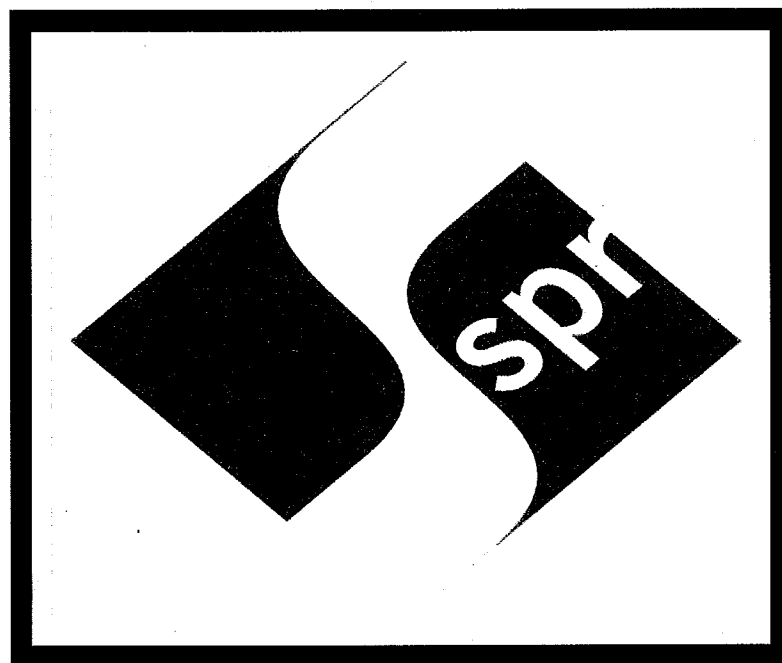


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# FLOW INSTABILITIES AND STEADY CHANNEL FLOW OF THE FULL PHAN-THIEN—TANNER FLUID

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## Abstract

The analytical solution for the steady-state channel flow of the complete Phan-Thien—Tanner fluid with a linear stress coefficient is derived. The channel flow is found to be unstable when the pressure gradient exceeds a critical value determined by a maximum shear rate at the wall. Expressions are also given for the viscometric viscosity and the normal stress difference coefficients  $\Psi_1$  and  $\Psi_2$  in steady plane shear flow. Here, the shear stress was found not to be a monotonically increasing function of the shear rate as strong shear-thinning sets in. The critical condition for the maximum shear stress in the rheogram is presented and is linked to the condition for existence of steady-state solutions in the channel flow.

## 1. Introduction

The Phan-Thien—Tanner (PTT) constitutive equation was developed from the Lodge-Yamamoto network theory [1,2] and is given in Eq. (1)

$$\left(1 + \frac{\varepsilon\lambda}{\eta} \text{tr } \tau\right) \tau + \lambda \overset{\circ}{\tau} = 2\eta \mathbf{D} \quad (1)$$

where the stress coefficient adopted is the linearised form adequate for weak channel flows,  $\tau$  and  $\mathbf{D}$  are the extra stress and rate of deformation tensors,  $\lambda$  is a relaxation time,  $\eta$  is the constant viscosity coefficient and  $\varepsilon$  imposes an upper limit to the elongational viscosity which becomes inversely proportional to  $\varepsilon$ .  $\overset{\circ}{\tau}$  is the Gordon-Schowalter convected derivative usually defined as

$$\overset{\circ}{\tau} = \frac{D\tau}{Dt} - \tau \cdot \nabla \mathbf{u} - \nabla \mathbf{u}^T \cdot \tau + \xi(\tau \cdot \mathbf{D} + \mathbf{D} \cdot \tau) \quad (2)$$

and introducing a third parameter  $\xi$ .

## 2. Material functions in steady plane shear flow

First, we consider a simple plane shear flow aligned with  $x$  (shear rate  $\dot{\gamma} = u_{,y}$ ) for which the constitutive Eq. (1) reduces to

$$\left(1 + \frac{\varepsilon\lambda}{\eta} \tau_{ii}\right) \tau_{xx} = \lambda(2 - \xi) \dot{\gamma} \tau_{xy} \quad (3)$$

$$\left(1 + \frac{\varepsilon\lambda}{\eta} \tau_{ii}\right) \tau_{yy} = -\lambda\xi \dot{\gamma} \tau_{xy} \quad (4)$$

$$\left(1 + \frac{\varepsilon\lambda}{\eta} \tau_{ii}\right) \tau_{xy} = \eta \dot{\gamma} + \lambda \left(1 - \frac{\xi}{2}\right) \dot{\gamma} \tau_{yy} - \frac{\lambda\xi}{2} \dot{\gamma} \tau_{xx} \quad (5)$$

The ratio of Eqs. (3) and (4) shows the relationship between both normal stresses to be

$$\tau_{yy} = -\frac{\xi}{2 - \xi} \tau_{xx} \quad (6)$$

Combining Eq. (3) for  $\tau_{xy}$ , Eq. (6) for  $\tau_{yy}$  and Eq. (5), we arrive at

$$\tau_{xx}^3 + a_1 \tau_{xx}^2 + a_2 \tau_{xx} + a_3 = 0 \quad \text{with coefficients}$$

$$a_1 = \frac{\eta(2 - \xi)}{\varepsilon\lambda(1 - \xi)}; a_2 = \frac{\eta^2(2 - \xi)^3 \dot{\gamma}^2 \xi}{4\varepsilon^2(1 - \xi)^2} + \frac{\eta^2(2 - \xi)^2}{4\varepsilon^2 \lambda^2 (1 - \xi)^2}; a_3 = -\frac{\eta^3(2 - \xi)^3 \dot{\gamma}^2}{4\varepsilon^2 \lambda (1 - \xi)^2} \quad (7)$$

The real solution of this cubic equation is the following explicit function of  $\dot{\gamma}$

$$\tau_{xx}(\dot{\gamma}) = \sqrt[3]{-\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \frac{\alpha^3}{27}}} + \sqrt[3]{-\frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} + \frac{\alpha^3}{27}}} - \frac{a_1}{3} \quad \text{with}$$

$$\alpha = a_2 - \frac{a_1^2}{3}; \beta = a_3 - \frac{a_1 a_2}{3} + \frac{2a_1^3}{27} \quad (8)$$

The shear stress is given by Eq. (5), written in terms of  $\tau_{xx}$  for compactness

$$\tau_{xy}(\dot{\gamma}) = \frac{\eta - \lambda\xi \tau_{xx}}{2\varepsilon\lambda(1 - \xi)} \dot{\gamma} \quad (9)$$

$$1 + \frac{\eta(2 - \xi)}{\varepsilon\lambda(1 - \xi)} \tau_{xx}$$

and the material functions are finally obtained from their definitions:

$$\mu(\dot{\gamma}) \equiv \frac{\tau_{xy}}{\dot{\gamma}} = \frac{\eta - \lambda\xi \tau_{xx}}{2\varepsilon\lambda(1 - \xi)} \quad (10 \text{ a,b,c})$$

$$1 + \frac{\eta(2 - \xi)}{\varepsilon\lambda(1 - \xi)} \tau_{xx}$$

$$\Psi_1(\dot{\gamma}) \equiv \frac{\tau_{xx} - \tau_{yy}}{\dot{\gamma}^2} = \frac{2\tau_{xx}}{(2 - \xi)\dot{\gamma}^2}; \quad \Psi_2(\dot{\gamma}) \equiv \frac{\tau_{yy} - \tau_{zz}}{\dot{\gamma}^2} = \frac{-\xi \tau_{xx}}{(2 - \xi)\dot{\gamma}^2}$$

For  $\xi = 0$  the SPTT model is recovered which predicts  $N_2 = 0$ . The SPTT model has a shear-thinning behaviour in both  $\mu(\dot{\gamma})$  and  $\Psi_1(\dot{\gamma})$ , with constant values at low shear rates as shown in Figs 1 and 2. The stresses are independent of  $\varepsilon$  for low values of  $\varepsilon$ , as expected [3], and for  $\xi \neq 0$  provided  $\varepsilon < \xi$ .

For  $\xi \neq 0$  there is a non-monotonic behaviour for  $\mu(\dot{\gamma})$  akin to that of a pure Johnson-Segalman fluid [4] and all shear flow characteristics are affected by  $\xi$ . Above a critical shear rate ( $\dot{\gamma}_c$ ), defined by the point where the shear stress attains its maximum value, shear-thinning intensity becomes so strong that the slope  $d \ln \mu / d \ln \dot{\gamma}$  becomes smaller than -1. This  $\dot{\gamma}_c$  depends on both  $\xi$  and  $\varepsilon$ , and was found here to be given by

$$\lambda \dot{\gamma}_c = \frac{\varepsilon(1 - \xi) + \xi(2 - \xi)}{[\xi(2 - \xi)]^{3/2}} \quad (11)$$

and is marked in Fig 2 by vertical lines. Equation (11) is an important result which will be derived in a different way from the analysis for the channel flow presented below.

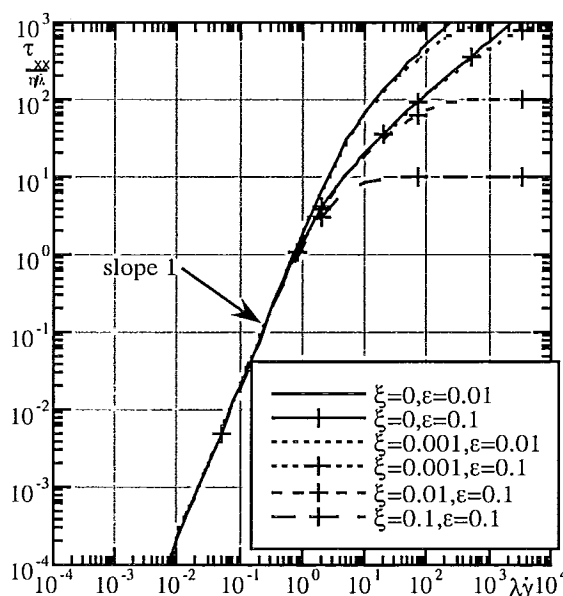


Figure 1- Variation of the shear stress with shear rate in steady Couette flow for a PTT fluid with linear stress coefficient. Vertical lines:  $\lambda\dot{\gamma}_c$  of Eq. (11).

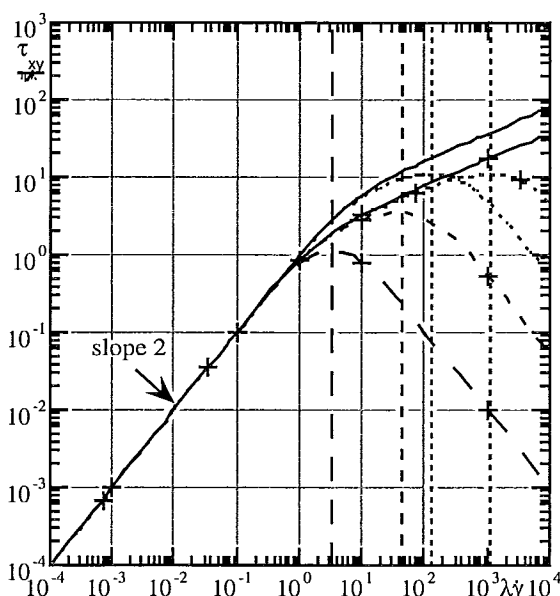


Figure 2- Variation of the streamwise normal stress with shear rate in steady Couette flow for a PTT fluid with linear stress coefficient. Caption as in Fig 1.

### 3. Solution for fully-developed channel flow

The channel has half-width  $H$ , streamwise coordinate  $x$  and velocity  $u$ , cross-stream coordinate  $y$  and velocity  $v$  with  $y=0$  on the centreline. The  $x$ - momentum equation is independent of the fluid and integrates to the well known linear variation of shear stress

$$\tau_{xy} = p_{,x}y \quad (12)$$

The constitutive equations (3-5), as well as Eq. (6), remain valid, so

$$\left(1 + \frac{\epsilon\lambda}{\eta} \tau_{ii}\right) = 1 + \frac{2\epsilon\lambda(1-\xi)}{\eta(2-\xi)} \tau_{xx} \quad (13)$$

Division of Eq. (5) by Eq. (3) results in a second order equation for  $\tau_{xx}$

$$\lambda\xi\tau_{xx}^2 - \eta\tau_{xx} + \lambda(2-\xi)\tau_{xy}^2 = 0 \quad (14)$$

which has only one physically realistic solution: at  $y=0$ , Eq. (12) gives a zero shear stress which implies  $\tau_{xx} = 0$  from Eq. (3). This is only possible in the solution of Eq. (14) having the minus sign before the discriminant, that is

$$\tau_{xx} = \frac{\eta}{2\lambda\xi} \left[ 1 - \sqrt{1 - \frac{4\lambda^2\xi(2-\xi)\tau_{xy}^2}{\eta^2}} \right] = \frac{\eta}{2\lambda\xi} \left[ 1 - \sqrt{1 - (ay')^2} \right] \quad (15)$$

where  $a \equiv -2\lambda p_{,x}H\sqrt{\xi(2-\xi)}/\eta$  is used for compactness and  $y' \equiv y/H$  is the dimensionless cross-stream coordinate. The pressure gradient is negative and since  $\xi \leq 2$  the dimensionless parameter  $a$  must be real and positive.

Equation (15) indicates the need for  $ay' \leq 1$  in order to obtain a real solution for normal stress. For the moment, it suffices to accept the need for  $a \leq 1$  since  $0 \leq y' \leq 1$ .

Expressions for the three normalised stress components are useful and readily obtained after scaling with the wall shear stress for Newtonian (or UCM) fluid. This introduces the Deborah number, here defined as  $De = \lambda U/H$  where  $U$  is the bulk velocity

$$\begin{aligned} T_{xx} &\equiv \frac{\tau_{xx}}{3\eta U/H} = \frac{1 - \sqrt{1 - (ay')^2}}{6\xi De} \quad ; \quad T_{yy} \equiv \frac{\tau_{yy}}{3\eta U/H} = -\frac{1 - \sqrt{1 - (ay')^2}}{6De(2 - \xi)} \\ T_{xy} &\equiv \frac{\tau_{xy}}{3\eta U/H} = -\frac{ay'}{6De\sqrt{\xi(2 - \xi)}} \end{aligned} \quad (16 \text{ a,b,c})$$

The whole stress field (Eqs. (12), (15) and (6)) can be substituted into Eq. (3) to give the transverse variation of the shear rate, which may then be integrated to yield the velocity profile. To improve readability of the equations parameter  $\chi \equiv \xi(2 - \xi)/[\varepsilon(1 - \xi)]$  which combines  $\varepsilon$  and  $\xi$  in a significant way.

The transverse profile of shear rate in non-dimensional form is

$$\Gamma(y') \equiv \frac{\dot{\gamma}}{3U/H} = \frac{-1}{3De\sqrt{\xi(2 - \xi)}} \left[ \left(1 + \frac{2}{\chi}\right) \left( \frac{1}{ay'} - \frac{\sqrt{1 - (ay')^2}}{ay'} \right) - \frac{ay'}{\chi} \right] \quad (17)$$

The variation of the viscosity across the duct can be determined from its definition

$$\mu(\dot{\gamma}) \equiv \frac{\tau_{xy}}{\dot{\gamma}} \Rightarrow \frac{\mu(\dot{\gamma})}{\eta} = \frac{(a^2 y')^2}{2 \left[ \left(1 + \frac{2}{\chi}\right) \left( \frac{1}{y'} - \frac{\sqrt{1 - (ay')^2}}{y'} \right) - \frac{a^2 y'}{\chi} \right]} \quad (18)$$

and the shear rate profile can be integrated to yield the nondimensional velocity profile

$$\frac{u(y)}{U} = -\frac{6}{a^2} \frac{U_N}{U} \left[1 + \frac{2}{\chi}\right] \left\{ \ln \frac{1 + \sqrt{1 - (ay')^2}}{1 + \sqrt{1 - a^2}} + \sqrt{1 - a^2} - \sqrt{1 - (ay')^2} \right\} - \frac{3}{\chi} \frac{U_N}{U} [1 - y^2] \quad (19)$$

$U_N$  is defined as  $U_N \equiv -p_x H^2 / 3\eta$  and represents the bulk velocity for the flow of a Newtonian fluid subjected to the same pressure gradient. Then, the ratio  $U_N/U$  can be viewed as a dimensionless pressure gradient and it is a new unknown of the problem. It can be obtained from integration of the velocity profile over the channel leading to

$$\left(\frac{U_N}{U}\right)^{-1} = \frac{6}{a^2} \left[1 + \frac{2}{\chi}\right] \left\{ 1 - \frac{\pi}{4a} + \frac{1}{2a} \arctan \left( \frac{\sqrt{1 - a^2}}{a} \right) - \frac{\sqrt{1 - a^2}}{2} \right\} - \frac{2}{\chi} \quad (20)$$

However, Eq. (20) does not provide an explicit expression for  $U_N/U$  since  $a$  itself depends on  $U_N/U$ . In fact, it is easy to show that  $a$  and  $U_N/U$  are related by

$$a = 6 \frac{U_N}{U} De \sqrt{\varepsilon(1 - \xi)} \sqrt{\chi} \quad (21)$$

and this expression suggests that we take as two independent parameters a modified Deborah number (defined by  $De^* \equiv De \sqrt{\varepsilon(1 - \xi)}$ ) and  $\chi$ . These parameters are only unsuitable in the limiting case of  $\varepsilon = 0$  where one has to revert back to  $De$  and  $\xi$ .

Eq. (20) represents a highly non-linear function of two non-dimensional parameters  $(De^*, \chi)$  for the determination of  $U_N/U$  which can be easily solved by standard numerical means.

#### 4. Flow instability

Inspection of Eq. (15) showed the need for  $a \leq 1$ . The equality  $a = 1$  leads to a condition corresponding to the onset of a constitutive instability akin to that investigated for the Johnson-Segalman model by [4]. For  $a > 1$  the flow is no longer steady, it may become unstable and time-dependent, or it may happen that no solution exists.

The maximum shear rate  $\dot{\gamma}_{\max}$  in the duct cross-section takes place at the wall and is found from Eq. (17). Further, imposing the limiting stability condition  $a = 1$  gives

$$\lambda \dot{\gamma}_{\max,c} = \frac{1 + \frac{1}{\chi}}{\sqrt{\xi(2-\xi)}} \Leftrightarrow \lambda \dot{\gamma}_{\max,c} = \frac{\varepsilon(1-\xi) + \xi(2-\xi)}{[\xi(2-\xi)]^{3/2}} \quad (22)$$

which is identical to Eq. (11). Since the instability is associated with the maximum in the shear stress-shear rate relationship it will always be there provided  $\xi$  is nonzero as Fig 1 has shown. Fig 3 represents  $\lambda \dot{\gamma}_c$  as a function of  $\xi$  and  $\varepsilon$  and shows the stabilizing effect of  $\varepsilon$  and the destabilizing effect of  $\xi$ . Interestingly, the corresponding critical pressure gradient is independent of  $\varepsilon$

$$\frac{(-p_{,x})_c H}{\eta/\lambda} = \frac{1}{2\sqrt{\xi(2-\xi)}} \quad (23)$$

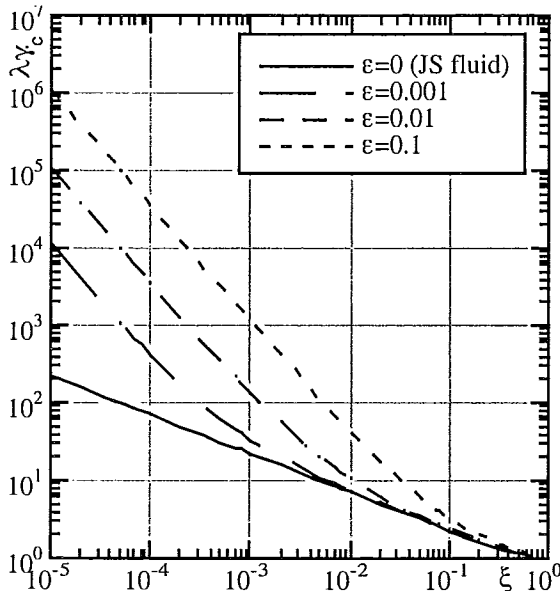


Figure 3- Variation of the critical shear rate with the material parameters  $\varepsilon$  and  $\xi$ .

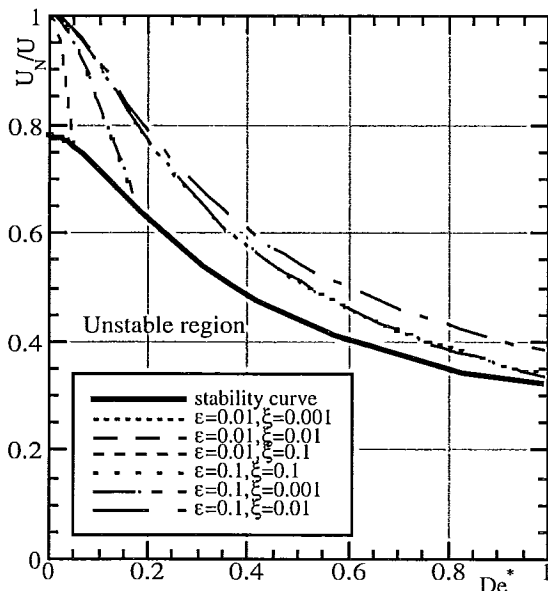


Figure 4- Stability map of the PTT model fluid.

The "stability" condition can be recast as a relation between bulk flow quantities and nondimensional numbers. The pressure drop for a given flow rate ( $U_N/U$ ) depends on two parameters,  $De^*$  and  $\chi$  and the limiting "stability" condition ( $a=1$ ) implies that there remains only a single independent parameter which we may take as  $De^*$  or  $\chi$ , whichever is more convenient. Equating Eqs. (20) and (21), subjected to the stability restraint, gives

$$\left(\frac{U}{U_N}\right)_c \sqrt{\frac{1}{(10-3\pi)} \left[ \left(\frac{U}{U_N}\right)_c - 6 \left(1 - \frac{\pi}{4}\right) \right]} = 6De_c^* \quad (24)$$

Eq. (24) is plotted in Fig 4 as a thick line and the Figure includes curves representing the relation between the pressure gradient ( $U_N/U$ ),  $De$ ,  $\xi$  and  $\varepsilon$  in the stable region. When  $De^*$  tends to zero (negligible elasticity), the critical velocity ratio tends to  $(U_N/U)_c \rightarrow 1/6(1 - \pi/4) \approx 0.7766$  and, as  $De^* \rightarrow \infty$ ,  $(U_N/U)_c$  asymptotes to zero. It is also useful to cast  $De_c^*$  or  $De_c^*$  in terms of model parameters only and this gives

$$De_c^* = \frac{1}{\sqrt{\chi}} \left[ \left(1 - \frac{\pi}{4}\right) + \frac{1}{3\chi} \left(5 - \frac{3}{2}\pi\right) \right] \text{ or } De_c = \frac{1}{\sqrt{\xi(2-\xi)}} \left[ \left(1 - \frac{\pi}{4}\right) + \frac{1}{3\chi} \left(5 - \frac{3}{2}\pi\right) \right] \quad (25)$$

indicating the widening of the range of  $De$  for a stable flow with a reduction of  $\chi$ .

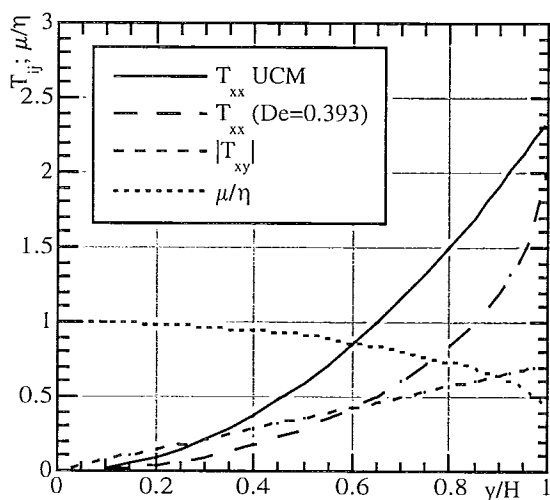


Figure 5- Transverse profiles of stresses and viscosity in channel flow for conditions just below critical at  $De = 0.393$  with  $\xi = 0.2$  and  $\varepsilon = 0.1$ .

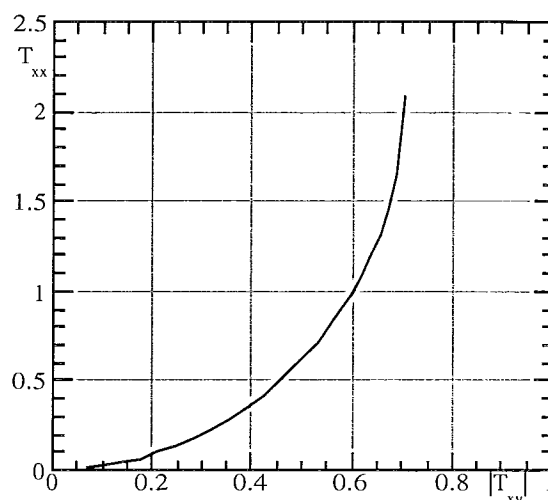


Figure 6- Variation of the normal stress with shear stress across the channel for a situation just below critical ( $\xi = 0.2$ ,  $\varepsilon = 0.1$ ,  $De = 0.393$ ).

It is interesting to observe the stress profiles for a situation on the verge of a critical state in Figs. 5 and 6 ( $De = 0.393$ ;  $De_c^* \approx 0.11116$ ). Typical values of the parameters  $\xi = 0.2$  and  $\varepsilon = 0.1$  give  $\chi = 4.5$  and the critical state is at  $De_c^* = 0.11121$  ( $De_c = 0.39318$ ). Although the velocity and shear stress profiles seem absolutely normal, the normal stress  $T_{xx}$  increases sharply near the wall. This differentiated rate of growth of the normal and shear stresses is much clearer in the plot of Fig 6: the critical situation corresponds to a slope of  $T_{xx}$  with  $T_{xy}$  tending to infinity, which could have been foreseen from Eq. (15), for if we do  $(\partial\tau_{xx}/\partial\tau_{xy}) \rightarrow \infty$ , we get Eq. (22).

## References

- [1] Phan-Thien, N. and Tanner, R. I. 1977. *J. Non-Newt. Fluid Mech.* **2**, 353-365.
- [2] Phan-Thien, N. 1978. *J. Rheol.* **22**, 259-283.
- [3] Larson, R. G., 1988. *Constitutive Equations for Polymer Melts and Solutions*, Butterworths, Boston.
- [4] Georgiou, G. C. and Vlassopoulos, D. V. 1998. *J. Non-Newt. Fluid Mech.*, **75**, 77- 97.