Estimating bivariate integer-valued moving average models with the generalized method of moments

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Abstract

In this paper an estimation method for the parameters of Bivariate INteger-valued Moving Average (BINMA) model, based on Generalized Method of Moments (GMM) is proposed. The performance of the GMM estimator and its small sample properties will be investigated in a simulation study.

Keywords: BINMA models, Count time series, Generalized Method of Moments, Parameter estimation.

1. Introduction

Time series of counts arise when the interest lies on the number of certain events occurring during a specified time interval. In many situations the collected time series are multivariate in the sense that there are counts of several events observed over time and the counts at each time point are correlated. Several approaches and diversified models that explicitly account for the discreteness of the data have been considered, among which are the INteger-valued AutoRegressive Moving Average, INARMA, models (McKenzie, 2003). These models are constructed by replacing the multiplication in the conventional ARMA models by an appropriate random operator that preserves the discreteness of the counting process. The most popular of such operator is the **binomial thinning operator** (Steutal and Van Harn, 1979) defined as $\alpha \circ Y = \sum_{j=1}^{Y} B_j$, where Y is a non-negative integer-valued random variable, $\alpha \in [0,1]$, and B_j, designated by counting series, is a sequence of independent and identically distributed (i.i.d.) Bernoulli random variables, independent of Y, such that $\mathrm{P}(B_j = 1) = 1 - \mathrm{P}(B_j = 0) = \alpha$.

Moving average (MA) models are widely used in econometric data. For example, inventories are often considered to be well described by MA processes. In fact, the periodic resetting of inventories to optimal levels implies an upper limit on the number of periods in which shocks can affect inventory levels, leading to autocorrelations functions characteristic of this type of processes. The **INteger-valued Moving Average** (INMA) process was introduced by McKenzie (1986) and Al-Osh and Alzaid (1988). A **Bivariate INMA** model that assumes independence between and within the thinning operations and that allows for both positive and negative correlation between the counts was proposed by Quoreshi (2006).

In this work, a BINMA model based on the same dependence structure of Al-Osh and Alzaid (1988) between thinning operations in the same equation is presented (Torres *et al.*, 2012), by generalizing the model proposed by Quoreshi (2006). In section 2, the model is described and emphasis is placed on models with Bivariate Poisson and Bivariate Negative Binomial innovations. The Method of Moments (MM) and the Generalized MM (GMM) are proposed in Section 3 in order to estimate the unknown parameters of these models. The methods will be compared and their performances will be investigated in a simulation study.

2. Bivariate integer-valued moving average models – BINMA(q_1, q_2)

Let $\boldsymbol{X}_t = \left[X_{1,t} \quad X_{2,t}\right]'_{t\in Z}$ be a non-negative integer-valued random vector. Then $\boldsymbol{X}_t$ can be defined as a BINMA(q_1, q_2) model, proposed by Torres *et al.* (2012), if satisfies the following equations

$$X_{1,t} = \varepsilon_{1,t} + \beta_{1,1} \circ \varepsilon_{1,t-1} + \cdots + \beta_{1,q_1} \circ \varepsilon_{1,t-q_1}$$

$$X_{2,t} = \varepsilon_{2,t} + \beta_{2,1} \circ \varepsilon_{2,t-1} + \cdots + \beta_{2,q_2} \circ \varepsilon_{2,t-q_2}$$

where $\circ$ denotes the binomial thinning operator, $\beta_{j,i} \in]\ 0,\ 1\ [$ for $j=1,\ 2;\ i=1,\ldots,\ q_j$, and $\{\varepsilon_{j,t}\},\ j=1,2;\ t\in \mathrm{Z}$, are i.i.d. sequences of non-negative integer-valued random variables, designated as innovations. Dependence between the two series that comprise the BINMA(q_1, q_2) models is introduced by allowing for dependence between $\{\varepsilon_{1,t}\}$ and $\{\varepsilon_{2,t}\}$. It is assumed the same dependence structure proposed by Al-Osh and Alzaid (1988) between thinning operations in the same equation. In this way, each element $\{\varepsilon_{j,t}\}$ can be "*active*" in the system q_j+1 time units, the $\beta_{j,i}$ are the probabilities that an element of $\{\varepsilon_{j,t}\}$ will be "*active*" in the system at time $t+i$, independent of the other elements of the system, and then $\beta_{j,i} \circ \varepsilon_{j,t-i}$ represents the number of elements of generation $t-i$ which are "*active*" in the system at time *t*.

2.1. Poisson BINMA(q_1, q_2) model

In the Poisson BINMA model the innovations of the two series follow jointly a bivariate Poisson distribution, denoted by $BP(\lambda_1,\ \lambda_2,\ \phi)$, for details see Kocherlakota and Kocherlakota (1992). Marginally each random variable follows a Poisson distribution with parameters $\lambda_1+\phi$ and $\lambda_2+\phi$, respectively. The parameter ϕ is the covariance between the two random variables.

2.2. Negative binomial BINMA(q_1, q_2) model

In the negative binomial BINMA model the innovations of the two series follow jointly a bivariate negative binomial distribution, denoted as $BNB(\lambda_1,\ \lambda_2, \tau)$, for details see Cheon *et al.*

(2009). Marginally each random variable follows a negative binomial distribution with mean λ_1 and λ_2 and variance $\lambda_1(1+\lambda_1\tau)$ and $\lambda_2(1+\lambda_2\tau)$, respectively. The covariance between the two random variables is $\lambda_1\lambda_2\tau$.

3. Parameter estimation

3.1. The method of moments

The MM estimates the population parameters by matching population (or theoretical) moments with corresponding sample moments. Suppose $\{X_{j,t}, j=1,2; t=1,\ldots,T\}$ is an observed sample from a BINMA(q_1, q_2) model with true $q_j \times 1$ parameter vector $\boldsymbol{\theta}_0$ and let $\boldsymbol{\theta}$ be the parameter vector estimator. The **moment conditions** are defined by $\mathrm{E}\lfloor m(X_{j,t},\boldsymbol{\theta}_o)\rfloor = 0$, where $m(X_{j,t},\boldsymbol{\theta})$ is a continuous $p \times 1$ vector function of $\boldsymbol{\theta}$ and the expected value exists and is finite for all t and $\boldsymbol{\theta}$ (Mátyás, 1999).

The estimator is obtained by solving the system of equations given by $\mathrm{E}\lfloor m(X_{j,t},\boldsymbol{\theta})\rfloor = 0$, with p equations for $p = q_j$ unknowns. However, $\mathrm{E}\lfloor m(X_{j,t},\boldsymbol{\theta})\rfloor$ is not observed and the analogous sample moment conditions defined by

$$m_T(\boldsymbol{\theta}) = \frac{1}{T}\sum_{t=1}^{T} m(X_{j,t},\boldsymbol{\theta})$$

is considered.

3.2. The generalized method of moments

The GMM estimation method was first introduced by Hansen (1982), into the econometrics literature and since then it has been widely applied to analyze economic and financial data. The GMM estimation is also based on population moment conditions, under the same conditions defined for the MM estimator. In this estimation method, the moment conditions are a set of p equations with $q_j < p$ parameters and the estimators are obtained by minimizing the quadratic form

$$Q_T(\boldsymbol{\theta}) = m_T(\boldsymbol{\theta})'\, W_T\, m_T(\boldsymbol{\theta}),$$

where $W_T = (\mathrm{cov}(m_T(\boldsymbol{\theta})))^{-1}$ is a positive definite weighting matrix. With any consistent estimator of W_T, the GMM estimator $\hat{\boldsymbol{\theta}}_W$ is consistent and

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0\right) \to N(\boldsymbol{0},\boldsymbol{\Sigma}_W),$$

where $\boldsymbol{\Sigma}_W = \left(\boldsymbol{D}'\boldsymbol{\Omega}^{-1}\boldsymbol{D}\right)^{-1}$, for $\boldsymbol{D} = \partial m_T(\boldsymbol{\theta})/\partial\boldsymbol{\theta}'$ and $\boldsymbol{\Omega} = \lim_{T\to\infty} T\, W_T$.

3.3. Considerations on the simulation study

The aim of the simulation study will be to illustrate the small sample properties of the two estimators previously described and compare their behaviour, regarding bias and mean squared error. Furthermore, this study will address two issues on GMM estimation which are the choice of moment conditions and the estimation of the covariance matrix.

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