

# FORCED CONVECTION IN A CHANNEL WITH IMPOSED CONSTANT WALL TEMPERATURE FOR THE SIMPLIFIED PHAN-THIEN—TANNER FLUID

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## Abstract

The fully-developed thermal and hydrodynamic steady laminar channel flow of the SPTT fluid is investigated for a constant wall temperature boundary condition, neglecting axial conduction. Two limiting conditions were identified and their solutions are presented: (1) the equilibrium between axial convection and transverse conduction of thermal energy, and (2) equilibrium of transverse conduction of energy and energy production by viscous dissipation. For the first problem the solution was obtained by a successive approximation method, yielding results within 0.3% of the exact solution, for the second problem the final result is exact.

## 1. INTRODUCTION

The simplified Phan-Thien—Tanner (SPTT) constitutive equation, given in Eq. (1), is a reduced version of the full PTT model of Phan-Thien and Tanner [1]

$$\left(1 + \frac{\varepsilon\lambda}{\eta} \text{tr}\tau\right) \tau + \overset{\nabla}{\tau} = 2\eta\mathbf{D} \quad (1)$$

where  $\overset{\nabla}{\tau}$  stands for Oldroyd's upper convected derivative of the stress tensor  $\tau$ , as defined by Eq. (2),  $\lambda$  is a relaxation time,  $\eta$  is a viscosity coefficient,  $\mathbf{D}$  is the rate of strain tensor and the stress-coefficient function, which has an exponential form (Phan-Thien [2]) was here linearised because the channel flow is a weak flow according to Tanner's [3] classification.

$$\overset{\nabla}{\tau} = \frac{D\tau}{Dt} - \tau \cdot \nabla\mathbf{u} - \nabla\mathbf{u}^T \cdot \tau \quad (2)$$

In Eq. (1)  $\varepsilon$  is a parameter that imposes an upper limit to the elongational viscosity that is proportional to the inverse of  $\varepsilon$  (p. 227 in [4]), and the upper-convected Maxwell model, which has an unbounded elongational viscosity in simple extensional flow, is recovered when  $\varepsilon = 0$ .

The analytical hydrodynamic solution of the SPTT channel flow was derived by Oliveira and Pinho [5] who subsequently performed the analysis of the corresponding heat transfer problem for imposed constant wall heat flux (Pinho and Oliveira [6]) accounting for viscous dissipation. The present work extends the analysis of Pinho and Oliveira [6] to the relevant thermally and hydrodynamically fully-developed, steady, laminar channel flow with an imposed constant wall temperature: (1) that of fully developed thermal conditions in the presence of viscous dissipation, and (2) the asymptotic problem in a channel in the absence of viscous dissipation.

## 2. FORMULATION OF THE PROBLEM

Fluid properties are taken as constants and isotropic, independent of temperature and kinematics, and Fourier's law of heat conduction is assumed to hold. These are standard assumptions in heat transfer of non-Newtonian fluids (Bird et al [7], Tanner [3] and Agassant et al [8]) and consequently the hydrodynamic and thermal problems become fully decoupled.

### 2.1 Hydrodynamic solution

The velocity profile for the channel flow was derived by Oliveira and Pinho [5]

$$u^* \equiv \frac{u}{U} = \frac{3}{2} \frac{U_N}{U} [1 - y^{*2}] \left\{ 1 + 9\epsilon We^2 \left( \frac{U_N}{U} \right)^2 [1 + y^{*2}] \right\} \quad (3)$$

where starred lengths are scaled with the channel half-width ( $y^* = y/H$ ). The Weissenberg number  $We = \lambda U/H$  is a measure of the level of elasticity in the fluid and is based on the cross-sectional average velocity  $U$ .  $U_N$  is the average velocity for a Newtonian fluid flowing under the same pressure gradient  $dp/dx$  [8]

The radial profiles of the shear stress  $\tau_{xy}$  and the corresponding shear rate  $du/dy$  are:

$$\tau_{xy}^* \equiv \frac{\tau_{xy}}{3\eta U/H} = -\frac{U_N}{U} y^* ; \quad \frac{du/dy}{3U/H} = -\frac{U_N}{U} y^* \left[ 1 + 18\epsilon We^2 \left( \frac{U_N}{U} \right)^2 y^{*2} \right] \quad (4)$$

It will be convenient for the analysis to define a modified nondimensional group as  $a \equiv 9\epsilon We^2 (U_N/U)^2$ , a measure of both the extensional and the elastic properties of the fluid.

### 2.2 Heat transfer procedure

The equation to be solved is the nondimensional energy transport equation for the plane channel flow with viscous dissipation, no internal heat sources and negligible axial conduction

$$\frac{\partial}{\partial y^*} \left( \frac{\partial T^*}{\partial y^*} \right) - 4Br\tau_{xy}^* \frac{du^*}{dy^*} = \frac{Pe}{2} u^* \frac{\partial T^*}{\partial x^*} \quad (5)$$

where the normalisation of temperature was based on  $T_w$  and  $\bar{T}_i$ , representing the wall temperature and the bulk temperature at the inlet, respectively.

$$T^* \equiv \frac{T_w - T}{T_w - \bar{T}_i} \quad (6)$$

The relevant boundary conditions are symmetry at the channel centreline and an imposed constant temperature at the wall, given in nondimensional form by Eq. (7)

$$\left. \frac{\partial T^*}{\partial y^*} \right|_{y^*=0} = 0 ; \quad T^*_{y^*=1} = 0 \quad (7)$$

In Eq. (5)  $Pe$  is the Peclet number defined by  $Pe \equiv Pr.Re = \rho c U 2H/k$ , and the Brinkman number  $Br$  (following the original definition of Dryden, see [9]) is  $Br \equiv \eta U^2 / k(T_w - \bar{T}_i)$ .

For Newtonian fluids the solution of Eq. (5) requires transformation of variables followed by such techniques as separation of variables or L ev eque analysis, and was obtained by Brinkman [10] and Brown [11], amongst others. For the SPTT fluid, the solution of Eq. (5) is rather more complex, because of the dependence on the Weissenberg number, and is not attempted here. We

concentrate instead on obtaining two asymptotic solutions. For that it is convenient to normalise differently the axial convective term, and we introduce a nondimensional temperature  $\theta$  as

$$\theta = \frac{T_w - T}{T_w - \bar{T}} = \frac{T^*}{\bar{T}^*} \quad (8)$$

where  $\bar{T}$  is the bulk temperature, so that the axial temperature gradient becomes:

$$\frac{\partial T^*}{\partial x^*} = \bar{T}^* \frac{\partial \theta}{\partial x^*} + \theta \frac{d\bar{T}^*}{dx^*} \quad (9)$$

One possible advantage of using  $\theta$  is that  $\partial\theta/\partial x^* = 0$  in fully-developed thermal flow situations. Using this normalised temperature an alternative dimensionless energy equation is

$$\frac{\partial}{\partial y^*} \left( \frac{\partial T^*}{\partial y^*} \right) = \frac{Pe}{2} u^* \theta \frac{d\bar{T}^*}{dx^*} + 4Br\tau_{xy}^* \frac{du^*}{dy^*} \quad (10)$$

One of the important quantities to be obtained is the heat transfer coefficient under the form of a Nusselt number, which will also be useful to verify convergence of the iterative procedure. The normalised heat transfer coefficient is

$$Nu \equiv \frac{-2 \left. \frac{\partial T^*}{\partial y^*} \right|_{y^*=1}}{\bar{T}^*} \quad (11)$$

With  $Br = 0$  Eq. (10) is solved with a mixed analytical/numerical procedure which requires application of a successive approximation method (see p. 96, Kays and Crawford [12]). It starts, as a first approximation, with the  $\theta$  profile for constant wall heat flux (Eq. (32) in Pinho and Oliveira [6]), which is substituted on the right-hand-side of Eq. (10) together with the profiles of  $u^*$  and  $\tau_{xy}^*$ . Eq. (10) is then readily integrated for  $T^*$  to obtain a corrected temperature profile and the determination of  $\theta$  and  $Nu$ . The new  $\theta$  is fed into Eq. (10) for a new integration and this procedure is stopped when the Nusselt number variation from successive approximations falls below a prescribed tolerance. This constitutes the first asymptotic solution in this paper, termed the "fully-developed thermal flow with negligible viscous dissipation". A second asymptotic case is obtained when  $d\bar{T}^*/dx^*$  vanishes in Eq. (10), whereby viscous dissipation is balanced by radial conduction only. The solution to this problem is easier and is also given as the "equilibrium viscous dissipation flow".

### 3. RESULTS AND DISCUSSION

#### 3.1 Fully-developed thermal flow with negligible viscous dissipation

The successive approximation method was applied until differences from consecutive iteration Nusselt numbers were less than 0.5%. This happened at the end of the third iteration which differed from the second by less than 0.06% in the Weissenberg number range from zero to infinity. For a Newtonian fluid ( $a=0$ ), the Nusselt number at the third iteration simplifies to 7.541, which agrees with the literature (7.541 in p. 155 of [9]). The final expressions for the temperature and Nusselt number are presented below and the corresponding variations analysed from plots. The normalised temperature profile is a long polynomial in  $y^*$  and  $a$  given by

$$T^* = \frac{Pe}{2} \frac{d\bar{T}^*}{dx^*} \left[ 0.6694 \sum_{i=0}^4 \delta_i a^i + 0.8604 y^{*2} \sum_{i=0}^4 \left[ \alpha_i a^i \times \sum_{j=0}^{11} \beta_{ij} y^{*2j} \right] \right] \bigg/ \sum_{i=0}^4 \gamma_i a^i \quad (12)$$

where the coefficients  $\alpha_i$ ,  $\gamma_i$  and  $\delta_i$  are presented in Table 1 and  $\beta_{ij}$  in Table 2.

Table 1- Coefficients  $\alpha_i$ ,  $\gamma_i$  and  $\delta_i$  for channel flow

	0	1	2	3	4
$\alpha_i$	0.7348	3.1757	5.1461	3.7051	1
$\gamma_i$	0.6391	2.8563	4.7902	3.5728	1
$\delta_i$	0.6680	2.9532	4.8985	3.6132	1

Table 2- Coefficients  $\beta_{ij}$  for channel flow.

$j \ i$	0	1	2	3	4
0	-1	-1	-1	-1	-1
1	$4.019 \times 10^{-1}$	$3.580 \times 10^{-1}$	$3.121 \times 10^{-1}$	$2.641 \times 10^{-1}$	$2.138 \times 10^{-1}$
2	$-1.316 \times 10^{-1}$	$-8.722 \times 10^{-2}$	$-4.240 \times 10^{-2}$	$2.902 \times 10^{-3}$	$4.873 \times 10^{-2}$
3	$2.646 \times 10^{-2}$	$5.108 \times 10^{-3}$	$-1.454 \times 10^{-2}$	$-3.235 \times 10^{-2}$	$-4.818 \times 10^{-2}$
4	$-4.688 \times 10^{-3}$	$1.248 \times 10^{-3}$	$5.271 \times 10^{-3}$	$7.325 \times 10^{-3}$	$7.346 \times 10^{-3}$
5	$5.547 \times 10^{-4}$	$-7.411 \times 10^{-4}$	$-1.071 \times 10^{-3}$	$-5.030 \times 10^{-4}$	$8.851 \times 10^{-4}$
6	$-5.047 \times 10^{-5}$	$1.343 \times 10^{-4}$	$2.968 \times 10^{-5}$	$-2.769 \times 10^{-4}$	$-6.938 \times 10^{-4}$
7	$1.769 \times 10^{-6}$	$-1.733 \times 10^{-5}$	$1.772 \times 10^{-5}$	$6.124 \times 10^{-5}$	$7.178 \times 10^{-5}$
8	0	$7.567 \times 10^{-7}$	$-5.087 \times 10^{-6}$	$-4.497 \times 10^{-6}$	$1.066 \times 10^{-5}$
9	0	0	$3.078 \times 10^{-7}$	$-1.072 \times 10^{-6}$	$-3.972 \times 10^{-6}$
10	0	0	0	$1.177 \times 10^{-7}$	0
11	0	0	0	0	$4.137 \times 10^{-8}$

The Nusselt number was found to be given by

$$Nu = 6.8567 \frac{\left[ 1 + \frac{6}{5} a \right] \sum_{i=0}^4 \gamma_i a^i}{\left[ a^5 + 4.4810 a^4 + 8.0360 a^3 + 7.2093 a^2 + 3.2355 a + 0.58111 \right]} \quad (13)$$

The normalised temperature  $\theta$ , defined by Eq. (8), is independent of axial location for thermally fully-developed flow. Its denominator, the bulk temperature, is obtained as usual by integration. The final expression for  $\bar{T}^*$  is mathematically equivalent to

$$\bar{T}^* = \frac{-1}{Nu} \frac{Pe}{2} \frac{d\bar{T}^*}{dx^*} \quad (14)$$

Fig 1 shows the resulting variation of Nusselt number with  $\epsilon We^2$ . The dashed lines help to show the iterative progression of Nusselt number and they show that the final Nusselt number ( $Nu_3$ ) number differs by less than 0.06% from  $Nu_2$ . The Nusselt number varies between two asymptotes: at the limit of vanishing  $\epsilon We^2$  it gives the well-known Newtonian value of 7.541 and, as  $\epsilon We^2$  goes to infinity, it tends to 8.228, thus giving a maximum increase of 9%.

The variation of  $\theta$  with radius is given in Fig. 2 for various values of the parameter  $\epsilon We^2$ . The variation with  $\epsilon We^2$  is mild since both its numerator and denominator depend similarly on  $\epsilon We^2$ , a conclusion similar to that reached in the previous work for the case of imposed wall heat flux [6].

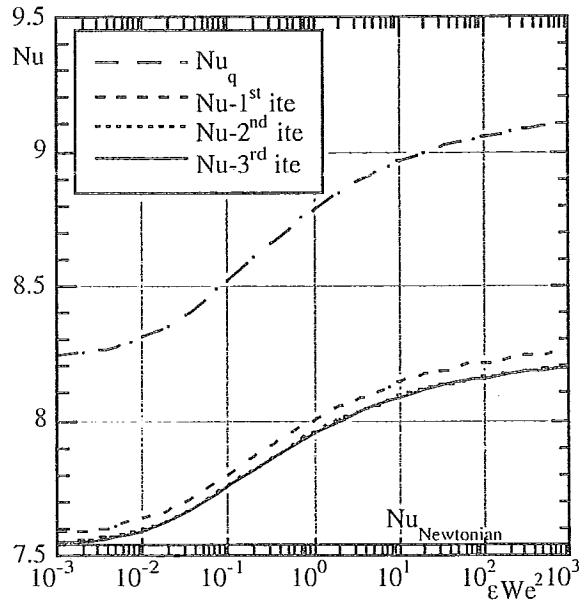


Figure 1- Variation of the Nusselt number with  $\epsilon We^2$ .

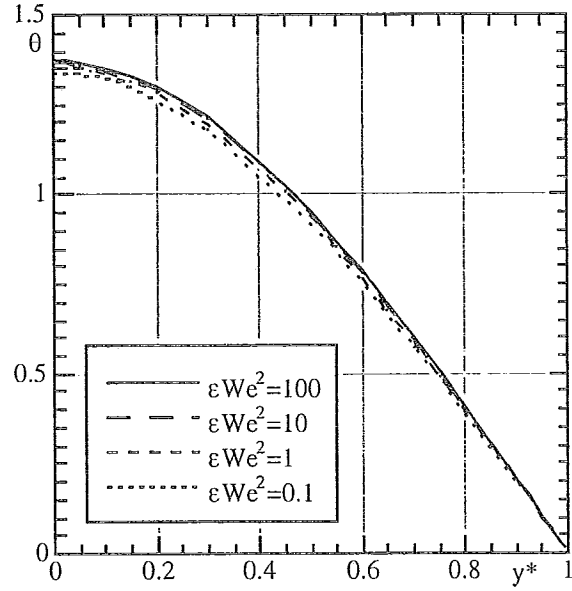


Figure 2- Radial profiles of  $\theta$  as a function of fluid elasticity.

### 3.2 Equilibrium viscous dissipation flow

The double integration of Eq. (5) with  $d\bar{T}^*/dx^* = 0$ , and imposition of the symmetry and wall temperature boundary conditions gives the temperature distribution

$$T^* = 3Br \left( \frac{U_N}{U} \right)^2 \frac{4a(y^{*6} - 1) + 5(y^{*4} - 1)}{20}; \quad \left. \frac{dT^*}{dy^*} \right|_{y^*=1} = 4Br \frac{U_N}{U} \quad (15)$$

and the corresponding dimensionless bulk temperature and heat transfer coefficient become

$$\bar{T}^* = -24Br \left( \frac{U_N}{U} \right)^3 \frac{54a^2 + 110a + 55}{1925}; \quad Nu = \frac{1925 \left( 1 + \frac{6}{5}a \right)^2}{2(54a^2 + 110a + 55)} \quad (16)$$

The Nusselt number expression gives the correct Newtonian value ( $Nu = 35/2$  for  $a=0$ ) [9] and, as with Newtonian fluids, the Nusselt number does not depend on the Brinkman number. As in the equivalent constant wall-heat-flux problem [6], viscous dissipation has a strong impact upon the heat transfer characteristics. For a Newtonian fluid the Nusselt number more than doubles, from 7.541 to 17.5, and this increase, shown in Fig. 3, is accentuated by viscoelasticity.

Radial temperature profiles for these situations are plotted in Fig. 4. An increase in Brinkman number raises the temperature level linearly in the pipe section. Although the influence of viscous dissipation increases with flow elasticity, see Fig. 3, the difference between the local and wall temperatures (recall that  $-\bar{T}^* \propto T - T_w$ ) decreases with  $\epsilon We^2$ , because the flow becomes increasingly shear-thinning and consequently viscous dissipation becomes progressively more localised in the wall region and less pronounced in the core of the flow. Thus, since the heat is generated closer to the wall, it is more easily evacuated without the need to heat the bulk of the flow. The decrease in  $-\bar{T}^*$  with viscoelasticity induces a wider core of uniform temperature.

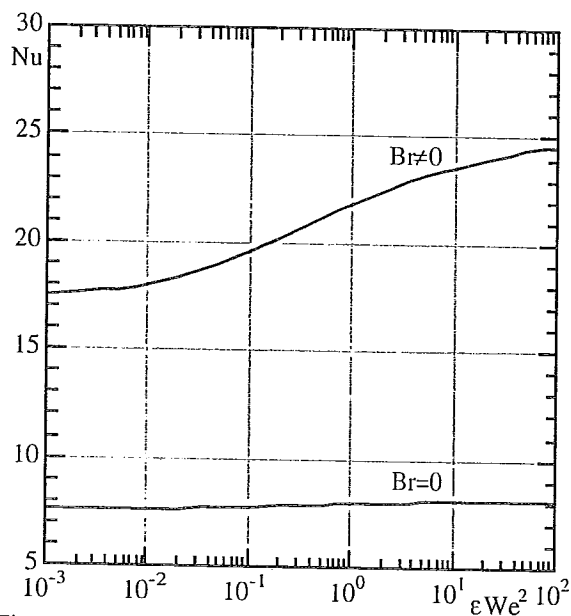


Figure 3-Comparison between the Nusselt numbers for negligible viscous dissipation and the limiting equilibrium viscous dissipation condition, as a function of fluid elasticity ( $\epsilon We^2$ ).

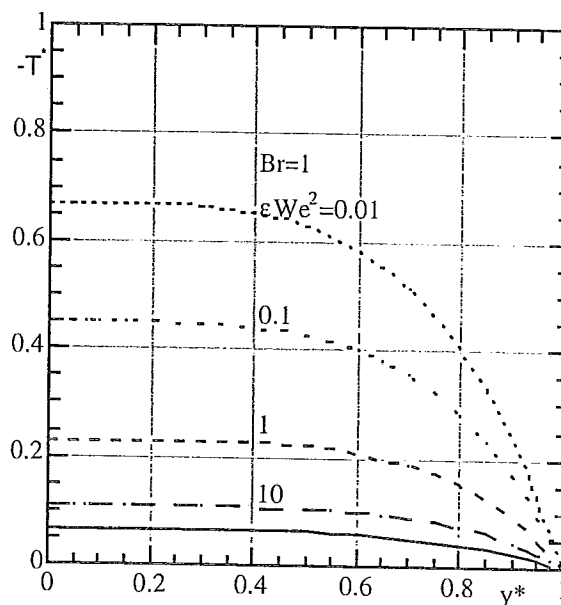


Figure 4- Effect of viscoelasticity and Brinkman number on the radial temperature profiles in the equilibrium viscous dissipation problem. All curves for  $Br=1$ , but the full line is for  $Br=0.1$  and  $\epsilon We^2 = 0.01$ .

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