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In Defence of the Centrifugal Force and the Geometric Law of Motion

An old discussion for some new perspectives

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ABSTRACT

After a brief introduction and some discussion on the academic conflict that has forever surrounded the idea of a *centrifugal force*, there will be presented in this text the mathematical concepts that may lead us to believe in the real existence of this force. Some of the probable implications arising from this fact are then addressed, in particular the possibility of describing the laws of motion as a single *Geometric Law of Motion*.

1. The centrifugal dilemma

More than 300 years have passed since Huygens unveiled his famous equation for the *centrifugal force* ($F_c = m.v^2/r$), and no consensual opinion has yet been reached by the academic community about the true meaning of such a force. Newton used it as a fundamental tool in order to derive his Theory of Gravitation, where he manipulated it as a force[1]. However, by his second law of mechanics, which generally states $\underline{\boldsymbol{F}} = d(m.\underline{\boldsymbol{v}})/dt$, it is a fact this force should not exist[2]. It seems there is, somehow, an incongruence in Newton himself concerning this fundamental issue, which was never resolved by his school of thought. Newton considered Huygens force as the force that compensates the gravitational attraction between bodies when they describe orbits, but, on the other hand, Newton says such a force is not real and must therefore be seen as a kind of "fictitious" force, or "apparent" force, coming from the inertia of the moving body. Nevertheless, Newton did not seem "against" the centrifugal force in the same way Newtonians of our days obviously are. The main reason for this, however untold, seems simple: Huygens force obviously challenges the general principle established by Newton's second law, as we will soon explain. Thus, the only logical option to legitimise such an excellent theory of mechanics (no doubt that it is) is to definitely "negate" the existence of the centrifugal force. Newtonian scholars and schools have therefore not only assumed their laws as unquestionable, but even seem to have invested in lobbying against the term "centrifugal force" in the classrooms and universities around the world, pointing those who do not agree with them as merely pathetic. Indirectly associated to such a subject was also the discussion on the existence or non-existence of a universal inertial frame of reference, in which Newton seemed to believe in - an idea challenged by the arguments of Ernest Mach and George Berkeley, amongst others, who were more convinced that inertia was due to some gravitational effect induced in each mass by the outer part of the universe. Einstein himself was

[1] An example of how Newton was deriving his Gravitational force may be found in: J. Manuel Feliz-Teixeira, "Deducing Kepler and Newton from Avicenna (ابن سينا), Huygens and Descartes", first published at http://www.fe.u.pt/~feliz/ and YouTube, April 2010.

[2] Herein we will represent vectors as bold underlined letters, a substitute for the usual arrow. So, vector v = $\underline{\mathbf{v}}$.

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expecting his theory of General Relativity to confirm Mach's principle, but to date, this has not happened. Of course, there is a lot of literature about this, even on the Internet. But let us definitely show why Huygens forces seem not to be compatible with Newton's beliefs: Newton's second law equation states:

$$\underline{F} = d(m.\underline{v})/dt \qquad (1)$$

or, if we consider *m* to be *constant* in time, as one usually does in mechanics of solids:

$$\underline{F} = m\, d\underline{v}/dt$$
$$\underline{F} = m.\underline{a} \qquad (2)$$

which is the well known relation between *force* and *mass*. This means, however, that when the total forces $\underline{F}$ acting on the body are null, then $d\underline{v}/dt$ must also be null, which is the same as saying that vector $\underline{v}$ must be constant in time. But, since an orbiting $\underline{v}$ is obviously not a *constant* vector, as it constantly changes direction, Newton infers that the resultant force in an orbiting body cannot be null. If it would be null, the body would simply run along a straight-line[3], as also expected by Galileo Galilei, not along a curve. Thus, in Newton's perspective, imagining a Huygens force compensating the gravitational force in the case of an orbit would be not only against the basis of his theory but also against Galileo's expectations. The way of resolving the conflict was to declare it "apparent", calling centrifugal forces fictitious, and stating that a body while orbiting is simply being subjected to the *centripetal force* needed to curve its natural rectilinear trajectory (in the next images we will denote the gravitational acceleration by $\underline{g}$ and the Huygens centrifugal acceleration by $\underline{c}$). Some centuries later, *centrifugal forces* have even been stigmatized, even though they are believed to be responsible for the Earth's bulge and many other effects, for example, the drying of clothes by centrifugation in our washing machines. The next figure basically represents Newton's model for an orbiting body.

[3] In fact, in our perspective the body can still be seen moving in a straight-line as predicted by Newton, but the direction of such a "straight-line" is also constantly changing according to the constant change in direction of the gravitational force. These directions are always kept perpendicular. So, in each instant of time the centrifugal force does not contradict Newton. Instead, is there to support him.

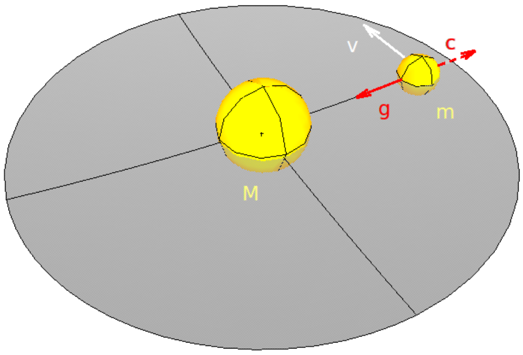

Fig. 1 Newton's model for a body of mass *m* orbiting another massive body (*M*) with a velocity $\underline{v}$. The scale of the drawing is obviously exaggerated. By Newton, $\underline{g}$ is the real acceleration due to the attraction between the two bodies (gravitational), while $\underline{c}$ is a centrifugal fictitious acceleration due the inertia of *m*.

As classical mechanics evolve to more generalist formalisms with Lagrange and Hamilton, who do not explicitly use the concept of force, but instead of energy associated with certain degrees of freedom, or generalised "directions" of movement, the centrifugal force conflict seems also to become diluted. After all, these formalisms also don't use vectors, as we try to represent in the next figure.

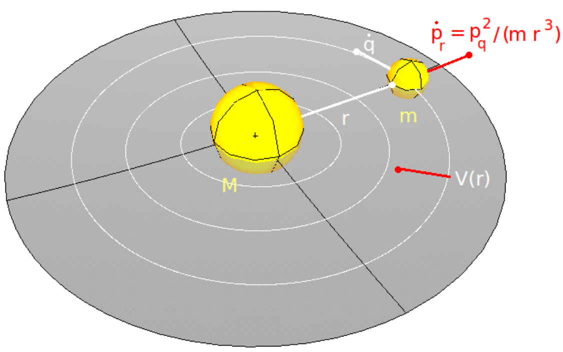

Fig. 2 Lagrange and Hamilton describe the system in terms of kinetic energy and potential energy *V(r)*, and compute momenta (*p*) and forces *[d/dt]p* from it. Here, the generalised coordinates *q* and *r* are used. A dot over a letter means time derivative = *[d/dt]*. So, $[d/dt]p_r$ is in fact the centrifugal force.

In the example of the previous figure (Fig. 2), *q* represents the generalised coordinate which is the angle of rotation in the orbit (usually denoted by θ), while *V(r)* is the gravitational potential "due" to mass *M*, and *[d/dt]q* the speed of *m* along *q*. Therefore, the quantity $[d/dt]p_r = p_q^2 /(m.r^3)$ is Huygens centrifugal force, where p_q^2 is the angular momentum of the mass *m*. This force, however, appears naturally from the natural manipulation of cylindrical coordinates, which of course are indirectly associated with the spatial type of "constraints" of geometry imposed by the gravitational field.

But, what did Einstein think about this issue? We really don't precisely know. But Einstein knew there were some curious facts that seem to contradict the Newtonian model. For example, accelerometers in free fall do not measure any acceleration. And orbits seem independent of the mass orbiting (in this case *m*). Einstein was well aware of this, therefore we suppose this was what made him decide looking at gravitation as being due to a deformation of space, not a force. He avoided Newton. For Einstein, the gravitational field results from a deformation of the *space-time* centred at each mass (Fig. 3), therefore masses fall into each other naturally along such a deformation. The intensity of the deformation is obviously proportional do the mass of the body.

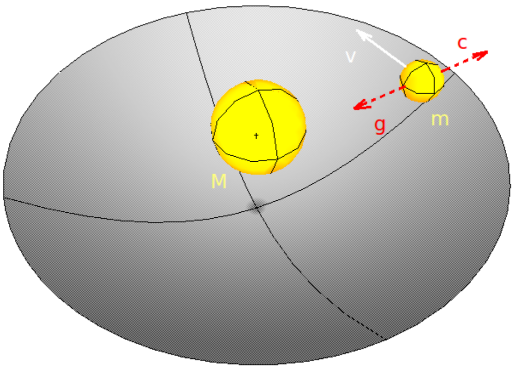

Fig. 3 Einstein's curvature of *space-time* due to mass *M*. Notice that in this model both **g** and **c** could be seen as "fictitious" accelerations resulting from *space-time* deformations. We may in fact say that Einstein's principle of equivalence positions these forces in the same category.

In this sense, the gravitational acceleration may also be seen as a "fictitious" acceleration, although it can perfectly be compared with any other kind of acceleration; for example, with the acceleration of a spacecraft in the empty space[4]. Free fall is, for Einstein, the same as orbiting, even the same as floating in the empty and free space, since in any of these cases the body feels no forces acting on it. That is why accelerometers in free fall or in orbit measure no gravitational force at all. If there would be no atmosphere, a person falling under gravity would simply feel as floating (till the moment of the crash, of course). Forces are obviously dependent on the state of motion of the observer. And Einstein seems to classify forces by their effects on bodies, not by their probable origins.

Thus, talking of centrifugal forces is acceptable again, since Einstein has legitimised them while comparing them even to gravitational forces. Notice, by the way, that Einstein was somehow suspecting the *inertia* that keeps masses moving in straight-lines in the absence of external forces could even result from the gravitational effect coming from all the mass existing in the outer part of the universe. So, Einstein seemed to believe that external masses might act upon the internal masses too. And this would also contradict what was claimed by Newton, who elegantly demonstrated that the field inside a hallow sphere must be null. So, any of these effects expected by Mach and Einstein would have to be null. The mathematical demonstration of a null field inside a sphere, however, starts by another assumption taken as unquestionable: *masses attract each other*. Could such a principle be somehow not perfectly defined yet? We expect to return to this subject some other time. For now, let us focus on the description of motion.

2. Radial and angular coordinates

By observing our reality it is easy to recognise that motion generally tends to be constrained by a certain circular geometry, that is, motion is usually made around an axis, which, in reality, happens to be the axis of the angular momentum. A closer look into the motion of the bodies in space will reveal that even rectilinear motion may in fact be described in terms of an angular movement with a radial component, where angular momentum is still preserved[5]. We therefore think it is in good compliance with the laws of nature to describe motion based on a referential with the two basic components: radial and angular. We will call them parallel $_{//}$ to the observer and perpendicular $_{\perp}$ to the

[4] You may refer to Einstein's Equivalence Principle at: http://en.wikipedia.org/wiki/Equivalence_principle

[5] An simple demonstration of this can also be found in: "Deducing Kepler and Newton from Avicenna (ابن سينا), Huygens and Descartes".

observer, respectively:

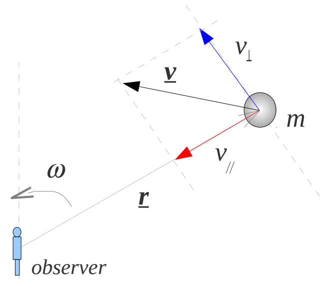

Fig. 4 Vector $\underline{v}$ expressed in terms of its (parallel, perpendicular) coordinates, belonging to the plan defined by $\underline{r}$ an $\underline{v}$. Notice that the angular momentum relatively to the *observer* (who does not have to be the center of curvature), is given by the cross product $\underline{L} = \underline{r} \times m\underline{v}$, which is a pseudo-vector perpendicular to the same plan.

Of course any vector can be represented in these "natural" coordinates (plus a coordinate *z* for the three-dimensional space, seen as a constant in many gravitational systems); but also kinetic energy, being a scalar, can be seen in terms of a parallel energy and a perpendicular energy. That is, energy can somehow be associated with a certain direction, as in the case of the Hamiltonian[6], for example. In effect, from the figure we have:

$$\underline{v} = (v_\perp , v_{//}) \qquad (3)$$

$$v^2 = \underline{v} \,.\, \underline{v} = (v_\perp , v_{//}).(v_\perp , v_{//})$$

$$v^2 = v_\perp^2 + v_{//}^2 \qquad (4)$$

Multiplying both members by ½ *m* to compute the kinetic energy, we get:

$$½\, m.v^2 = ½\, m.v_\perp^2 + ½\, m.v_{//}^2$$

$$K = K_\perp + K_{//} \qquad (5)$$

On the other hand, if we multiply (4) by *m/r* in order to compute Huygens force (F_c):

$$m.v^2/r = m.v_\perp^2/r + m.v_{//}^2/r$$

$$F_c = F_{c\perp} + F_{c//} \qquad (6)$$

Could this mean that even the centrifugal force (usually considered always pointing along $\underline{r}$) can be the result from two contributions?: the contribution from the perpendicular component of the velocity, and the contribution from the parallel component of the velocity. This let us wonder about the concept of *centrifugal force* in a wider perspective: a force due to the movement itself, perhaps inertial, therefore intrinsic, which could be either positive or negative (repulsive/attractive) in respect to the observer, depending on its radial ($_{//}$) contribution. Even if the angular contribution ($_\perp$) is assumed to be always positive, as it points towards the increase of $\underline{r}$, in a wider sense the force applied to elevate a spacecraft into the atmosphere at a constant velocity could also be seen as a centrifugal force, relatively to the centre of the Earth. Thus, a centrifugal force could exist even with no angular motion at all. Could this be the force responsible for the existence of non circular orbits, for example? And for the movement in a straight-line under null external forces? Could Aristotle[7] somehow be also right, even if Galileo and Newton were right? We will address these questions soon. For now, all motion must be seen in relation to an observer, assumed located at the centre of the coordinate system, and we will always count on that.

3. The angular law: angular momentum

Let us consider $\underline{r}$, *m* and $\underline{v}$ as the fundamental "entities" for the description of motion. It would be also possible to consider linear momentum $\underline{p} = m.\underline{v}$, for example, as a fundamental "entity", but it is our preference to maintain the separation between *mass*, considered an inertial property, in the present context, from *velocity*, which is a cinematic entity. We will also consider here mass as a constant in time, since we are obviously dealing with motion from the perspective of solid objects. Once we establish this, we should be able to derive the laws governing the state of motion based only on these fundamental "entities", plus an eventual interference which we will describe as the total force $\underline{F}$. The only way for changing the state of motion of our system, which in part can be identified by the constancy of its angular momentum $\underline{L}$, will be through the action of such a force. The next figure will help us to visualize the idea:

[6] In the Hamiltonian $H = p_q \,.\, [d/dt]q - L$, where L is the Lagrangian, the terms $p_q .[d/dt]q$ can be seen as 2 times the kinetic energy in the direction of the generalised coordinates q.

[7] We may remember that Aristotle defended "something" should be responsible for the movement of a body through the "free" space, a kind of "engine" intrinsic to the body, that should push it forward in the space. As we know, Galileo rejected such a view and stated that no force at all is needed for a body to move at a constant velocity. Around 50 years later this turned to be the first law of Newton.

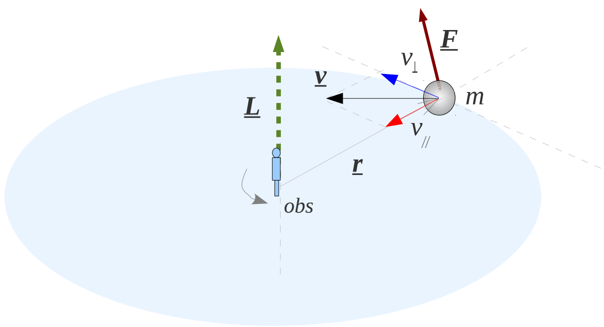

Fig. 5 The fundamental "entities" *m*, $\underline{r}$ and $\underline{v}$ expressing the present state of motion of mass *m*, and the respective angular momentum $\underline{L}$ as an "entity" representative of such a state. A generic force $\underline{F}$ applied to *m* is the only way for such a state to be changed.

We know from the very beginning that the general law of conservation of angular momentum must hold. This means that in the absence of a component of $\underline{F}$ perpendicular to the observer, angular momentum $\underline{L}$ will not be changed. That is, we need $\underline{r} \times \underline{F} = (r \,.\, F_\perp) \,.\, \hat{\underline{u}}$ not to be null if we want to induce a change into the angular momentum of the system. Notice that $\hat{\underline{u}}$ is a *versor* perpendicular to the plane defined by $\underline{r}$ and $\underline{F}$. We may of course synthesize all this through the well known equation of torque:

$$\underline{r} \times \underline{F} = d\,\underline{L}\,/dt \qquad (7)$$

or, more explicitly:

$$\begin{aligned} \underline{r} \times \underline{F} &= d\{\underline{r} \times m.\underline{v}\}/dt \qquad (8)\\ &= [d\underline{r}/dt] \times m.\underline{v} + \underline{r} \times [d(m.\underline{v})/dt] \\ &= m\,.[d\underline{r}/dt] \times \underline{v} + m\,.\,\underline{r} \times [d\underline{v}/dt] \\ &= m\,.[d\underline{r}/dt] \times \underline{v} + m\,.\,\underline{r} \times \underline{a} \end{aligned}$$

Obviously, $\underline{a}$ is the acceleration; and the term $[d\underline{r}/dt] \times \underline{v}$ does not have to be null, so it must be kept in the equation. In a pure circular motion, however, this term is $m.\underline{v} \times \underline{v} = \underline{0}$ and this equation reduces to:

$$\underline{r} \times \underline{F} = m\,.\,\underline{r} \times \underline{a} \qquad (9)$$

Since only the perpendicular components of $\underline{F}$ and $\underline{a}$ and the variation of $\underline{r}$ perpendicular to $\underline{v}$ contribute for these cross products, we may obviously write:

$$r\,.\,F_\perp = m\,.\,[dr/dt]\,.\,v\,.\,\sin\beta + m\,.\,r\,.\,a_\perp \qquad (10)$$

Where β is the angle between $d\underline{r}/dt$ and $\underline{v}$. Finally, by dividing by $r \neq 0$, we will deduce the general expression:

$$F_\perp = m.(\sin\beta).\,[dr/dt].v\,/\,r \;+\; m\,.\,a_\perp \qquad (11)$$

Force equals mass times acceleration. That is, we may expect the angular motion to accelerate not only by an amount $a_\perp$ due to some angular force, but also by the amount $(\sin\beta).[dr/dt].v/r$ when moving the mass perpendicular to its present velocity vector. This last behaviour results in an *angular* force produced by a radial movement (somehow the opposite effect of a centrifugal force being produced by an angular movement). This is obviously part of the mechanism of conservation of angular momentum. By means of this term, the two dimensions ($_{//}$) and ($_\perp$) become interconnected, thus dependent on each other. One can from now on exchange radial energy with angular energy and vice-versa. In a pure circular motion this force will be null, of course.

From this we understand that angular momentum conservation allows us to sense how the plane state of motion defined by $\underline{r}$ and $\underline{v}$ may be stable. It depends on the mass involved, of course. Masses seem not to have any special property responsible for defining orbits, but masses do obviously have something that makes them maintain those orbits, or their states of motion. We suspect this is why the idea of attributing the mass as the origin of gravitational forces have forever been questionable and, in certain cases, replaced by the idea of a constraint or deformation of space or *space-time*. Could it be that masses have no such property as that of attracting each other?

4. The radial law: deriving the centrifugal force

From the last section we have understood the effects of the perpendicular component of the total force $\underline{F}$ applied to our system, while considering $\underline{r}$, *m* and $\underline{v}$ the only "entities" needed for describing its motion. We may now wonder about the effects of the other component of $\underline{F}$, parallel to the observer. Such effects, however, in order to be consistent with the previous considerations, should be also expressed based on the same fundamental "entities", that is $\underline{r}$, m and $\underline{v}$. Also, they should express the same law of physics, even if this time "projected" onto the radial dimension. We assume, therefore, that both the angular law and the radial law must in effect come from a more general and single law of motion, which in the first case is projected through the "angular" dimension, while in the second case onto the "radial" dimension. And, since the mathematical principles should be the same, also the equations expressing such principles should be the same. In the "radial" case such an equation must represent a "projection" onto $\underline{r}$, an operation which is obviously related to the dot product, therefore we simply decide to use again equation (8) but this time with the cross product replaced by the dot product. That is, with the "rotational" effect replaced by a "projectional" effect. Our radial law will consequently be written in the form:

$$\underline{r} \,.\, \underline{F} = d\{\underline{r} \,.\, m.\underline{v}\}/dt \qquad (12)$$

Which can be interpreted as: the projection of $\underline{F}$ along the radial dimension will be responsible for the variation of the projection of the linear momentum ($m.\underline{v}$) along the same dimension. It does not sound bad, at least. So, let us now develop this equation using the same steps of the previous case.

$$\begin{aligned} \underline{r} \,.\, \underline{F} &= [d\underline{r}/dt] \,.\, m.\underline{v} + \underline{r} \,.\, [d(m.\underline{v})/dt] \qquad (13) \\ &= m \,.\, [d\underline{r}/dt] \,.\, \underline{v} \;+ m \,.\, \underline{r} \,.\, [d\underline{v}/dt] \\ &= m \,.\, [d\underline{r}/dt] \,.\, \underline{v} \;+ m \,.\, \underline{r} \,.\, \underline{a} \end{aligned}$$

This, of course, is a scalar equation that we may also write in the following form, in order to use our definition of perpendicular and parallel components:

$$r \,.\, F_{//} = m.(\cos\beta).\, [dr/dt].v \;+ m \,.\, r \,.\, a_{//} \qquad (14)$$

Where β is again the angle between $d\underline{r}/dt$ and $\underline{v}$. We would dare to interpret this as: the total energy of the mass m along $\underline{r}$ is the potential energy ($m.r.a_{//}$) plus a term related to the kinetic energy given by $m.(\cos\beta).\ [dr/dt].v$, which reduces to $m.v^2$ in the case of a circular motion ($\beta=0$). Somehow this reminds us of Hamiltonian. Also, such an equation and such a definition seem to suggest that kinetic energy could have the form $m.v^2$ instead of the standard $½m.v^2$. It could be an interesting proposal too: the total energy contained in a mass m suddenly transformed into light, therefore travelling at the speed of light (c), would simply be given by:

$$E = m \,.\, c^2$$

But let us now divide equation (14) by $r \neq 0$, as we did in the previous case, to obtain:

$$F_{//} = m.(\cos\beta).\, [dr/dt].v \,/\, r \;+\; m \,.\, a_{//} \qquad (15)$$

From which we can obviously deduce that the total parallel force comes from two contributions: a radial force $m.(\cos\beta).[dr/dt].v/r$ coming from an angular displacement in respect to $\underline{r}$, and a force due to an acceleration parallel to $\underline{r}$. Both terms can of course be either positive or negative, thus centrifugal forces can also be perceived as negative. In the case of gravitation, the second term is obviously negative, and due to Newton's gravitational force (centripetal).

5. Some other interesting consequences

The first surprise resulting from equation (15), which seems to contradict both Galileo and Newton, is the fact that a force may exist even in the absence of a real acceleration imposed to the mass. The simple fact of moving at a constant velocity, no matter where to, implies the existence of a force acting on the body along its position vector with an intensity given by $m.v^2/r$. This force is in truth an *energy per unit of length* moving along the direction connecting the body to the observer. Could Aristotle be somehow right too? Notice that this force could also be positive or negative, depending on the relative intensities of $v_\perp$ and $v_{//}$, as can be found if we count with the fact that $v^2 = v_\perp^2 + v_{//}^2$ and rewrite equation (15) like this:

$$F_{//} = m \,.\, v_\perp^2/r \;\pm\; m \,.\, v_{//}^2/r \;\pm\; m \,.\, a_{//} \qquad (16)$$

Interesting is also the fact that for an observer located at infinity, these "centrifugal" components vanish and equations (15) and (16) reduce to Newton's second law. This makes us think that perhaps Newton's second law is somewhat incomplete in the sense that it is a simple derivative of the linear momentum ($m.\underline{v}$), thus valid only in the case of a very distant observer, that is, in those cases where angular momentum reduces to linear momentum due to the "lack of $\underline{r}$".

A third consequence, for example, is related to the case of absence of total forces ($F_{//}= 0$), which can be achieved in the situation of orbiting; that is, when the centrifugal force compensates the centripetal force resulting in:

$$0_{//} = m \,.\, v^2/r + m \,.\, a_{//} \;=>\; m \,.\, v^2/r \;=\; - m \,.\, a_{//}$$

This, of course, justifies the sensation of absence of gravity in orbiting bodies. No accelerometer will measure any acceleration while orbiting. When a body starts an orbit, the body around which it is orbiting automatically "disappears". In a certain way, orbiting is like getting safe from falling into that body. Thus, orbiting means tranquillity, we wonder.

6. The Geometric Law of Motion

From these results we notice that motion can be seen in a certain geometric perspective, which here we try to represent with the help of the next figure. In a certain sense, motion seems to follow the geometric space of a spheric surface crossed by a rotating plane, with the angular momentum as its axis of symmetry.

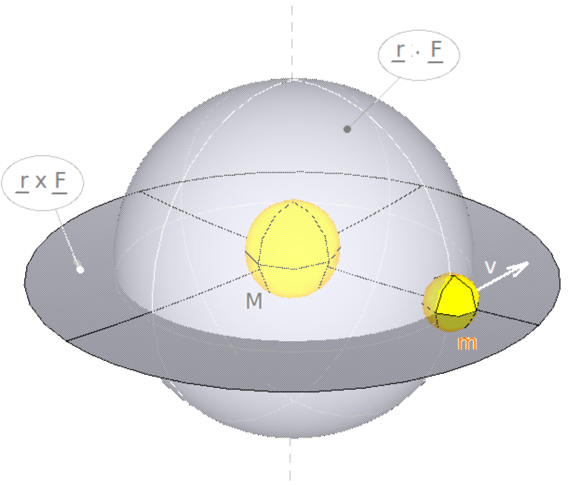

Fig. 6 The intrinsic geometry of motion, based on what we have expressed in the previous sections: a spheric surface driven by the term $\underline{r} \,.\, \underline{F}$, crossed by a rotating plan dominated by the term $\underline{r} \times \underline{F}$. The two dimensions are, however, interconnected by the centrifugal force and the angular momentum. This may perhaps be seen as a single Geometric Law of Motion.

Notice that the sphere shrinks or expands as mass m approaches or moves away from the center of rotation. But this also tends to be accomplished by an increase or a decrease in the speed of rotation depending on the case. What drives the changes in the perpendicular (angular) dimension is $\underline{r} \times \underline{F}$. What drives the changes in the parallel (radial) dimension is $\underline{r} \,.\, \underline{F}$. However, the two dimensions are obviously not independent, they are interconnected by the *centrifugal* and the *angular* forces, which transfer angular dimension into radial dimension and vice-versa. This happens to be a coupled system, where both dimensions are thus coupled by means of the mechanism of angular momentum. Thus, motion is a single thing, happening at the same time.

We may also infer from this that a spinning object will tend to be more stable than a non spinning one, since its radial dimension is a constant and its structure is protected against any changes by its own angular momentum. From this we infer that the two laws previously referred should be joined together. We may try to write them as the conjunction of the two equations:

$$\begin{cases} \underline{r} \times \underline{F} = d\{\underline{r} \times m.\underline{v}\}/dt \\ \underline{r} \,.\, \underline{F} = d\{\underline{r} \,.\, m.\underline{v}\}/dt \end{cases} \quad (17)$$

Or, in another manner, if we want to see it as the components of a new abstract "modifier" vector:

$$\underline{\mathcal{M}} = (\mathcal{M}_{\perp} , \mathcal{M}_{//}) = (\underline{r} \times \underline{F} \;,\; \underline{r} \,.\, \underline{F}) \quad (18)$$

If we use Geometric Algebra, however, all this can in fact be described as the *geometric product* of vector $\underline{r}$ by vector $\underline{F}$, which is simply written as[8]:

$$\underline{\mathcal{M}} = \underline{r}\,\underline{F} = \underline{r} \,.\, \underline{F} + \underline{r} \wedge \underline{F} \quad (19)$$

Where the cross product has been replaced by the outer product only for convenience. Curiously, those expert in Geometric Algebra consider this a single entity which they call a "*spinor*": the conjunction of a *scalar* and a *bivector*, which is a n-dimensinal *rotator* of vectors, in this case a two dimensional rotator. So, if we now express this equation in terms of our

[8] You may learn more about the geometric product in:
Jaap Suter, (March 12, 2003), "*Geometric Algebra Primer* ": http://www.jaapsuter.com/2003/03/12/geometric-algebra/

fundamental "entities", could the following equation represent the *Geometric Law of Motion*?:

$$\underline{r}\,\underline{F} = d\{\underline{r}\,.\,m.\underline{v}\}/dt \;+\; d\{\underline{r} \wedge m.\underline{v}\}/dt$$
$$\underline{r}\,\underline{F} = d\{\underline{r}\,.\,m.\underline{v} \;+\; \underline{r} \wedge m.\underline{v}\}/dt \qquad (20)$$

Or, in a more compact form, using only the *geometric product* of vectors:

$$\underline{r}\,\underline{F} = d\{\underline{r}\,m.\underline{v}\}/dt \qquad (21)$$

7. Conflicting concepts of centrifugation?

From what we have said, everything suddenly seems to turn beautiful and clear. But we must, even so, remain sufficiently critical in order to test this against other concepts. Even if broking the peace between Aristotle, Galileu and Newton in a 300 years war would sound great, some questions arise, for instance concerning the usual definition of the centrifugal force due to Huygens. Is the centrifugal force really given by $m.v^2/r$? Or is there any other expression more precise for it? Huygens force comes from empirical science. Could it express only a particular aspect of the effects of centrifugation forces? Here we present two different perspectives leading to different conclusions, and we could not yet decide which to choose.

Frequently the centrifugal force is described in vectorial terms as a double crossed product related to the angular speed vector $\underline{\omega}$:

$$\underline{F}_{c1} = -\,m\,.\,\underline{\omega} \times \{\underline{\omega} \times \underline{r}\} \qquad (22)$$

Since $\underline{\omega}$ is a pseudo-vector, $\{\underline{\omega} \times \underline{r}\}$ is a real vector, so, $\underline{\omega} \times \{\underline{\omega} \times \underline{r}\}$ is again a pseudo-vector. The centrifugal force defined this way seems not to be a real thing. But if we try representing it in terms of our fundamental "entities" of motion, which are unambiguous vectors, we will find out, with little manipulation, that a centrifugal force can also be expressed as a *real vector* by means of the equation:

$$\underline{F}_{c2} = \underline{v} \times \{\underline{r} \times m.\underline{v}\}\,/\,r^2 \qquad (23)$$

$$\underline{F}_{c2} = \underline{v} \times \underline{L}\,/\,r^2 \qquad (24)$$

However, by this definition $\underline{F}_{c2}$ will always be perpendicular to $\underline{v}$, that is, pointing away from the center of curvature of the "space" in that moment, supposing $\underline{v}$ as being always tangential to such a "curvature". So, this is a centrifugal force in respect to the center of curvature (CC), independent of the observer, as shown in the next figure. Its intensity is the same as the previous one, and given by:

$$F_{c2} = F_{c1} = m\,.\,v.v_{\perp}\,/\,r \qquad (25)$$

Notice that this force $\underline{F}_{c2}$ can have components not only along the $\underline{r}$ direction (contrary to the common sense, and $\underline{F}_{c1}$) but also along the angular dimension. This seems to express quite interestingly the idea of depending solely on the "space curvature" induced in the movement by whatever types of forces or constraints. The direction of this centrifugal force is only dependent on the curvature, and not on the location of the observer, for a fixed distance to the observer. This is at odds with the previous formulation. Of course both descriptions only match Huygens force intensity when the *observer* is located at the CC. It seems there is a conflict somewhere...

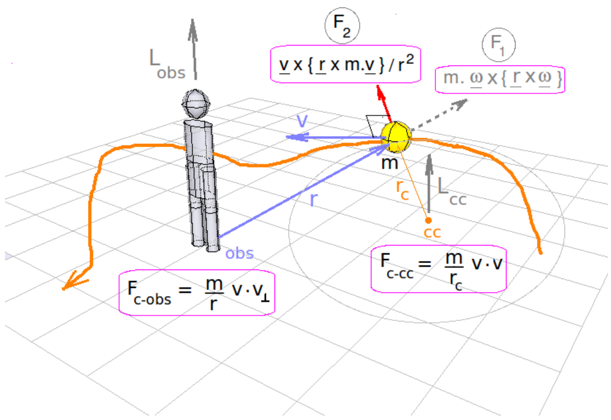

Fig. 7 Conflicting concepts of a centrifugal force?

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