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The Ellipse of Elasticity and Mohr Circle-based graphic dynamic modal analysis of torsionally coupled systems

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ABSTRACT: A graphic dynamic method of modal analysis of elastic torsionally coupled systems is presented, which complements the graphic recursive approach referred to as Ellipse of Elasticity Modal Analysis (EEMA) (Faggella 2013) with the introduction of a single-step Mohr Circle Modal Analysis (MCMA). The approach allows a fully graphic computational visualization of all the parameters of interest for dynamic analysis, such as the modal frequencies, the effective modal stiffness and masses, the intensity and line of actions of the elastic and inertial forces. It provides insight into the behavior of torsionally coupled systems and represents a basis for the definition of a graphic dynamic response analysis of irregular structures subjected to horizontal forces such as earthquake and wind.

KEY WORDS: torsionally coupled structures; graphic dynamics; graphic modal analysis; Ellipse of Elasticity; Mohr Circle.

1 INTRODUCTION

In-plan irregularity and torsional coupling of structural systems are aspects of primary importance in seismic structural design and assessment. Torsional effects can often result in unbalanced demand on structural components leading to collapse or poor earthquake performance.

This paper illustrates two graphic computation approaches that can be referred to as graphic dynamic, [1], which retain the accuracy of numerical analyses, [2], and the elegance of traditional graphic static methods. The methods are based on the assumption of rigid floor diaphragm behavior, and on the rotational kinematic representation of the displacements and accelerations through the modal centers of rotation, i.e. the modal ‘nodes’, [3]. They provide useful insight into the dynamic behavior of torsionally coupled systems and can be further explored to develop graphic methods of earthquake response analysis, [4], [5].

Two graphic dynamic methods of modal analysis of one-way asymmetric single-story structures are illustrated in this paper: 1) the Ellipse of Elasticity Modal Analysis (EEMA), a recursive graphic modal analysis approach based on the antipolarity with respect to the Ellipse of Elasticity and to the mass circle of gyration; and 2) the Mohr Circle Modal Analysis (MCMA), a single-step graphic modal analysis based on the properties of the Mohr Circle of the stiffness matrix.

2 ELLIPSE OF ELASTICITY MODAL ANALYSIS (EEMA)

Consider the general one-way eccentric single-story elastic structure of Figure 1. Both the center of mass G and the center of stiffness K of the lateral load-resisting elements are located on a principal axis at a distance equal to the eccentricity $e_x$. The ellipse of elasticity is centered in $K_x$ and is defined by the two principal semi-diameters given by in terms of the ratios of the rotational stiffness $k_\theta$ about the center of stiffness $K$, and of $k_x$ and $k_y$, which are the stiffness in the $x$ and in the $y$-direction respectively.

The degrees of freedom are the displacement of the center of mass in the $y$-direction $u_y$ and the floor rotation angle $\theta$, and can be expressed in the nodal form as a function of the rotation $\theta$ and of the position of the center of rotation $x_c$.

$$\mathbf{u} = \begin{bmatrix} u_y \\ \theta \end{bmatrix} = \begin{bmatrix} -x_c \\ 1 \end{bmatrix} \theta = \sum_{n=1}^{n} \begin{bmatrix} -x_{c,n} \\ 1 \end{bmatrix} \theta_n(t)$$

Figure 1. Torsionally coupled system with a one-way eccentricity between the center of mass G and the center of stiffness K of the lateral load-resisting elements.

The mass circle of gyration is centered in the center of mass G, and is defined through the gyration radius $\rho$ defined as the ratio of the polar moment of inertia $I_p$ and the floor mass $m$.

$$\rho_x = \frac{k_0}{k_x}, \quad \rho_y = \frac{k_0}{k_y}, \quad \rho = \sqrt{\frac{I_p}{m}}$$

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The system’s free vibrations are governed by the equilibrium between the elastic and inertial forces \( \mathbf{F}_e \) and \( \mathbf{F}_m \):

\[
\mathbf{F}_m + \mathbf{F}_e = \mathbf{m} \ddot{\mathbf{u}} + \mathbf{k} \mathbf{u} = \mathbf{0}
\]  

(3)

where \( \mathbf{k} \) and \( \mathbf{m} \) are the stiffness and mass matrices

\[
\mathbf{m} = \begin{bmatrix} 1 & 0 \\ 0 & \rho^2 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 1 & \mathbf{e}_x \\ \mathbf{e}_x & \mathbf{d}_x^2 \end{bmatrix}
\]  

(4)

\( d_x \) is a ‘coupled stiffness gyrator’, and corresponds to the distance of the center of mass \( G \) from the end point of the semi-diameter \( \rho_x \), rotated along the vertical line through \( K \), as it is shown in Figure 2. The ratio of the coupled stiffness gyrator to the mass gyrator \( q = d_x / \rho \) describes the ‘translationality’ of the system

\[
d_x = \sqrt{\mathbf{e}_x^2 + \rho^2}, \quad q_x = d_x / \rho
\]  

(5)

The expressions of the vector of DOFs \( \mathbf{u} \) in terms of the modal centers of rotation can be used to determine the lines of action of the elastic and inertial forces, which are also identified by the position \( x_{c,j} \) of the modal centers of rotation. In fact these distances correspond also to the distances of the line of action of the corresponding forces from the center of mass. The EEMA is based on the graphic determination of \( C_1 \) and \( C_2 \) through the two antipolarities expressed by the equations

\[
(x_{c,j} - \mathbf{e}_x) = -\rho_j^2 / (x_{c,j} - \mathbf{e}_x), \quad x_{c,j} = -\rho_j^2 / x_{c,j}
\]  

(6)

The EEMA procedure can be seen as a graphic realization of the Ritz method for torsionally coupled 2DOF systems. Starting from a trial node, it consists of recursive generations of directions of elastic forces of the trial node, and of subsequent determination/update of the node by applying the antipolarity with respect to the masses.

Figure 2. Plan schematic of the structure floor and lateral load-resisting elements with the Ellipse of Elasticity and the Mass circle of Gyration.

Figure 3. a) Recursive EEMA determination of the modal centers of rotation \( C_n \), b) corresponding lines of action of the vertical modal forces.

The converged positions \( x_{c,j} \) of the modal centers of rotation correspond also to the directions of the modal elastic and inertial forces \( \mathbf{F}_{e,j} \) and \( \mathbf{F}_{m,j} \). The remaining parameters of interest for dynamic free vibration analysis are then expressed as a function of \( x_{c,j} \) as

\[
\frac{k_{e,j}}{k_y} = \frac{x_{c,j} - \mathbf{e}_x}{x_{c,j} - x_{c,j}}, \quad \frac{m_{e,j}}{m} = \frac{x_{c,j}}{x_{c,j} - x_{c,j}}, \quad \omega^2 = \omega^2 = \frac{x_{c,j} - \mathbf{e}_x}{x_{c,j}}
\]  

(7)

3 MOHR CIRCLE MODAL ANALYSIS (MCMA)

Although the EEMA is effective for deriving graphic dynamic parameters, the reliance on iterations and the determination of the modes and frequencies based on ratios between distances represents a disadvantage. This aspect can be overcome through the Mohr Circle Modal Analysis (MCMA) graphic procedure, which provides a single step graphic solution, and a more straightforward visualization of the frequencies and of the modal nodes based on a single measure of lengths. The systems of differential equations (3) can be put in the form

\[
\mathbf{u} + \frac{1}{m} \mathbf{k} \mathbf{u} = \mathbf{0}
\]  

(8)

expressing the DOFs in terms of a new vector of the degrees of freedom \( \mathbf{u}_p \) made of two translational components

\[
\mathbf{u}_p = \begin{bmatrix} \mathbf{u}_y \\ \rho \theta \end{bmatrix} = \sum_{n=1}^{N} \begin{bmatrix} -x_{c,n} \\ \rho \theta \end{bmatrix} \theta_n(t)
\]  

(9)

these entries are respectively the displacement of the center of mass \( \mathbf{u}_y \) and the translational displacement \( \mathbf{u}_y = \rho \theta(t) \) at a distance from the center of mass equal to the gyration radius. These new equations correspond to a classical eigenproblem...
This eigenproblem can be solved through a Mohr Circle, through the MCMA procedure, which is illustrated in Figure 4. It consists of superimposing the non-dimensional schematic of the plan view of the structure per unit-radius of gyration onto the Mohr plane so that the intersection of the mass circle of gyration with the vertical axis through G coincides with the pole P of the circle.

\[
\mathbf{k}_e = \begin{bmatrix}
1 & \varepsilon_x \\
\varepsilon_x & q^2_x \\
\end{bmatrix}
\]

where \(\varepsilon_x = \theta_i / \rho\) \(\text{ (10)}\)

The MCMA performs a single-step determination of the modal nodes and of the modal frequencies, and allows a more straightforward determination of the eigenvalues in terms of geometric distances. The remaining parameters of interest, i.e. the lines of action of the modal elastic and inertial forces, and the effective modal mass and stiffness ratios are derived in terms of the Mohr circle construction.

The two procedures represent overall an effective graphic method of modal analysis of one-way torsionally coupled systems, and yield visual insight on the structural behavior.

The results of GMA procedures can be further explored and incorporated into graphic earthquake response analysis procedures.

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REFERENCES


