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RESEARCH AND VALIDATION OF GLOBAL MPP IN THE RELIABILITY ANALYSIS OF COMPOSITE STRUCTURES

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ABSTRACT

The more realistic analysis of structure failures under uncertainties is associated with the use of probabilistic methods. One of the main problems in the structural reliability analysis of composite laminate structures is the possible existence of multiple MPP (Most Probable Failure Point). In this work, we propose a numerical inverse technique for the global MPP search as a function of the anisotropy of the laminated composites. A Bayesian method to estimate the probability of failure based on Monte Carlo simulation performs the results validation. The validation process demonstrates that the proposed methodology is adequate to estimate the probability of failure of the laminated composite structures.

Keywords: uncertainty, reliability, composite structures, inverse optimization, Bayesian estimation

INTRODUCTION

The more realistic analysis of structure failures under uncertainties is associated with the use of probabilistic methods. The probabilistic analysis of structural integrity, or structural reliability, considers the uncertainties in the variables and in the design parameters of structures. Furthermore it is studied also the impact of these uncertainties on the characterization of the failure of these physical systems. Approximate reliability methods, such as first-order methods (FORM) or second order (SORM), aim to obtain the so-called Most Probable Failure Point (MPP). MPP research is equivalent to solving a problem of optimization that until recently has been performed using gradient-based techniques. When the search methods are gradient based there is the possibility of existence of multiple failure points, that is, multiple local optima. In this case, one needs to use simulation techniques, such as the Monte Carlo Method.

FUNDAMENTAL PROBLEM OF STRUCTURAL RELIABILITY

The fundamental problem of structural reliability consists in solving the following integral:

$$p_f = P(G(\mathbf{X}) \leq 0) = \int_{G(\mathbf{x})\leq 0} f_{\mathbf{X}}(\mathbf{x})\, dx \tag{1}$$

where $\mathbf{X} = (X_1, X_2, \cdots, X_n)^T$ is a random vector defined in the probability space, $(\Omega, \mathcal{F}, P)$ where Ω is the sample space, and can be identified as the set of all the structural design scenarios, $\mathcal{F}$ is the σ-algebra of the subsets of Ω and P is the probability measure. The vector $\mathbf{X}$ represents the random variables that contain information about the uncertainties that affect the safety of the structure under consideration and $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of **X**. The basic variables usually describe the randomness in geometry, in material properties or applied loads.

For each realization of **X** the state of the structure is determined by quantities such as displacements, deformations, stress, damage measures, etc. The state of a structure can be of failure or safety. The failure state occurs when a realization of **X** is in the failure domain defined by $D_f = \{\mathbf{x} \in \mathbb{R}^n : G(\mathbf{x}) \leq 0\}$. The function $G\big(X, \Psi(\mathbf{X})\big)$ is denoted by the limit state function and depends on the input variables, **X**, and the response, $\Psi(\mathbf{X})$. For one simplification, only uses the **X** notation referring to all the random variables involved in the analysis. The set $\{\mathbf{x} \in \mathbb{R}^n : G(\boldsymbol{x}) = 0\}$ defines the limit state surface, representing the boundary between the two states.

Two sets are then considered:

$D_f = \{\mathbf{x} \in \mathbb{R}^n : G(\boldsymbol{x}) \leq 0\}$ which is called the failure domain;

$D_s = \{\mathbf{x} \in \mathbb{R}^n : G(\boldsymbol{x}) > 0\}$ which is called the safety domain.

The safety state is the state for which a structure, or part of it, can fulfill functions for which it was designed. On the other hand, the limit state is seen as the state beyond which a structure, or part of it, is no longer in safety state.

In the analysis of the reliability of composite structures, the random variables are uncorrelated, defining the vector **X**. In this particular case, the random variables are the mechanical properties of the composite laminate structure and the limit state function is,

$$G(\mathbf{x}) = \bar{R} - 1 \tag{2}$$

where $\bar{R}$ is the most critical Tsai number defined by

$$\bar{R} = min\big(R_1, \cdots, R_k, \cdots, R_{N_S}\big) \tag{3}$$

being N_s the total number of points where the stress vector is evaluated. The *Tsai number*, $\bar{R}$, which is a strength/stress ratio, is obtained from the Tsai-Wu interactive quadratic failure criterion and calculated at the *k-th* point of the structure solving equation

$$1 - \big(F_{ij} s_i s_j\big) R_k^2 - (F_i s_i) R_k = 0 \quad i, j = 1, 2, 6 \tag{4}$$

where s_i are the components of the stress vector, and F_{ij} and F_i are the strength parameters associated with unidirectional reinforced laminate defined from the macro-mechanical point of view.

If the distribution of the basic variables, $\mathrm{X_i}$, and the limit state function, $G(\mathbf{x})$ are known, the probability of failure can be employed as a measure of reliability. Using the Lind Hasofer method, one will get an approximate value for this probability.

LIND HASOFER METHOD

In 1974, Lind and Hasofer proposed a method (Hasofer and Lind, 1974) that implicitly assumes the normality and independence of the input variables associated with the randomness or uncertainty of the problem.

The problem originally defined in the normal space **x** and transformed into a problem defined in standardized normal space **u** of coordinates $u_1, u_2, \cdots, u_n$ through the transformation

$$U_i = \frac{X_i - \mu_i}{\sigma_i} \tag{5}$$

Let $U_1, U_2, \cdots, U_n$ be uncorrelated standardized normal random variables. Consider the limit state function projected in this space. Let $M = g(U_1, U_2, \cdots, U_n)$.

Let us rewritten the probability of failure in the form,

$$p_f = \int_{g(\mathbf{u}) \leq 0} \varphi(\mathbf{u})\, d\mathbf{u} \tag{6}$$

where $\varphi(\mathbf{u})$ is the joint standardized normal density function. This function has a maximum at the origin and decays exponentially with $\| \mathbf{u} \|^2$. So, the points that most contribute for the previous integral are those of the limit state surface nearest to the origin.

Thus, the first step consists in determining the point on the limit state surface nearest the origin.

In the simplest case, where the limit state function is linear, this function can be written in the form $g(\mathbf{u}) = \beta - \boldsymbol{\alpha}^T \mathbf{u}$, where β is the distance from the hyperplane $g(\mathbf{u}) = 0$ to the origin and $\boldsymbol{\alpha}$ is a unitary vector normal to the hyperplane and directed outwards. It is proved that $\varphi(\mathbf{u})$ has a maximum at $\mathbf{u}^* = \beta\boldsymbol{\alpha}$. Therefore, $\mathbf{u}^*$ is called the most probable failure point (MPP). This point on the limit state surface, nearest the origin, represents the worst combination of the random variables (Melchers, 1999).

The random variable, $= \beta_{LH} - \boldsymbol{\alpha}^T \mathbf{U}$, is normally distributed with mean value $E(M) = \beta$ and variance $V(M) = 1$ and the probability of failure given by

$$p_f = P(\beta - \alpha^T \mathbf{U} \leq 0) = \phi(-\beta) \tag{7}$$

In the case where $g(\mathbf{u})$ is not linear, the search of the MPP can be formulated as a constrained optimization problem

$$\mathbf{u}^* = \min_{g(\mathbf{u})=0} \|\mathbf{u}\| \tag{8}$$

The above problem solution uses appropriate methods. The classical method of variational calculus is to solve the equivalent problem

$$\min L(\mathbf{u}, \mu) = \|\mathbf{u}\| + \mu g(\mathbf{u}) \tag{9}$$

where μ is the Lagrange multiplier associated with the limit state equation $g(\mathbf{u}) = 0$.

If the number of basic variables is high or if the limit state function is complicated, one needs an iterative scheme to solve this problem.

After determining the MPP, the corresponding reliability index is defined,

$$|\beta| = \|\mathbf{u}^*\| \tag{10}$$

The second step consists of approximate the surface state limit.

In FORM, one considers the limit state function linearized in the MPP, $\mathbf{u}^*$. Such linearization corresponds to approximate the limit state surface $g(\mathbf{u}) = 0$ by the normal hyperplane to the vector $\mathbf{u}^*$. This hyperplane equation is $-\boldsymbol{\alpha}^T \mathbf{u}$, where the unit vector $\boldsymbol{\alpha}$ is given by $\frac{\nabla g|_{\mathbf{u}^*}}{\|\nabla g|_{\mathbf{u}^*}\|}$.

The study is then similar to the case where the limit state function is linear. So, the probability of failure is given by the equation:

$$p_f = \int_{g(\mathbf{u})\le 0} \varphi(\mathbf{u})\, d\mathbf{u} \approx p_{fFORM} = \int_{\beta-\boldsymbol{\alpha}^T\mathbf{u}\le 0} \varphi(\mathbf{u})\, d\mathbf{u} = \phi(-\beta) \tag{11}$$

The first-order approximation provides good results if the curvatures of the limit state surface are not high (Der Kiureghian, 2005). If the limit state surface is strongly nonlinear, uses a second-order approximation aiming to improve the results. The second order approximation, obtained by developing $g(\mathbf{u})$ in a second-order Taylor polynomial around $\mathbf{u}^*$, is

$$g(\mathbf{u}) \approx \nabla g|_{\mathbf{u}^*}(\mathbf{u}-\mathbf{u}^*) + \frac{1}{2}(\mathbf{u}-\mathbf{u}^*)^T \nabla^2 g|_{\mathbf{u}^*}(\mathbf{u}-\mathbf{u}^*) \tag{12}$$

where $\nabla^2 g = \left[\frac{\partial^2 g}{\partial u_i \partial u_j}\right]$ is the Hessian of the limit state function and $g(\mathbf{u}^*) = 0$.

If input variables are not normal, or are not independent, Rackwitz and Fiessler (Melchers, 1999) suggest that the problem originally defined in x-space is transformed into a problem defined in u -space.

The transformation of x-space to u -space depends on the available statistical information if the input variables are normal or not and whether they are independent or not.

MONTE CARLO SIMULATION (BAYESIAN FORMULATION) IN RELIABILITY

To estimate the probability of failure, the previous approximate methods use the so-called most probable failure point (MPP). Research from this point is an optimization problem that involves techniques based on gradients, evolutionary research or others. When using the gradient-based methods, we face the possibility of multiple points, which are multiple local optima. This problem is exacerbated, when the number of random variables involved is large, or when the degree of non-linearity of the limit state functions associated with the response of the structural system is high. In this case, there is no guarantee to access correctly the structural reliability. To overcome these difficulties, complementary approaches are proposed, such as the Monte Carlo method. These complementary approaches are used essentially to provide values for the validation of the results obtained by previous methods.

According to the Bayesian formulation, in the estimation of a parameter θ, the *a posteriori* distribution $h_{\theta|\boldsymbol{x}}(\theta|\mathbf{x})$ is the product of the likelihood of the sample $f_{\mathbf{x}|\boldsymbol{\theta}}(\mathbf{x}|\theta)$ by the *a priori* distribution $g_\theta(\theta)$ with an appropriate constant. For the particular case in which the parameter to be estimated is the probability of failure, p_f, by the Bayes Theorem, the *a posteriori* distribution of p_f, $h_{p_f|k,N}(p_f|k,N)$ is given by (Guérin et al., 2007).

$$h_{p_f|k,N}(p_f|k,N) = \frac{f_{k,N|p_f}(k,N|p_f)g_{p_f}(p_f)}{\int_0^1 f_{k,N|p_f}(k,N|p_f)g_{p_f}(p_f)dp_f} \tag{13}$$

where $g_{p_f}(p_f)$ is the *a priori* distribution and must belong to a family such that the *a posteriori* distribution $h_{\theta|\mathbf{x}}(\theta|\mathbf{x})$ is from the same family. For the likelihood, we have the binomial model

$$f_{k,N|p_f}(k,N|p_f) = \binom{N}{k}(p_f)^k(1-p_f)^{N-k}, k = 0,1,\cdots,N \tag{14}$$

from which the *a priori* distribution must be a Beta distribution.

The *a posteriori* distribution to a binomial likelihood and *a priori* Beta is

$$h_{p_f|k,N}\left(p_f\middle|k,N\right)=\begin{cases}\dfrac{1}{B(a+k,N-k+b)}\left(p_f\right)^{a+k+1}\left(1-p_f\right)^{N-k+b-1}, & 0\le p_f\le 1\\ 0 \qquad , & other\ values\end{cases} \tag{15}$$

which is a $Beta(k+a,N-k+b)$ distribution with mean value $E\left(p_f\middle|k,N\right)=\frac{k+a}{N+a-b}$ and variance $V\left(p_f\middle|k,N\right)=\frac{(a+k)(b+N-k)}{(a+b+N)^2(a+b+N+1)}$

A confidence interval with confidence γ is given by $\left(p_{fmin},p_{fmax}\right)$ with p_{fmin}such that

$$\int_0^{p_{fmin}} Beta(k+a,N-k+b)dp_f=\frac{\gamma}{2} \tag{16}$$

and p_{fmax} is such that

$$\int_0^{p_{fmax}} Beta(k+a,N-k+b)dp_f=1-\frac{\gamma}{2} \tag{17}$$

Let us consider two situations here:

- $\boldsymbol{g_{p_f}(p_f)}$ **is non-informative**

In this case, the appropriate model for $g_{p_f}(p_f)$ follows a uniform distribution $p_f{\sim}Beta\ (1,1)$. The *a posteriori* distribution is a Beta distribution $(k+1,N-k+1)$. A confidence interval, with confidence γ is given by $\left(p_{fmin},p_{fmax}\right)$ with p_{fmin} such that

$$\int_0^{p_{fmin}} Beta(k+1,N-k+1)dp_f=\frac{\gamma}{2} \tag{18}$$

and p_{fmax} is such that

$$\int_0^{p_{fmax}} Beta(k+1,N-k+1)dp_f=1-\frac{\gamma}{2} \tag{19}$$

- $\boldsymbol{g_{p_f}(p_f)}$ **is informative**

If the source of information is the knowledge of a previous Monte Carlo simulation (N' simulations and k' failures), the *a priori* distribution is a Beta distribution $(k'+1,N'-k'+1)$.

The *a posteriori* distribution is a Beta distribution $(k+k'+1,N-k+N'-k'+1)$.

A confidence interval, with confidence γ is given by $\left(p_{fmin},p_{fmax}\right)$ with p_{fmin} such that

$$\int_0^{p_{fmin}} Beta(k+k'+1,N-k+N'-k'+1)dp_f=\frac{\gamma}{2} \tag{20}$$

and p_{fmax} is such that

$$\int_0^{p_{fmax}} Beta(k+k'+1,N-k+N'-k'+1)dp_f=1-\frac{\gamma}{2} \tag{21}$$

INVERSE RELIABILITY PROBLEM

An approach based on the design of composite structures to achieve a specified reliability level is proposed, and the corresponding maximum load is outlined. The objective function describing the performance of the composite structure is defined as the square difference between the structural reliability index, β_S, and the prescribed reliability index, β_a. The design variables are the ply angle, a, and load factor, λ. The random variables are the elastic and strength material properties.

Thus, the function that describes the performance of the structural system is

$$F(\lambda, a, \mathbf{x}) = [\beta_S(\lambda, a, \mathbf{x}) - \beta_a]^2 \tag{22}$$

where $\mathbf{x}$ is the realization of random variable. The vector of applied loads is defined as: $\mathbf{L} = \lambda \mathbf{L}^{ref}$, where $\mathbf{L}^{ref}$ is the reference load vector.

The minimization of functional defined in equation (22) corresponds to a conventional RBDO inverse optimization problem (António and Hoffbauer, 2009). Indeed, for each ply angle, there will be an optimal load factor λ^* associated with a prescribed structural reliability index β_a.

The prescribed reliability index, β_a, is thus the objective to be achieved by the structural reliability index.

RESULTS

Let's consider an aircraft wing-like composite panel as shown in Figure 1. The panel thickness is equal to 0.015 m. The structure is clamped along linear side (AB) and free along opposite side. One vertical load with perpendicular direction relatively to OXY plan is applied on point C. The structure is built by one laminate made of a carbon/epoxy composite system. A balanced angle-ply laminate with eight layers and stacking sequence $[+a/-a/+45^o/-45^o]_s$ is considered in a symmetric construction. Ply angle a, is referenced to the x-axis of the reference coordinate, as detailed in Figure1. All plies have same thickness.

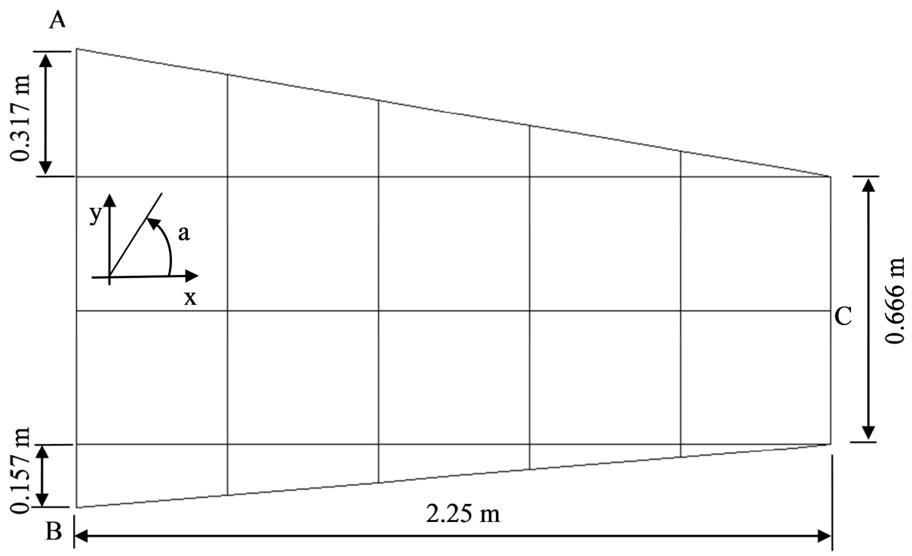

Fig. 1 - Geometric definition of aircraft wing-like composite panel.

Ahmad degenerated shell finite element (Ahmad et al., 1979) is used here for structural analysis. To assess reliability, the previously described procedure in equation (22) is applied considering the vector of random variables $\boldsymbol{X} = \left[E_1, E_{2,}, Y, S\right]$ where

$$E_1 \sim \mathcal{N}(181.00{,}10.860)GPa$$
$$E_2 \sim \mathcal{N}(10.30, 0.618)GPa$$
$$Y \sim \mathcal{N}(40.00, 2.400)MPa$$
$$S \sim \mathcal{N}(68.00, 4.080)MPa$$

The target reliability index is $\beta_a = 3$ and the coefficient of variation of each random variable is set to $CV(\boldsymbol{X}) = 6\%$. The corresponding maximum load is plotted in Figure 2 and it is used as the reference load.

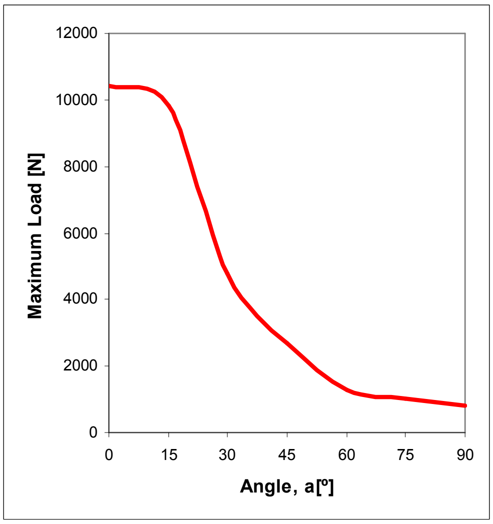

Fig. 2 - Maximum load for $\beta_a = 3$, solving the inverse RBDO problem for aircraft wing-like composite panel.

Monte Carlo Simulation method is used to study the Tsai number. Analyzing 10000 simulations, histograms descriptive measures and of the Tsai number are obtained and presented in Figure 3 and Table 1 for all angles. Histograms suggest a normal distribution for all angles.

Table 1 - Descriptive measures of the Tsai number (10000 Monte Carlo simulations)

Ply angle	Mean	C.V. (%)	Minimum	Maximum	Number of failures
0º	1.244	6.80	0.953	1.557	13
15º	1.133	3.87	0.948	1.290	13
30º	1.159	4.71	0.939	1.361	20
45º	1.216	6.32	0.910	1.519	13
60º	1.224	6.29	0.949	1.522	11
75º	1.222	6.10	0.951	1.491	13
90º	1.229	6.26	0.966	1.515	12

The results presented in Figure 2 can be validated using a Bayesian method to estimate the probability of failure based on Monte Carlo simulation, calculating a two sided 95% confidence interval of p_f, associated with the obtained maximum load, for each ply angle, a.

A prior and *a posterior* distributions used for Bayesian estimation are defined in Table 2 and their graphic representations are shown in Figure 4. The source of information is the knowledge of a previous Monte Carlo simulation ($N' = 2000$ simulations).

In this case, the maximum load is applied to the composite structure for each value of ply angle, a, and the Monte Carlo simulations used, together with Bayesian inference, to confirm the prescribed reliability index of $\beta_a = 3$. The confidence intervals for p_f are shown in Figure 5, where the red points indicates the failure probability $p_f = \phi(-3)$, calculated using equations (20) and (21).

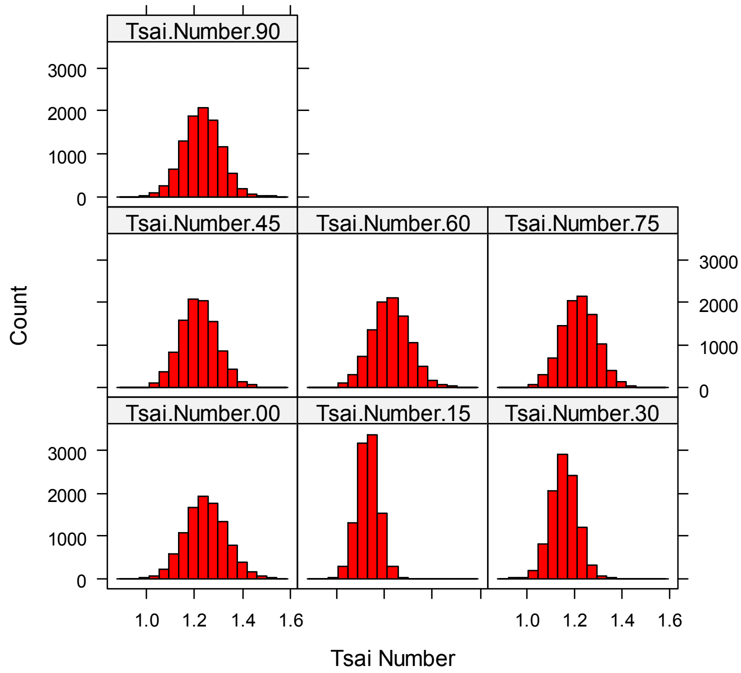

Fig. 3 - Distribution of the Tsai number.

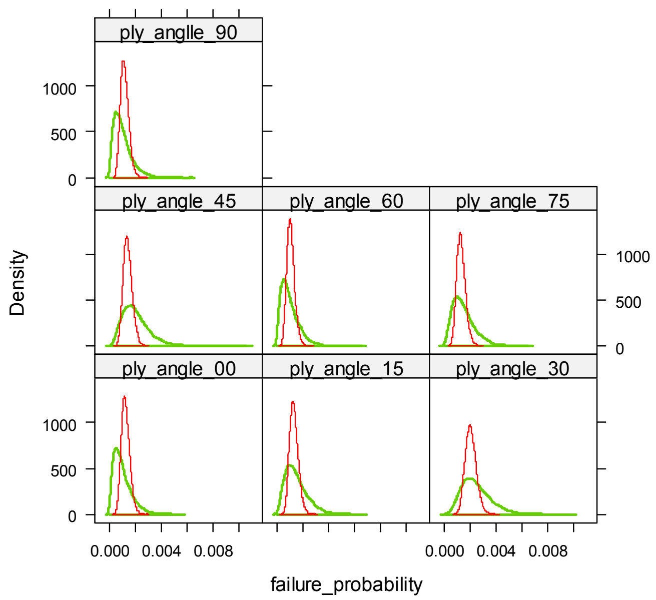

Fig. 4 - *A prior* and *a posterior* distributions used for Bayesian estimation.

Table 2 - Confidence intervals of failure probability

Ply angle	a priori distribution	a posteriori distribution	CI (95%)
0º	Beta(2,2000)	Beta(15,11987)	(0.00069972, 0.0019565)
15º	Beta(3,1999)	Beta(16,11986)	(0.00076224, 0.0020607)
30º	Beta(5,1997)	Beta(25,11977)	(0.00134855, 0.0029741)
45º	Beta(4,1998)	Beta(17,11985)	(0.00082540, 0.0021642)
60º	Beta(2,2000)	Beta(13,11989)	(0.0005769, 0.00174600)
75º	Beta(3,1999)	Beta(16,11986)	(0.00076224, 0.0020607)
90º	Beta(2,2000)	Beta(14,11988)	(0.00063792, 0.0018517)

The red points are inside of the confidence interval, showing the agreement of the results obtained using the Lind-Hasofer method with those from the Monte Carlo simulation together with Bayesian inference.

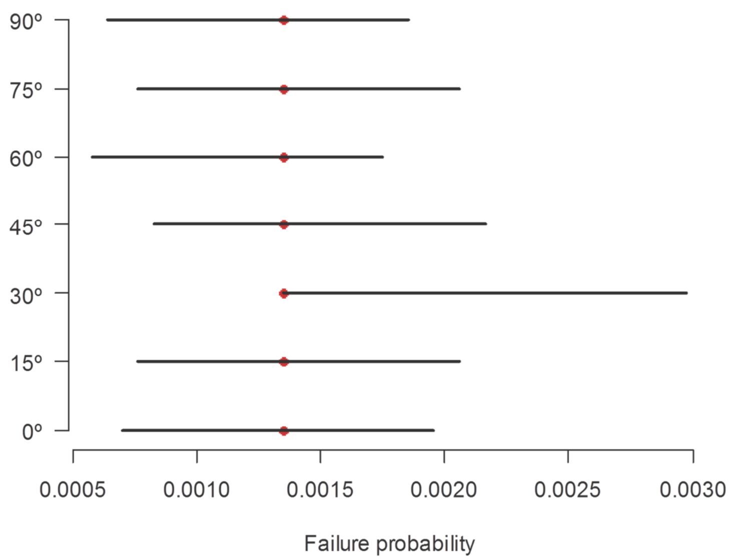

Fig. 5 - Confidence intervals of failure probability (bayesian inference).

CONCLUSION

Depending on a certain reliability index imposed on the composite structure, the maximum load is obtained, as a function of the anisotropy of the laminated composites.

The results obtained for the maximum load were validated using the Monte Carlo simulation with Bayesian methodology to estimate the probability of failure. The validation process shows that the proposed methodology is adequate to estimate the probability of failure of the laminated composite structures.

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