STRENGTH AND LIFE ANALYSIS IN PLASTIC STRAIN RANGE. NEUBER VERSUS STRAIN ENERGY CONSERVATION PRINCIPLE

Luciano Brambilla(*)
Consultant at FACC, St. Martin im Innkreis, Austria
(*)Email: l.brambilla@facc.com

ABSTRACT
The Science of Construction had a remarkable improvement with the mathematical theory of elasticity. A safety criterion was defined, in order to guarantee the integrity of the structure under the operating load: “Stress at each point of the structure to be <= of yield stress(residual strain 0.002) “. This type of analysis can be carried out simply assuming the linear correlation between stress and strain (Hooke’s law).

The next step was to calculate the structure collapse mechanism and load (=max load supported by structure up to catastrophic collapse). Analysis becomes more complex since it requires the knowledge of the plastic flow segment of the material stress-strain curves up to the strain failure. Collapse analysis must comply with the following requirements: applied and reaction load equilibrium in each point/subcomponent of the structure, compatibility of element deformations, and in addition plastic strain energy balance.

A tragic event occurred at Versailles on May 8, 1842. Two locomotives and seventeen cars were involved due to the rupture of an axle of the first locomotive. Almost one hundred people died. The enquiry draws the conclusion that failure was due to “fatigue”, for the first time. Fatigue failure due to load cycling, became a major issue for safety and for structure life estimation

Keywords: energy, fatigue, Neuber, plasticity, static.

INTRODUCTION
Plasticity has a robust influence on life and crack onset. Neuber hyperbola [1], is currently used as a “qualitative” approach for fatigue and ultimate load static analysis. When the calculated stress via Hooke’s law, is violating the material stress-strain curve, a local relaxation occurs relocating the unrealistic linear elastic stress-strain on the material plastic flow range.

The methodology presented on this paper, assumes that “local relaxation” is driven by the Energy Conservation Principle: plastic energy = linear elastic energy. The rationale behind this approach is that, for peaky stress gradient due to notches, the relaxation remains confined on the notches itself, without modifying the overall structure equilibrium.

The validation of the Energy Conservation approach, is done by comparing the analytical results with FEM non linear analysis, Neuber hyperbola and strain gage from test data.
PAPER PURPOSE

This paper compares the accuracy of Neuber Strain Energy methodology on calculation of plastic stress, from “peaky” stress gradient due to a linear analysis. Both methods are then qualified against notched test specimen data, equipped with strain gages [2].

“Peaky” stress relaxation applies to following structural conditions:

1. Structure static analysis, as long as the stress relaxation doesn’t modify local and/or overall force balance (= Free Body Diagram). (Methods not viable for structure collapse analysis).

2. Fatigue stress as product of stress concentration factor \( K_t \) and reference stress. The \( K_t \) [4] is defined assuming a local linear stress field based on notched geometry and material infinitely elastic.

Fig. 1 - Neuber Hyperbola and Strain Energy approach
STRAIN ENERGY FORMULATION

Strain energy formulation is based on complementary energy as shown on figure below:

![Complementary Energy Calculation](image)

\[ U_{tot} = \sigma_p \cdot \varepsilon_p \]

\[ U = \int_{\sigma_0}^{\sigma_p} \varepsilon(\sigma) \, d\sigma \]

\[ U^C = \int_{\sigma_0}^{\sigma_p} \sigma(\varepsilon) \, d\varepsilon = U_{tot} - U \]

Ramberg-Osgood Ref. 0

\[ \varepsilon_p = \frac{\sigma_p}{E} + 0.002 \left( \frac{\sigma_p}{\sigma_y} \right)^n \]

\[ U_{tot} = \sigma_p \cdot \varepsilon_p = \sigma_p \left[ \frac{\sigma_p}{E} + 0.002 \left( \frac{\sigma_p}{\sigma_y} \right)^n \right] = \frac{\sigma_p^2}{E} + \frac{0.002 \cdot n \cdot \sigma_y^n \cdot \sigma_y^{n+1}}{(n+1)\sigma_y} \]

\[ U = \int_{\sigma_0}^{\sigma_p} \varepsilon(\sigma) \, d\sigma = \frac{\sigma_p^2}{2E} + \frac{0.002 \cdot n \cdot \sigma_y^n \cdot \sigma_y^{n+1}}{(n+1)\sigma_y} \]

\[ U^C(\sigma) = U_{tot} - U = \frac{\sigma_p^2}{2E} + \frac{0.002 \cdot n \cdot \sigma_y^n \cdot \sigma_y^{n+1}}{(n+1)\sigma_y} \]
ALGORITHM

The stress and strain $\sigma_p, \epsilon_p$ (in plastic flow range) is calculated by assuming the equivalence of linear elastic strain energy ($U_{LE}$) and the complementary energy ($U^c$):

$$U_{LE} = \frac{1}{2} \sigma_{LE}^2 \epsilon_{LE} = \frac{\sigma_{LE}^2}{2E}, \quad U^c = \frac{\sigma_p^2}{2E} + \frac{0.002 \times n}{(n + 1) \sigma_y^n} \sigma_p^{n+1}$$

System equation

1) $\frac{\sigma_p^2}{2E} + \frac{0.002 \times n}{(n + 1) \sigma_y^n} \sigma_p^{n+1} = \frac{\sigma_{LE}^2}{2E}$

2) $\epsilon_p = \frac{\sigma_p}{E} + 0.002 \left( \frac{\sigma_p}{\sigma_y} \right)^n$

HYPERBOLA - ENERGY : MAJOR DIFFERENCES

The following remarks apply:

1. Stress variation (572., 580.) in plastic range between Neuber and Energy methods is negligible, if compared with the strain variation (0.0051, 0.0074)

2. Neuber method is estimating higher strain values, extremely conservative in term of margin of safety, neglecting a consistent strain plastic energy reservoir. Higher estimation of plastic strain is also questionable for F&DT analysis.

Fig. 3 - Neuber hyperbola and Energy - major differences.
NEUBER and ENERGY VERSUS TEST DATA: COMPARISON

Energy Conservation Principle results and Neuber approach, are compared with the notched test results taken from Ref. [2].

Test specimen data:

1. Material: St-52 and AlMgSi 1
2. Cross section: 8 x 40 mm
3. Hole in the center as a notch: 10.0 mm (Kt = 2.42 )
4. Inside the hole two strain gages were applied as shown on Figure 4 below

Fig. 4 - Test setting and Stress-Strain material curve
St-52 Test Data versus Analysis: Comparison

Calculation results (ALGORITHM)

<table>
<thead>
<tr>
<th>Measured Test Data</th>
<th>stress [Mpa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.001992</td>
<td>166.7</td>
</tr>
<tr>
<td>0.003024</td>
<td>253.4</td>
</tr>
<tr>
<td>0.003252</td>
<td>272.8</td>
</tr>
<tr>
<td>0.003528</td>
<td>295.7</td>
</tr>
<tr>
<td>0.003768</td>
<td>311.6</td>
</tr>
<tr>
<td>0.004032</td>
<td>328.8</td>
</tr>
<tr>
<td>0.004344</td>
<td>348.2</td>
</tr>
<tr>
<td>0.004656</td>
<td>363.0</td>
</tr>
<tr>
<td>0.005076</td>
<td>380.1</td>
</tr>
</tbody>
</table>

Note that calculated Neuber curve is matching the curve reported on Ref. 0.

Fig. 1 - St 52 Test Data and Analysis - comparison
AlMgSi 1 Test Data versus Analysis: Comparison

Fig. 6 - AlMgSi 1 Test Data and analysis comparison
ALGORITHM
The calculation can be performed by using a spreadsheet system (Excel). Goal-Seek plug in to calculate the $\sigma_p$ by solving the energy equation.

### MATERIAL DATA: St 52

<table>
<thead>
<tr>
<th>$E$ (Mpa)</th>
<th>$\sigma_Y$ (Mpa)</th>
<th>$n$ (-)</th>
<th>$Kt$</th>
<th>$\sigma_{dp}$ (Mpa)</th>
<th>$\varepsilon_{dp}$ (-)</th>
</tr>
</thead>
<tbody>
<tr>
<td>210000</td>
<td>570.00</td>
<td>52.00</td>
<td>2.42</td>
<td>514.8</td>
<td>0.002451</td>
</tr>
</tbody>
</table>

1) Linear elastic stress and strain curve
2) Elastic strain energy = complementary energy: $U_{LE} = U^c$
3) Goal Seek plug in and macro, to solve the energy balance equation $\sigma_p$
4) Ramberg-Osgood equation to calculate the relevant strain $\varepsilon_p$
5) Calculate $\sigma_{neuber}$ and $\varepsilon_{hyperbola}$
6) $\sigma_{neuber} = \frac{\sqrt{E \cdot \sigma_h \cdot \varepsilon_h}}{Kt}$
7) $\sigma_{energy} = \frac{\sqrt{2 \cdot E \cdot U^c}}{Kt}$

![Fig. 7 - Algorithm](image)

---

-308-
MATERIAL CURVE REFERENCED TO DIRECT PROPORTIONAL STRESS

The formulation of the stress-strain curve is based on Ramberg-Osgood equation (Ref. [3]):

\[ \varepsilon = \frac{\sigma}{E} + 0.002 \left( \frac{\sigma}{\sigma_y} \right)^n, \quad \sigma_y = \text{yield stress, E= Elastic modulus} \]

To simplify the numerical calculation, and split the curve in two segment (elastic and Elastic-Plastic), the stress-strain curve is rearranged in term by referring to Direct Proportional stress \( \sigma_{dp} \) (residual strain = 0.00001)

\[ \varepsilon = \frac{\sigma}{E} + 0.00001 \left( \frac{\sigma}{\sigma_{\phi}} \right)^n, \quad \sigma_{\phi} = \sigma_y \left( \frac{0.00001}{0.002} \right)^{\frac{1}{n}} \]

Fig. 8 - Ramberg-Osgood versus Direct Proportional Stress

The results of the energy approach is graphically shown on the diagram below for St-52 material. Linear elastic stress \( \sigma_{LE} = Kt \sigma_{ref} \) is the initial data. The values \( (\sigma_p, \varepsilon_p) \) on the stress-strain material curve, indicates the Strain Energy Equivalence point \( (U_{LE} = U^p) \).

Example

\[ \sigma_{ref} = 413 ~ [MPa] \]
\[ \sigma_{LE} = \sigma_{ref} * Kt = 413 \times 2.42 = 1000 ~ [MPa] \]
\[ U_{LE} = \frac{\sigma_{LE}^2}{2 \times E} = \frac{1000^2}{2 \times 210000} = 2.3809 \]
\[ \sigma_p = 574 ~ [MPa] \]
\[ \varepsilon_p = 0.00561 \]

-309-
CONCLUSION

Neuber hyperbola has been widely used over the past decades as a method to predict the effect of stress concentration on static strength and life.

The approach based on Strain Energy Conservation Principle, object of this paper, is meant to refine the calculation.

The outcomes of this approach are:

1. Accurate ultimate static Margin of Safety (based on strain rather than stress)
2. Reliable fatigue life estimation for strain cycling in plastic segment.

REFERENCES


