A non-linear low-Reynolds number $k$-$\varepsilon$ turbulence model for drag reducing viscoelastic pipe flow

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ABSTRACT

An anisotropic low Reynolds number $k$-$\varepsilon$ turbulence model is proposed for viscoelastic fluids and is used to predict drag reducing pipe flow. The Reynolds stress is given as an expansion of up to second order of the time average rate of deformation and vorticity tensors and its good performance is assessed by analysing limiting behaviour and comparisons with literature data. The model includes variable turbulent diffusion for $k$ and $\varepsilon$, and is able to predict enhanced turbulence anisotropy with increasing strain-hardening of the Trouton ratio and shear-thinning of the shear viscosity.

INTRODUCTION

Turbulent flows of viscoelastic fluids occur in many relevant industrial situations, as in oil-well drilling. Early turbulence models for viscoelastic fluids, were based on the use of non-universal laws of the wall or on ad-hoc modifications of damping functions, but lacked generality and a relationship with fluid rheology [1]. Research in the 1980's and 1990's had a decisive role in establishing the link between drag reduction, and its corresponding flow characteristics, with rheology, in particular the extensional viscosity [2,3]. This is the driving force behind recent developments of new turbulence models for viscoelastic drag reducing fluids [1,4] which the current work continues.

Here, we propose a non-linear formulation of the $k$-$\varepsilon$ model for viscoelastic fluids and we apply it to predictions of fully-developed pipe flow for which there are adequate experimental data.

GOVERNING EQUATIONS AND TURBULENCE MODEL

For fully developed pipe flow the momentum equation to be solved is written as

$$\frac{1}{r} \frac{d}{dr} \left[ r \left( \frac{P}{\rho} \frac{dU}{dr} + 2 \mu' s_{xr} - \tau_{uv} \right) \right] = \frac{\tau_{xr}}{\rho} = 0$$

(1)

where capital letters or overbars denote time-averaged quantities and fluctuations are represented by small letters or a prime. In Eq. (1) $s_{xr}$ stands for the fluctuating $xr$ component of the rate of deformation tensor and it appears combined with the fluctuating viscosity. This new stress was recently modeled by Cruz et al [7]. The Reynolds shear stress is given by the following non-linear expression

$$-\rho \mu \mu_{ij} = 2 \nu \tau S_{ij} - \left( f_{1} \beta_{2} + C_{w} f_{3} \beta_{2,wall} \right) \times k \left( S_{ik} S_{kj} - \frac{1}{6} S_{ij} \delta_{ij} \right) - \left( f_{2} \beta_{3} + C_{w} f_{3} \beta_{3,wall} \right) \times k \left( W_{ik} S_{kj} - \frac{1}{6} S_{ij} W_{ij} \right) - \frac{2}{3} k \delta_{ij}$$

(2)

where $S_{ij}$ and $W_{ij}$ are the average rate of deformation and vorticity tensors, respectively.

The eddy viscosity $\nu_{T}$ is given by the Prandtl-Kolmogorov model, modified for wall and low Reynolds number effects, $\nu_{T} = C_{\mu} \mu \kappa^{2} / \varepsilon$ and imposes the need to calculate $k$ and $\varepsilon$ via adequate transport equations. The transport equation of $k$ is

$$\frac{1}{r} \frac{d}{dr} \left[ \rho \frac{dU}{dr} \right] + \frac{2}{3} \frac{dU}{dr} - \frac{dU}{dr} - \frac{dU}{dr}$$

$$- \frac{D}{\rho} = 0$$

(3)

and the transport equation for the modified dissipation rate ($\tilde{\varepsilon}$) is given by

$$\frac{1}{r} \frac{d}{dr} \left[ \frac{\rho' f_{s}}{\rho_{s}} \frac{dU}{dr} \right] + \frac{\rho' f_{s} \tilde{\varepsilon}}{\rho_{s}} + \rho' C_{\varepsilon} \frac{\tilde{\varepsilon}}{k} - \frac{\rho' C_{\varepsilon} \tilde{\varepsilon}}{k} \tilde{\varepsilon}^{2}$$

$$+ \rho' \nu_{T} \left( 1 - f_{\mu} \frac{\alpha^{2} U_{\infty}}{\alpha^{2}} \right) + C_{\mu} \frac{\nu_{T}}{\alpha_{\kappa}} \frac{dU}{dr} \frac{dU}{dr} = 0$$

(4)
In Eqs. (3) and (4) the use of function \( f_\ell \) allows a better control of turbulent diffusion across the pipe [5]. Here, the function of Park and Sung [6] is adopted.

\[
f_\ell = 1 + 3.5 \exp \left( - \left( \frac{R_f / 50}{150} \right)^2 \right)
\]

(5)

This function goes together with an increase of \( \sigma_k \) and \( \sigma_\varepsilon \) to 1.1 and 1.3, respectively. Improved predictions are also obtained by using a lower value of \( C_\mu = 0.084 \). The turbulent Reynolds number is defined as \( R_f = k^2 (\bar{\nu} / \bar{c}_v) \).

The damping function \( f_\mu \) was modelled by Cruz and Pinho [1] for viscoelastic fluids and is used here with \( C= 70 \). The wall coordinate is based on the wall viscosity, ie, \( y^+ = u_c y / \nu \), \( n \) and \( K_y \) are the exponent and consistency indices of the power law fitted to the shear viscosity and \( p \) and \( K_e \) are the exponent and consistency indices of the power law fitted to the Trouton ratio.

The non-Newtonian stress \( 2 \mu s_{xR} \) is modelled according to Cruz et al [4]

\[
2 \mu s_{xR} = \bar{C} \frac{K_e K_y}{A_k^{p-1}} \left[ \frac{\rho C_{ij} f_{ij} k^2}{2 \bar{w}} \left( \frac{\partial U}{\partial x} \right)^2 \right]^{\frac{p+n-2}{2}}
\]

(6)

\[
\times \left[ \frac{C_f \nu k^2}{\bar{c}_v} \times \frac{1}{L_c} \times \left( \frac{\partial U}{\partial x} \right)^2 \right]
\]

with \( \bar{C} = (1 + C_p)^{n+2} \), \( L_c = u_R^3 / \bar{c}_v \), \( \alpha = 4 \) and

\[
u^2_R = k \left[ \exp \left( - \left( \frac{u_c^2}{T} \right)^\alpha \right) \right] - 1
\]

Other damping functions are \( f_1 = 1.0 \) and \( f_2 = 1 - 0.3 \exp \left( - \frac{y^+}{T} \right) \) and model parameters are listed in Table I.

Table I- Some parameters of the turbulence model

<table>
<thead>
<tr>
<th>( C_{ij} )</th>
<th>( C_f )</th>
<th>( C_{ij} )</th>
<th>( A_k )</th>
<th>( A_2 )</th>
<th>( C_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.45</td>
<td>1.90</td>
<td>1.0</td>
<td>10</td>
<td>0.45</td>
<td>2.4</td>
</tr>
</tbody>
</table>

The two new terms of Eq. (2) give the anisotropic behaviour of the model. The model is of second order in \( S_{ij} \) and \( W_{ij} \) [7], and is based on the non-linear model of Park et al [8], where all functions not discussed here can be found. The exceptions are the Helmholtz-type elliptic function \( f_w \) adopted from [9], and given by

\[
 f_w = 1 - \exp \left( \frac{-y^+}{42} \right)
\]

and the corrective functions \( f_{n1} \) to \( f_{n3} \).

The model is able to predict enhancement of anisotropies of normal Reynolds stresses with drag reduction even in the absence of the corrective functions, ie, with a decrease in \( n \) and an increase in \( p \). The quality of the predictions are significantly improved by the following modifications of coefficients \( \beta_1 \) and \( \beta_{i, wall} \).

\[
f_{n1} = \frac{u_2}{K_e} \left[ 0.9 u + 0.9 a \right]^{\frac{1}{1.7a}}
\]

(7)

\[
f_{n2} = \left[ f_{n,21} + f_{n,22} \right]^{3.6 + a}
\]

(8)

\[
f_{n3} = \frac{0.5a}{\frac{1 - 0.1a}{0.8(K_e + a) + 0.2}}^{\frac{1}{2}}
\]

(9)

\[
f_{n,21} = \left( 0.8 + 0.0048 R_f \right)^{0.7}
\]

(10)

\[
f_{n,22} = \frac{K_e a (\sigma_v^+)^{0.01}}{1.95 - p + n}
\]

(11)

\[
f_{n,21} = 8 \times 10^6 \left[ \frac{0.94}{n} \left( 0.85(p-n) + 0.01 \right) + \frac{0.15 p}{n} \right]^{12.6}
\]

(12)

The roles of these functions are: \( f_{n3} \) ensures realizability, while \( f_{n1} \) and \( f_{n2} \) correct the behaviour of the azimuthal and radial normal Reynolds stresses, respectively. The latter is accomplished separately near the wall and away from the wall with \( f_{n,21} \) and \( f_{n,22} \), respectively.

**RESULTS AND DISCUSSION**

The corrective functions were tuned against experimental data of Escudier et al [10] and Presti [11] for aqueous solutions of PAA, CMC, xanthan gum and a blend of CMC and xanthan gum as well as for theoretical fluids illustrating limiting behaviour (effect of \( n \) for \( p = 1 \) and effect of \( p \) for \( n = 1 \)). Here, only the results for 0.125% PAA and the analysis of the effect of \( p \) (strain-hardening of the Trouton ratio) are analysed.

For the PAA solution the model predicts \( f = 0.00681 \) at \( Re_p = 42900 \) against an experimental value of 0.00689 at \( Re_p = 42110 \), hence the same drag reduction of 69% relative to Blasius equation. This matches predictions by Cruz et al [4] and the corresponding mean velocity profiles in wall coordinates are shown in Fig. 1a, illustrating the quality of the agreement. The prediction of \( k \) is shown in Fig. 1b and is as good...
as previously [4], but now the anisotropy of normal Reynolds stresses is well captured.

\[
\begin{align*}
\bar{u}' &= 11.7 \ln y + 17.0 \\
\bar{u}' &= 2.5 \ln y + 5.5
\end{align*}
\]

Fig. 1. Comparison between predictions and experiments for 0.125% PAA at \(Re = 42900\): (a) mean velocity; (b) Normal Reynolds stresses.

Figs. 2a, 2b and 2c show the effect of \(p\) on the predictions of the normalised axial, radial and azimuthal Reynolds stresses, respectively. It is quite clear the large increase in \(u'^+\) when \(p\) increases from 1 to 1.55 (corresponding to maximum drag reduction), compared with the small increase in \(v'^+\) and \(w'^+\), hence the enhancement of Reynolds stress anisotropy with drag reduction. The increase in \(v'^+\) with \(p\) is less intense than the decrease in the friction velocity, so the dimensional \(v'\) really decreases as is expected.

REFERENCES


Fig. 2. Effect of \(p\) on the axial (a), radial (b) and azimuthal (c) normal Reynolds stresses for \(n= 1\) and \(Re = 42800\).