VARIATIONAL AND DIVERGENTIONAL SPH AND MLSM COMPARISON FOR HIGH SPEED IMPACT

Alexander V. Gerasimov(*)
Roman O. Cherepanov

Institute of Applied Mathematics and Mechanics (IAMM), Tomsk State University, Tomsk, Russia

(*) Email: ger@mail.tomsknet.ru

ABSTRACT
This work compares SPH and MLSM approximation in high speed impact modelling. Direct approximation of differential equations and weak variational formulation were analysed. Numerical tests for different node distribution were made. It is shown that in both cases the direct use of approximation formulas to the corresponding equations does not allow to achieve at the same time energy conservation and the correct calculation of the acceleration field in general.

Keywords: SPH, MLSM, meshless methods.

INTRODUCTION
Smoothed particle hydrodynamics (SPH) [1] is a popular numerical meshless method of hydrodynamics and solid mechanics. It is simple in realization, robust and able to treat large deformations with fragmentation efficiently. SPH is based on kernel approximation with smooth finite kernel function. Moving least square method [1] is other numerical meshless method, which uses least square method to build approximation of spatial derivatives on arbitrary set of nodes.

Basis of SPH [2] approximation is equation

\[ f^i \approx \int f(\vec{x}) W(\vec{r} - \vec{r}^i, h) dV \]  

(0.1)

where \( h \) is smoothing parameter, which define a radius of fluency for points, \( \vec{x} \) - is a space coordinate, \( W \) - smoothing function.

Spatial derivatives are defined via:

\[ f^i_{,\alpha} = \frac{\partial f^i}{\partial x_\alpha} \approx \int f(\vec{r}) W_{,\alpha}(\vec{r} - \vec{r}^i, h) dV \]  

(0.2)

Corresponding to (0.2) particle approximation is written as:

\[ f^i_{,\alpha} \approx \frac{\partial f^i}{\partial x_\alpha} \approx \sum_k f^k W_{,\alpha}^k \frac{m^k}{\rho^k} \]  

(0.3)
where $r^k, f^k, m^k, \rho^k$ - radius-vector, approximated function value, mass and density at $k$-th point, $w_{\alpha}^k = W(x^k - x^i, h)_{\alpha}$ and $h$ - smoothing parameter.

SPH approximation (0.3) has $C^0$ consistency [3], and some techniques were developed to restore particle consistency [3-5]. It leads to improved approximation, but in general approach stays the same.

MLSM uses approximation with minimal $L_2$ norm:

$$\sum_k v^k \left( f^k - f^p - f_{\alpha}^p \left( r_{\alpha}^k - r_{\beta}^p \right) \right)^2 = \text{min} \quad (0.4)$$

Eq. (0.4) lead to simple approximation of first order:

$$f_{\alpha}^p = T_{\alpha \beta}^p \cdot \sum_k f^k \left( r_{\alpha}^k - r_{\beta}^p \right) \quad (0.5)$$

where

$$T_{\alpha \beta}^p = \left[ \sum_k \left( r_{\alpha}^k - r_{\alpha}^p \right) \left( r_{\beta}^k - r_{\beta}^p \right) \right]^{-1} \quad (0.6)$$

Summation is carried out over the nodes located near the node $P_i$: $|p^k - \bar{p}^p| < 2h$.

Both approximations lead to simple and robust numerical techniques for modeling of elasto-plastic flows.

Equation of motion of elasto-plastic media has form:

$$\dot{\rho}^\delta = \sigma_\gamma^\delta \quad (0.7)$$

If described approximations are used to calculate divergence of stress

$$\sigma_\gamma^\delta = \sum_k \sigma_{\gamma \delta} W_{\gamma \delta}^k \frac{m^k}{\rho^k} \quad \text{SPH}[1]$$

$$\sigma_\gamma^\delta = \sum_k \left( \sigma_{\gamma \delta}^k + \sigma_{\delta \gamma}^k \right) W_{\gamma \delta}^k \frac{m^k}{\rho^k} \quad \text{symmetricSPH}$$

$$\sigma_\gamma^\delta = T_{\gamma \delta}^p \sum_k \sigma_{\gamma \delta} W_{\gamma \delta}^k \frac{m^k}{\rho^k} \quad \text{xSPH}[3]$$

$$\sigma_\gamma^\delta = T_{\gamma \delta}^p \sum_k \sigma_{\gamma \delta}^k \left( r_{\beta}^k - r_{\beta}^p \right) \quad \text{MLSM}[1]$$
and first order approximation is used for calculation of time derivative of velocity:

\[ \dot{v} = \frac{v^{n+1} - v^n}{\Delta t^n} \]  

(0.9)

we have forms of SPH or MLSM method.

It is shown [6] this approximation has some disadvantages for impact modeling. First of one is nonsymmetrical nodal forces. In SPH we start with approximation:

\[ \sigma_{ij}^p = \sum_k \sigma_{ij}^k w_{ij}^k v^k \]  

(0.10)

\[ F_a^{pk} = m^p v_a^p = V^p \sum_k \sigma_{ij}^k w_{ij}^k \cdot v^k \]  

(0.11)

\[ V^p \sigma_{ij}^k w_{ij}^k \cdot v^k \neq -V^k \sigma_{ij}^p w_{ij}^k \cdot V^p \Rightarrow F_a^{pk} \neq -F_a^{kp} \]  

(0.12)

Therefore momentum conservation is lost and some correction should be made to restore it.

Simple way to restore momentum conservation is to perform some symmetrization of approximation:

\[ F_i^{pk} = V^p \sum_k \left( \sigma_{ij}^k + \sigma_{ij}^p \right) w_{ij}^k \cdot v^k \]  

(0.13)

It can be treated as numerical analog of equation of motion in form:

\[ \dot{v}_i = \frac{1}{\rho} \left( \sigma_{ij,i} + \sigma_{ij} \nabla_j \right) \]  

(0.14)

In MLSM this method is usefulles, because of MLSM approximation of (0.14) is:

\[ \sigma_{ij,i}^p + \sigma_{ij}^p \nabla_j \right) = T_{ij}^{p} \sum_k \left( \sigma_{ij}^k + \sigma_{ij}^p \right) \left( r_i^k - r_i^p \right) \]  

(0.15)

One can see Eq (0.15) lead to nonsymmetrical interparticle force:

\[ V^p T_{ij}^{p} \left( \sigma_{ij}^k + \sigma_{ij}^p \right) \left( r_i^k - r_i^p \right) \neq -V^k T_{ij}^{k} \left( \sigma_{ij}^p + \sigma_{ij}^p \right) \left( r_i^k - r_i^k \right) \]  

(0.16)

and reason of this asymmetry is the fact that correction matrices \( T_{ij}^{p} \) and \( T_{ij}^{k} \) are different in general and there is no way to guarantee their equality. This matrices depends on nodal distribution, which is moving with media during impact.
The second problem is calculation of acceleration at constant press. Even for symmetrized equations we have:

\[
\dot{v}_i^p = V_i^p \cdot \sum_k \left( \sigma_{ij}^k + \sigma_{ji}^k \right) W_{ij}^{pk} \cdot V^k = 2\sigma_{ij} \cdot V_i^p \sum_k W_{ij}^{pk} \cdot V^k \quad (0.17)
\]

The summ in eq. (0.17) is not zero, if nodal distribution is not symmetric with respect to node \( P \). If we test SPH code with analytical solutions at cubical particle displacement or with high ratios of smoothing length to interparticle distance, we have near correct calculated acceleration at constant stress field, but if we try to calculate strong shock waves, at front of wave nodal distribution is strongly nonsymmetric, and acceleration calculation loses accuracy.

Third problem is energy conservation: we need global conservation of energy, but are trying to build local approximation of stress divergentia and velocity gradients, while all this values must be linked.

If approximations are applied to weak variational formulation of equations of motion:

\[
\int \rho \dot{\varepsilon}_{ij} dV + \int \sigma_{ij} \delta \varepsilon_{ij} dV + \int P_i \delta u_i dS_j = 0 \quad (0.18)
\]

and SPH or MLSM approximation used to calculate deformation rate tensor \( \varepsilon_{ij} \) and its variation \( \delta \varepsilon_{ij} \), then we obtain other form of SPH and MLSM method. In this case energy and momentum conservation are satisfied by construction but problem of incorrect acceleration calculation with constant stress field is still persist. It is important, that at computer realization of method we usually perform numerical tests for initially "good" conditions: nodes are placed at cubical mesh, they have equal masses, densities and volumes, shock wave propagation is tested on 1D problem, with small \( h / \Delta x \) ratio (when SPH/MLSM becomes equal to finite difference method). This lead to good tests, but high speed impact is not "good" task from this point of view.

RESULTS

Table 1 - Comparison of SPH and MLSM

<table>
<thead>
<tr>
<th></th>
<th>Energy conservation</th>
<th>Momentum conservation</th>
<th>Correct acceleration with constant stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPH</td>
<td>divergent</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>variational</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>SPH, restored</td>
<td>divergent</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>particle consistence</td>
<td>variational</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>MLSM</td>
<td>divergent</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td></td>
<td>variational</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>
In case of uses of weak variational formulation approximation must satisfy Gauss theorem to calculate accelerations of nodes correctly, but SPH and MLSM does not, and therefore, both method have troubles with acceleration calculation when gradient of density present or when nodes are placed asymmetrically.

Procedure of particle consistence restoring, used in SPH, introduces a correction matrix, calculated at each node to build $C^1$-approximation, but numerical integration and numerical differentiation in this case are not consistent. It should be noted that well known conditions for zeroth- and first-order completeness in $\mathbb{R}^3$ for $W(\vec{x}, h)$ are not sufficient to satisfy numerical Gauss theorem, and corrections of Liu's type [3] or Krongauz-Belytschko [4] or Randles-Liberskiy [5] does not satisfy Gauss theorem too. Special kind of smoothing functions [7, 8] satisfy discrete conditions for zeroth- and first-order completeness without special corrections, but doesn't satisfy numerical Gauss theorem, this corrected kernels still have inequality:

\begin{equation}
\int_V f_{\alpha} dV \approx \sum_{P \in V} f_{\alpha}^P v_P^P \neq \sum_{S \in S} f^k S_a^k \approx \int_S fdS \tag{0.19}
\end{equation}

CONCLUSION

While Galerkin approach provide clear framework to derive discrete equations with energy and momentum conservation, and special kernel or corrections guarantee first order of accuracy, there is a lack of final accuracy of SPH and MLMS due to inconsistency of numerical integration and numerical differentiation procedures. This inconsistency is expressed in violation of the Gauss theorem, and for Galerkin approach this violation is important, because basis of approach is weak variational formulation of equation of motion, which involves integration of gradients of velocity. To avoid such loses of accuracy a meshless approximation satisfying the Gauss theorem should be developed.

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation (RSF), project No. 16-19-10264.

REFERENCES


