SOME OBSERVATIONS FROM ANALYTICAL SOLUTIONS OF VISCOELASTIC FLUID MOTION IN STRAIGHT DUCTS

Paulo J. Oliveira[1], **Fernando T. Pinho**[2]

[1] Departamento Engª Elecetromecânica, Unidade MTD, Universidade da Beira Interior, 6200 Covilhã, Portugal
[2] Centro de Estudos de Fenómenos de Transporte, DEMEGI, Universidade do Porto, 4050-123 Porto, Portugal

ABSTRACT

We study the exact solutions for fully developed channel flow of model fluids obeying two typical constitutive equations: the Phan-Thien/Tanner and the Giesekus models. Particular attention is paid to the profiles of stress components across the channel and a peculiar behaviour is observed for the normal stresses: as the Deborah number increases up to unity, the level of stresses also increases, but for higher De the trend is reversed. We find that if stresses are scaled with the wall shear stress then the variation becomes monotonic.

KEYWORDS: NORMAL STRESSES, ANALYTICAL SOLUTION, PTT, GIESEKUS, FULLY DEVELOPED

INTRODUCTION

Analytical solutions for simple flows of complex viscoelastic model fluids can provide useful means of checking how adequately those models mimic real fluid behaviour and may also highlight some peculiar or unexpected particularity of the models. This type of analysis is, in a way, to revert back to classical Newtonian fluid mechanics, in which one tries to solve the known governing equations and learn from the solution. In rheology, due to the unknown behaviour of real fluids, it has been more customary to assume particular kinematics and derive from thereon the relevant material functions.

In this paper we study the exact solutions for fully developed channel flow of two typical model fluids: those obeying simplified forms of the Phan-Thien/Tanner [1] and the Giesekus [2] constitutive equations. We examine the profiles of the stress components across the channel and we find a peculiar behaviour for the normal stress variation. This abnormal behaviour is removed if the normal stresses are scaled with the shear stress value ate the wall. We speculate about possible implications.

SOLUTION FOR PTT MODEL

The analytical solution for fully developed flow of PTT fluids in channel and pipes is given in [3], for both the linear and exponential forms of the model. Here we take the linear form of the PTT model [1], namely,

$$(1 + (\epsilon\lambda/\eta)\ \mathrm{tr}(\boldsymbol{\tau}))\boldsymbol{\tau} + \lambda \overset{\triangledown}{\boldsymbol{\tau}} = 2\eta \boldsymbol{D} \tag{1}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{D}$ are the extra-stress and deformation rate tensors, λ is the relaxation time, η is the constant shear viscosity coefficient, ϵ is a model parameter and $\overset{\triangledown}{\boldsymbol{\tau}}$ denotes Oldroyd's upper convected derivative,

This constitutive model has often been used to represent experimental fluids in contraction flows, e.g. Quinzani et al [4], White and Baird [5], Carew et al [6], Baaijens [7], Azaiez et al [8], etc. One advantage over some other constitutive models is the easiness to incorporate extensional stress effects through the parameter ϵ and indeed, Quinzani et al [4] have found it to be the best model to represent the fluid used in their own experiments [9]. White and Baird [10] have also used a form of this model in the simulation study of their measurements [5] in the 4:1 planar contraction.

The relevant results from [3], for a channel with half-width H and average velocity U, are the expressions for the shear and normal stress profiles in the channel:

$$T_{xy} = -(U_{\mathrm{N}}/U)\ (y/H)\,. \tag{2}$$

$$T_{xx} = 6\, De\, (U_{\mathrm{N}}/U)^2\ (y/H)^2 \tag{3}$$

where y is the lateral coordinate, and also the velocity profile

$$u/U = \tfrac{3}{2}\ (U_{\mathrm{N}}/U)\ \Big(1-(y/H)^2\Big)\Big(\ 1\ + \\ +\ 9\ \epsilon\ De^2\ (U_{\mathrm{N}}/U)^2 (1+(y/H)^2)\Big). \tag{4}$$

In the above equations $T_{ij} \equiv \tau_{ij}/(3\eta U/H)$, the Deborah number is defined as $De \equiv \lambda U/H$, and $U_{\mathrm{N}} \equiv -(\mathrm{d}p/\mathrm{d}x)H^2/(3\eta)$ is a variable parameter with the meaning of the average velocity that would exist for the same pressure drop $(-\mathrm{d}p/\mathrm{d}x)$ if the fluid were Newtonian instead of a PTT fluid. This is readily apparent since $\epsilon = 0$ brings the above equations equal to the well known fully developed solution for a UCM fluid and if further $De = 0$, the Newtonian solution is obtained. The group (U_{N}/U) can also be viewed as a non-dimensional pressure gradient and was shown in [3] to be given by:

$$(U_{\mathrm{N}}/U) = (432)^{1/6}(\ \delta^{2/3} - 2^{2/3}\ )/(6 b^{1/2}\delta^{1/3}) \tag{5}$$

with:

$b \equiv \frac{54}{5}\epsilon De^2;\ \alpha \equiv 27\mathrm{b}+4;\ \delta \equiv \alpha^{1/2} + 3\times\ 3^{1/2}b^{1/2}.$

The effect of shear-thinning is illustrated by some

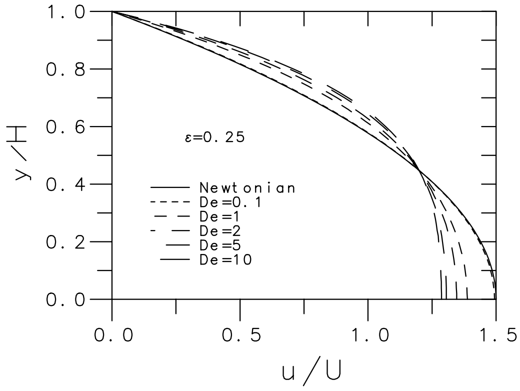

Figure 1. Velocity profiles in fully-developed channel flow of a PTT fluid with $\epsilon = 0.25$.

velocity profiles given in Fig. 1, for increasing value of De and fixed $\epsilon = 0.25$. At higher De the maximum velocity in the centreline decreases and the profiles become more flat.

Fig. 2 shows the variation across the channel width of the stress components, from Eqs. (2) and (3), for various De at $\epsilon = 0.25$. The value $\epsilon = 0.25$ was suggested by [9] and is appropriate for high density polymer solutions (or melts [6]). The curves in Fig. 3 are remarkable: the shear stress is seen to decrease, in absolute terms, as De is increased; the non-dimensional normal stress, on the other hand, exhibits a non-monotonic behaviour. For De up to 1, T_{xx} increases; but for higher elasticity levels ($De > 1$), the normal stress is reduced in the whole section of the channel and more pronounced so near the wall ($y/H \approx 1$).

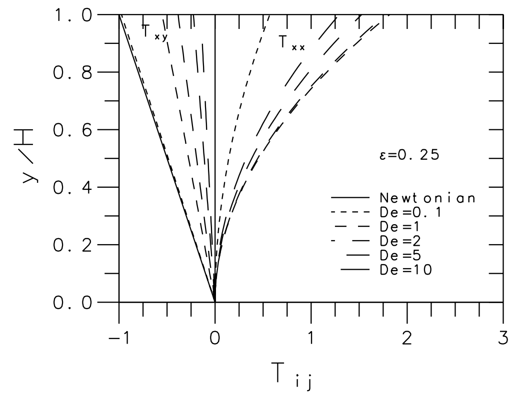

Figure 2. Stress profiles for the PTT fluid with $\epsilon = 0.25$.

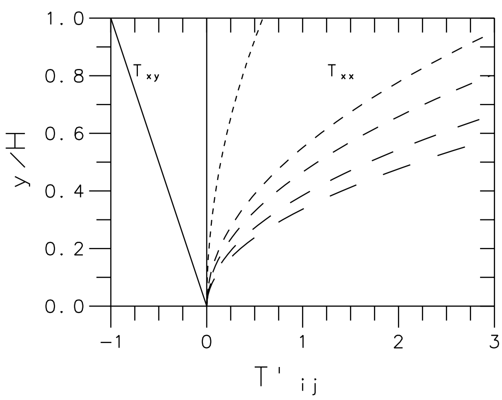

Figure 3. Stress profiles for the PTT fluid scaled with the PTT wall shear stress, $(\tau_w)_{\text{PTT}}$ from Eq (6): $T'_{ij} \equiv \tau_{ij}/(\tau_w)_{\text{PTT}}$ (lines as in Fig. 2).

If instead the stresses from Eqs. (3) and (4) are made non-dimensional with the shear stress value at the wall, i.e.

$$(\tau_w)_{\text{PTT}} \equiv -\tau_{xy}(y = H) = (\bar{u}_{\text{N}}/U)(3\,\eta U/H) \quad (6)$$

where U_{N}/U is given by Eq. (6), then we get:

$$T'_{xy} \equiv \tau_{xy}/(\tau_w)_{\text{PTT}} = -(y/H)\,. \quad (7)$$

$$T'_{xx} \equiv \tau_{xx}/(\tau_w)_{\text{PTT}} = 6\,De\,(U_{\text{N}}/U)(y/H)^2. \quad (8)$$

These profiles are plotted in Fig. 3 for the same $\epsilon = 0.25$ and the same De range as in Fig. 2. The shear stresses have collapsed onto a single curve (as anticipated, cf. Eq. 7) and the normal stresses now follow an expected behavioral pattern: higher normal stresses as De is increased.

SOLUTION FOR GIESEKUS MODEL

In order to check that the above effect is not specific to the PTT model, we examine another constitutive model. A semi-analytical solution for fully-developed channel flow also exists for the Leonov-like fluid, given by Lim and Schowalter [11]. The constitutive equation for this fluid is identical to that proposed by Giesekus [2] without solvent viscosity:

$$\boldsymbol{\tau} + (\alpha\lambda/\eta)\,\boldsymbol{\tau}\cdot\boldsymbol{\tau} + \lambda\,\overset{\triangledown}{\boldsymbol{\tau}} = 2\eta \boldsymbol{D} \quad (9)$$

and with the parameter $\alpha = 0.5$. In [11] the solution in written in terms of a Weissenberg number based on centreline velocity ($We \equiv \lambda U_0/H$) and the stresses are scaled with $\eta U_0/\text{H}$. Since U_0 varies with the flow rate U (that is, with our De), we express that solution by making use of:

$$We = De\,/\,(U/U_0) \quad (10)$$

$$U/U_0 = \left(2 + \tfrac{1}{\beta}\ln\tfrac{1-\beta}{1+\beta}\right)/\ln(1-\beta^2) \quad (11)$$

The parameter β embodies the unknown pressure gradient for a given flow rate U and was given by Lim and Schowalter as an implicit equation which is easily solved by iteration if written as:

$$\beta = \sqrt{1 - \exp(-2\beta We)}. \quad (12)$$

Starting with $\beta = 0.5$ (a value which was seen to work well) on the right-hand side of (12) and iterating through the last three equations (10-12), we end up with We and β for a given De. The expressions for the velocity and stress profiles are like those in [11] but the stresses are scaled with either $3\eta U/H$ or $(\tau_w)_{\text{GIE}}$. The wall value for the shear stress of the Giesekus fluid in a plane channel can be shown to be given by:

$$(\tau_w)_{\text{GIE}} = (\beta/3De)\,(3\,\eta\,U/H) \quad (13)$$

so, by comparing with Eq. (6) we see that $\beta/3De$ has the same meaning of our U_{N}/U.

The resulting stress profiles are shown in Fig. 4. If the stresses are scaled with the shear stress at the wall then we obtain the profiles given in Fig. 5. The

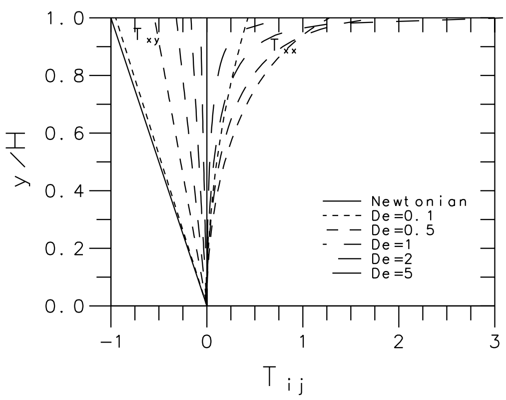

Figure 4. Stress profiles for the Giesekus fluid.

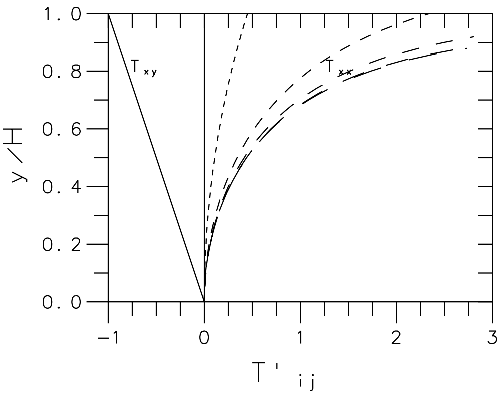

Figure 5. Stress profiles for the Giesekus fluid, scaled with its own wall shear stress, $(\tau_w)_{\text{GIE}}$ from Eq. (13).

qualitative behaviour is like that for the PTT above and the same comments and conclusions apply. The fact seen in Fig. 5 that for the highest Deborah numbers the normal stress becomes very high and seems to collapse to a single curve is related to a limiting behaviour found for the Giesekus fluid. For $De > 5$ (and $\alpha = 0.5$) the flow becomes plug-like, resembling that for a Bingham fluid; this is shown in the corresponding velocity profiles given in Fig. 6. As a consequence, for $De = 10$ we could not converge the iterative procedure described before, an indication that the velocity profile had become perfectly rectangular. This seems to show that the Giesekus equations with $\alpha = 0.5$ are not adequate to model the fluid behaviour of polymer solutions, at least for high elasticity flows. For polymer melts, some recent experimental results [12] show plug-like velocity profiles at high De and the Giesekus model may then be adequate.

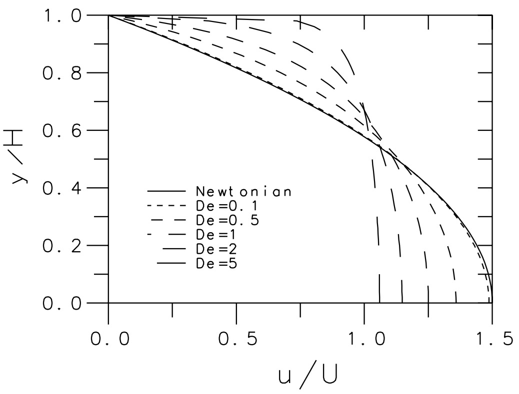

Figure 6. Velocity profiles for the Giesekus fluid.

DISCUSSION

We have shown that two different constitutive models, the PTT and Giesekus, although leading to qualitatively different velocity profiles, exhibit similar behaviour in terms of normal stress variation. In both cases, the normal stresses at a given lateral position are found to first increase for De up to 1 and then, for higher De, they decrease. This was rather unexpected since, from intuition, one would anticipate the normal stresses to grow monotonically with elasticity as measured by De. What we found interesting is that by scaling stresses with the wall shear stress, then the pattern of variation becomes as expected: higher normal stresses with higher De.

This type of behaviour is most probably linked with the shear-thinning characteristics of the fluid, although this is a rather vague explanation: is it shear-thinning of viscosity, of the N_1, or even a mixture of both? We don't know, all we know is that for a constitutive model with constant viscosity and shear-thinning N_1, like the simplified FENE-CR used by [13], the behaviour here reported is not observed.

If we extrapolate this behaviour of the normal stress in a channel to the normal stress along a planar contraction, then our findings are in line with the observations of White and Baird [5] regarding vortex patterns in contractions. They observed that vortex growth could be correlated with higher ratios of N_1 to wall shear-stress but not with the N_1 values alone. This is exactly analogous to the monotonic behaviour of τ_{xx}/τ_w with De, and non-monotonic of τ_{xx} with De, we found in the exact solution for channel flow. It may be argued that here the normal stress is of shear origin whereas in the centreline of the contraction they are from pure extensional origin. However it is fair to say that a fluid element subjected to a normal-stress difference does not know (or is indifferent to) the origin of that stress and also, we see from Fig. 2, that the "abnormal" behaviour of τ_{xx} is also present near the centreline of the channel.

As a contrasting example we take Xue et al [14] who chose $\eta_E/3\eta_0$ to characterise the vortex behaviour in the 4:1 contraction. It is straightforward to show that that parameter is proportional to the un-scaled normal stress difference and thus we expect un-conclusive results, as those authors show in their Fig. 6b.

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