PHAN-THIEN/TANNER FLOW IN CONCENTRIC ANNULI

F. T. Pinho[1], P. J. Oliveira[2]

[1] Centro de Estudo de Fenómenos de Transporte, Faculdade de Engenharia da Universidade do Porto, 4050-123 Porto, Portugal (fpinho@fe.up.pt)

[2] Universidade da Beira Interior, Departamento Engª Electromecânica, 6200 Covilhã, Portugal (pjpo@ubi.pt)

ABSTRACT

An analytical solution is derived for the fully developed flow of a class of non-linear viscoelastic fluids in concentric annuli. The rheological constitutive equation is the simplified Phan-Thien /Tanner model with linear stress coefficient. Velocity and stress profiles are given together with adequate equations for the solution of the direct and inverse problems relating flow rate and pressure drop.

Fluid elasticity coupled with finite extensibility are responsible for a reduction in fRe as large as 90% and this quantity is found to be a function of ϵDe^2 and independent of the radius ratio k if adequately scaled with the corresponding Newtonian value.

KEYWORDS: ANNULAR FLOW, ANALYTICAL SOLUTION, VISCOELASTIC FLUID, PTT.

INTRODUCTION

We consider the axial annular flow between two concentric cylinders. For the case of a Newtonian fluid, the exact solution to this problem can be found in classical textbooks (see eg. [1], pp. 51-54). For power law, inelastic, non-Newtonian fluids the problem was first studied by Fredrickson and Bird [2] who presented a relation between the flow rate and the pressure gradient in terms of an integral which required numerical quadrature. This was rather cumbersome to use in practical calculations and Hanks and Larsen [3] were later able to derive a simpler analytical solution to the flow rate/pressure drop relation. Later, Hanks [4] studied the case of the viscoplastic Herschel-Bulkley model and gave design charts for the computation of the flow rate in terms of pressure drop, or vice-versa. References to other studies with different inelastic rheological models can be found in this latter work, and some of those are discussed in [5] (pages 221 (Ellis fluid), 226 (helical flow of power-law fluid), 230 and 265 (power-law fluid), 267 (Bingham fluid)).

Literature is scarcer on annular flow of viscoelastic fluids. Theoretical/numerical work with simplified viscoelastic fluid models are less frequent; two works of that type [6,7] are with linear models. An extensive literature survey in preparation by Escudier et al. [8] has shown that there appears to be no published solutions for the same problem with elastic fluids obeying non-linear differential equations for the evolution of the stress tensor.

In the present work we selected one of the most often used non-linear viscoelastic model fluids and derived analytical expressions for the velocity and stress (shear and normal components) profiles in concentric annular flow. The solution includes relations between the normalised pressure drop and flow rate (friction factor and Reynolds number) as a function of the relevant dimensionless parameters (Reynolds number, Deborah number and elongational parameter in the rheological model). For the "direct" problem in which the pressure drop is given and the flow rate is unknown, the present solution is a fully explicit exact solution. For the more useful "inverse" problem, in which the flux is known and the pressure drop unknown, a coupling between two cubic equations does not allow an explicit analytical solution but simple iteration between those two equations allows a solution to be obtained without difficulty, as will be shown.

The model fluid considered in this analysis is that governed by the simplified Phan-Thien and Tanner [9] constitutive equation (hereafter denoted by PTT):

$$\left(1 + \frac{\epsilon\lambda}{\eta}\mathrm{tr}(\boldsymbol{\tau}) \right) \boldsymbol{\tau} + \lambda \overset{\nabla}{\boldsymbol{\tau}} = 2\eta \boldsymbol{D} \qquad (1)$$

where $\boldsymbol{\tau}$ and $\boldsymbol{D}$ are the extra-stress and deformation-rate tensors, λ is the relaxation time, η is a constant viscosity coefficient and $\overset{\nabla}{\boldsymbol{\tau}}$ denotes Oldroyd's upper convected derivative. The parameter ϵ in Eq. (1) is related to the elongational behaviour of the fluid, precluding the possibility of an infinite elongational viscosity in a simple stretching flow as would occur for an upper Maxwell model UCM, in which $\epsilon = 0$.

The PTT model has found widespread use in numerical simulations of the flow of polymer solutions and melts. It is shown in [10] to be an excellent simple differential model for the elongational properties of polymer solutions in entry flows thus giving support for the usefulness of the present study.

ANALYSIS

The problem under consideration is that of fully developed axial flow with cylindrical symmetry in the annular gap between two concentric cylinders, with inner and outer radius given respectively by R_i and R_o ($\delta \equiv R_o - R_i$). The axial velocity component, along coordinate z, is denoted u and it is only a function of the radial coordinate r. There are four non-dimensional parameters to this flow problem: the radius ratio $k = R_i/R_o$ (geometric parameter); the Reynolds number $Re = \rho U 2\delta/\eta$ (dynamic parameter); ϵ and the Deborah number $De = \lambda U/\delta$ (constitutive parameters). It is standard practice to base the Reynolds number on the hydraulic diameter $D_H \equiv 4A/P$, where A is the cross-section area and P the wetted perimeter, which for an annulus gives $D_H = 2\delta$.

Stress Relations

For the particular case here considered of unidirectional flow in cylindrical coordinates the axial momentum equation can be integrated to give:

$$\tau_{rz} = -p_{,z} \frac{R_*}{2} \left(\frac{R_*}{r} - \frac{r}{R_*} \right) \qquad (2)$$

Here $p_{,z} \equiv \mathrm{d}p/\mathrm{d}z$ is the constant pressure gradient and the natural boundary condition of no-slip at the inner cylinder ($u = 0$ at $r = R_i$) has been replaced, following [2] and [1] (p. 52), by a zero shear stress condition: $\tau_{rz} = 0$ at $r = R_*$. We define a characteristic velocity scale as

$$U_c \;=\; \frac{-p_{,z}\delta^2}{8\eta} \tag{3}$$

and write Eq. (2) in non-dimensional form ($y \equiv r/\delta$) as:

$$T_{rz} \equiv \frac{\tau_{rz}}{\eta\frac{U}{\delta}} \;=\; 4\, y_*\, \frac{U_c}{U}\; (\; \frac{y_*}{y} \;-\; \frac{y}{y_*}\;) \tag{4}$$

where $y_* \equiv R_*/\delta$ and U is the average velocity in the annulus $(U \equiv Q/\pi(R_o^2 - R_i^2))$. The constitutive equation for the only two non-vanishing stress components, from Eq. (1), give:

$$\tau_{rz} \;=\; \frac{\eta}{1+\frac{\epsilon\lambda}{\eta}\tau_{zz}}\dot\gamma \tag{5}$$

$$\tau_{zz} \;=\; \frac{2\lambda\eta}{(1+\frac{\epsilon\lambda}{\eta}\tau_{zz})^2}\dot\gamma^2 \tag{6}$$

where $\dot\gamma \equiv \mathrm{d}u/\mathrm{d}r$ is the velocity gradient and the trace of the stress tensor reduces to τ_{zz}. These can be combined to give the following nondimensional normal stress equation:

$$T_{zz} \;=\; 2\; De\; T_{rz}^2. \tag{7}$$

Kinematic Relations

The velocity gradient is made explicit from Eq. (5) to give

$$\Gamma \;\equiv\; \frac{\dot\gamma}{U/\delta} \;=\; T_{rz}\;(\;1 + \;\epsilon\;\; De\;\; T_{zz}\;). \tag{8}$$

Upon substitution of equations (4) and (7) into equation (8) we obtain an explicit equation for the velocity gradient in terms of the normalised radial coordinate:

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}y} \;&=\; 4\; y_*\; \frac{U_c}{U}\; (\; \frac{y_*}{y} \;-\; \frac{y}{y_*}\;) \times \\ &\left(1 + 2\epsilon De^2 (4\; y_*\, \frac{U_c}{U}(\frac{y_*}{y} - \frac{y}{y_*}))^2\right)\end{aligned} \tag{9}$$

with $v \equiv u/U$ denoting the non-dimensional axial velocity component. Eq. (9) can be integrated to give:

$$\begin{aligned}v = 4Xy_*^2 \Big\{ &\left(\ln\frac{y(1-k)}{k} - \frac{y^2 - [k/(1-k)]^2}{2y_*^2}\right) - 32\epsilon De_c^2 y_*^2 \\ &\times \left(3\ln\frac{y(1-k)}{k} + \frac{y^2 - [k/(1-k)]^2}{2y_*^2}\Big[\frac{y^2 + [k/(1-k)]^2}{2y_*^2} - 3\Big] \right.\\ &\left. + \frac{y_*^2}{2}\Big[\frac{1}{y^2} - \frac{(1-k)^2}{k^2}\Big]\right)\Big\}\end{aligned} \tag{10}$$

where $De_c \equiv \lambda U_c/\delta$ is a Deborah number based on the velocity scale U_c. This can be related to the usual definition by $De_c = DeX$, where $X \equiv U_c/U$ is an unknown of the problem with the physical meaning of a non-dimensional pressure gradient to be determined later.

Solution for $y_* = R_*/\delta$

The second boundary condition expresses no-slip at the outer cylinder, $v = 0$ at $y = 1/(1-k)$. If this condition is inserted into Eq. (10) and the various terms are grouped in powers of y^*, we end up with a cubic equation for $x \equiv y_*^2$:

$$x^3 \;+\; a_1 x^2 \;+\; a_2 x \;+\; a_3 \;=\; 0 \tag{11}$$

where:

$$\begin{aligned}a_1 \;&=\; \frac{6\ \ln k}{(1-k^2)((1/k)-1)^2} \\ a_2 \;&=\; \frac{3k^2}{(1-k)^4} \;-\; \frac{2\ \ln k}{(32\epsilon De_c^2)(1-k)^2((1/k)^2-1)} \\ a_3 \;&=\; -\frac{k^2}{(1-k)^4}\left(\frac{1}{32\epsilon De_c^2} + \frac{1+k^2}{2(1-k)^2}\right)\end{aligned} \tag{12}$$

For a Newtonian fluid, or an elastic fluid with infinite extensibility or devoid of elasticity, the null shear stress radial position reduces to the known result ([1], p.52):

$$(y_*)_N = \frac{(R_*)_N}{\delta} \;=\; \left(\frac{1+k}{2(1-k)\ln(1/k)}\right)^{1/2} \tag{13}$$

The real solution to the cubic Eq. (11) can be determined explicitly by the formulae for third order algebraic equations and depends on De_c, and thus on $X = U_c/U$.

The Average Velocity: Solution for $X = U_c/U$

From its definition, the cross-section average velocity is given by:

$$U \;=\; \frac{2(1-k)}{(1+k)} \int_{k/(1-k)}^{1/(1-k)} u(y)\; y\; \mathrm{d}y \tag{14}$$

If the velocity variation from Eq. (10) is substituted for $u(y)$ above and the integration is performed, the following equation is obtained for $X = U_c/U$:

$$\frac{1}{y_*^2(\frac{8(1-k)}{(1+k)})} \;=\; I_1\; X \;-\; 32\epsilon De^2 y_*^2\; I_2\; X^3 \tag{15}$$

with:

$$I_1 \;=\; \frac{\ln(\frac{1}{k})}{2(k-1)^2} + \frac{(k+1)(2y_*^2(k-1)-k-1)}{8y_*^2(k-1)^2}, \tag{16}$$

and

$$\begin{aligned}I_2 \;=\; &\left(\frac{y_*^2}{2} + \frac{3}{2(k-1)^2}\right)\ln\frac{1}{k} + \frac{(1+k)}{24y_*^4k^2(k-1)^4} \times \\ &\Big(6y_*^6(k-1)^5 + 18y_*^4k^2(k-1)^3 - \\ &9y_*^2k^2(k-1)^2(k+1) + k^2(k+1)(2k^2+1)\Big)\end{aligned} \tag{17}$$

Hence we see that X (Eq. 15) depends on $y_*^2 \equiv x$, which itself (Eq. 11) depends on X through $De_c = DeX$. Therefore an iterative procedure will be required to resolve this non-linear coupling.

Friction Factor

One important engineering parameter is the Fanning friction factor

$$f \equiv \frac{\Delta p}{4\frac{L}{D_H}\frac{\rho U^2}{2}} \;=\; \frac{-p_{,z}D_H}{2\rho U^2} \tag{18}$$

and in Newtonian flow the factor fRe is a function of the radius ratio only (see [1], p. 53):

$$(fRe)_N = 16 \frac{1}{\frac{(1+k^2)}{(1-k)^2} - \frac{(1+k)}{(1-k)} \frac{1}{\ln(1/k)}} \tag{19}$$

In the case of the present PTT fluid the factor fRe is obtained directly from its definition (18):

$$fRe = 16 \frac{U_c}{U} \tag{20}$$

and $U_c/U \equiv X$ is calculated from Eq. (15).

RESULTS

Some of the important features of the present solution for the velocity and stress variations are exemplified in Figs. 1 and 2, for varying De number at fixed $\epsilon = 0.25$ and $k = 0.5$. For low elasticity ($De = 0.1$) the velocity variation in Fig. 1 differs little from the Newtonian case, but as De is increased the profiles become more flattened indicating the effect of shear-thinning. Such effect is absent from a purely elastic fluid model with infinite extensibility (that is, when $\epsilon = 0$ as for the UCM) but is one of the important features of the PTT model.

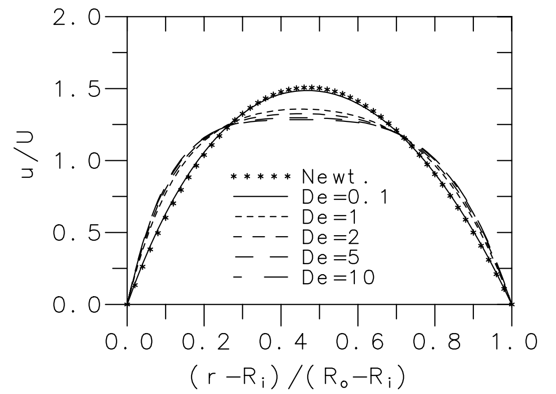

Fig. 1. Velocity profiles for various De, at fixed values of $\epsilon = 0.25$ and $k = 0.5$.

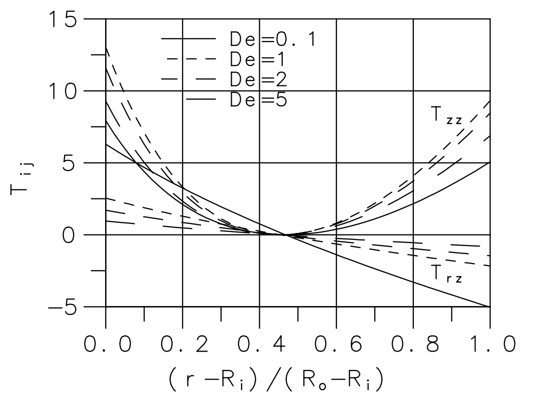

Fig. 2. Stress profiles for various De, at fixed values of $\epsilon = 0.25$ and $k = 0.5$.

The variation of shear and normal stress components across the annular gap are given in Fig. 2 in terms of the normalised radial distance $(r - R_i)/\delta$. Although the variation of T_{rz} is monotonic with increased values of De, the same does not hold for T_{zz}. The normal stress increases when De goes from 0 to 1.0, and then decreases. Such behavioral pattern is due to shear-thinning at the wall and can be removed if the stresses are scaled with the inner wall shear stress value.

In practical situations the parameter of most importance is the friction factor, which is better expressed in terms of fRe, shown in the analysis to be proportional to the solution parameter X. In order to see how the viscoelastic results depart from the Newtonian, it is better to examine the ratio of fRe to the corresponding Newtonian values, $(fRe)_N$. Fig. 3 shows a reduction in wall friction imparted by elasticity coupled with extensibility. The important finding from Fig. 3 is, however, that there is no influence of the radius ratio on the variation of the friction factor ratio $(fRe/(fRe)_N)$ with elasticity, as measured by $\epsilon^{1/2}De$ $-$ the collapse seen in the figure of all curves onto a single curve is perfect. This property could not be foreseen from the form of the solution which exhibits dependence on k in various term. If the Deborah number were defined in a different way, e.g. as $De = \lambda U/R_i$, then no collapse of the various curves would be obtained.

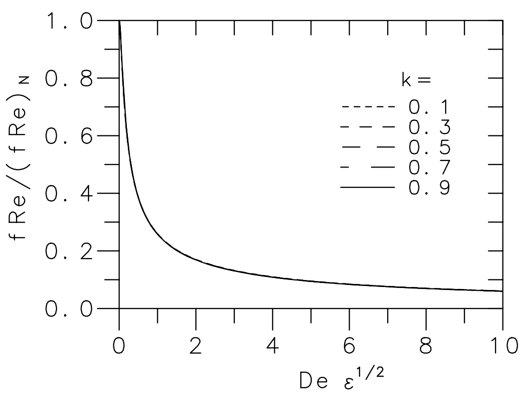

Fig. 3. Ratio fRe/fRe_N as a function of $\epsilon^{1/2}De$ for various k.

REFERENCES

1. Bird, R.B., Stewart, W.E., and Lightfoot, E.N. 1960 *Transport Phenomena*, John Wiley and Sons, New York.
2. Fredrickson, A.G., & Bird, R.B. 1958 *Ind. Eng. Chem.* **50**, 347-352.
3. Hanks, R.W., & Larsen, K.M. 1979 *Ind. Eng. Chem. Fundam.* **18**, 33-35.
4. Hanks, R.W. 1979 *Ind. Eng. Chem. Proc. Des. Dev.* **18**, 488-493.
5. Bird, R.B., Armstrong, R.C. and Hassager, O. 1977 *Dynamics of Polymeric Liquids. Volume 1: Fluid Mechanics*. John Wiley and Sons, New York.
6. Savins, J.G., & Wallick, G.C. 1966 *AIChE J.*, **12**, 357-363.
7. Dierckes, A.C., & Schowalter, W.R. 1966 *Ind. Eng Chem Fund.* **5**, 263-271.
8. Escudier, M.P., Oliveira, P.J., Pinho, F.T., & Smith, S. 2000 to be submitted to *J Non-Newt Fl Mech*.
9. Phan-Thien, N. & Tanner, R.I. 1977 *J. Non-Newtonian Fluid Mech.* **2**, 353-365.
10. Quinzani, L., Armstrong, R.C. & Brown, R.A. 1995 *J. Rheology* **39**, 1201-1228.