile forces due to increasing of friction. The measurements results for bar's length \( L=450 \text{ mm} \) and \( L=650 \text{ mm} \) are presented in figure Figure 8.

<table>
<thead>
<tr>
<th>Length ( H ), mm</th>
<th>650</th>
<th>450</th>
</tr>
</thead>
<tbody>
<tr>
<td>End-fixity factor ( r )</td>
<td>0.12</td>
<td>0.16</td>
</tr>
<tr>
<td>( H/R ) ratio</td>
<td>1.99</td>
<td>1.38</td>
</tr>
<tr>
<td>Experimentally obtained ( P_{cr} ), N</td>
<td>8.1</td>
<td>53.6</td>
</tr>
<tr>
<td>Theoretical critical load ( P_{cr}^* ), N</td>
<td>8.4</td>
<td>51.9</td>
</tr>
<tr>
<td>Discrepancy, %</td>
<td>3.7</td>
<td>3.2</td>
</tr>
</tbody>
</table>

Figure 7. Experimental determination of initial stiffness \( K_{r,0} \) and initial natural frequency of the tested system using three values of additional mass.
tensile force value with sufficient accuracy. However, the linear extrapolation allows determine the critical motion of the system due to friction influence. Measurements can be performed only for tensile load not its end and dimensionless curvature bar's dimensions, dimensionless fixity parameter on the one of end and with a hinge sliding along a circle on the other is at non-destructive testing, whereas at testing. As results of experimental investigation, performed for constant slider curvature and variation of bar's length:
- the character of bar's lateral stiffness dependency on tensile load and $H/R$ ratio has been validated – the value of the critical tensile load has been validated with sufficient accuracy in the diapason of sliding curvatures $R=(1.4–2.5)H$.

REFERENCES


5 CONCLUSIONS

The behavior of stretched bar with rotational spring on the one end and with a hinge sliding along a circle on the other is considered in the paper. Dependency of its lateral stiffness on bar's dimensions, dimensionless fixity parameter on the one of its end and dimensionless curvature $H/R$ has been determined in the present work. For non-destructive experimental determination of critical tensile force by dynamic method the linear approximation of lateral stiffness dependency on tensile load in the loads range from zero up to critical value has been proposed. Maximal deviations between approximating line and theoretical values depending on end-fixity factor and dimensionless curvature are presented in tabulated form.

Experimental measurement of critical tensile load of the stretched bar loaded by force directed to a pole has been performed for the first time. The developed configuration of the experimental set-up allows significantly reducing the friction translation resistance of slider. Consequently, it allows increasing the maximal tensile force at non-destructive testing.