Proceedings of the 7th International Conference on Mechanics and Materials in Design,
Albufeira/Portugal 11-15 June 2017. Editors J.F. Silva Gomes and S.A. Meguid.
Publ. INEGI/FEUP (2017)

PAPER REF: 6923

# AN AVERAGE-BASED DESIGN OPTIMIZATION ALGORITHM

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## ABSTRACT

This article introduces a metaheuristic algorithm to solve engineering design optimization problems. The algorithm is based on the concept of diversity and independence that is aggregated in the average design of a population of designs containing information dispersed through a variety of points, and on the concept of intensification represented by the best design. It is a population-based algorithm. The algorithm is validated using standard classical constrained test engineering optimum design problems reported in the literature. The results presented indicate that the proposed algorithm is a very simple alternative to solve this kind of problems. They compare well with the analytical solutions and/or the best results achieved so far. Two constrained problem analytical solutions not found in the literature are presented in annex.

***Keywords:*** heuristic, randomness, population-based.

## INTRODUCTION

With the advent of fast, cheap and reliable computing power over the last decades, in addition to the application of classic optimization to larger and larger size problems, new alternative algorithms operating in a different fashion have been developed. The classical optimization algorithms have shortcomings and are not suitable for all optimization problems. The new alternative algorithms allow attacking optimization problems either too costly or not applicable to classical algorithms.

The purpose of heuristic algorithms applied to optimization problems is to search a solution to them by trial-and-error in a satisfying amount of computing time. The optimum solution is not guaranteed but a near optimum solution is accepted as a good solution. Metaheuristics refers to higher-level algorithms combining lower-level techniques and tactics for exploration and exploitation of the design space. That is, these algorithms, on one hand, must be able to generate a range of points in the whole design space including potentially optimum ones; on the other hand, they intensify the search around the neighborhood of an optimum or near optimum points (Yang, 2008). Exploration and exploitation are the two important components of the metaheuristic algorithms. They are also called diversification and intensification (Glover, 1997). A good balance of these components is required. Too much weight in diversification risks slow convergence with the solutions jumping around the potentially optimum ones; too much weight in intensification restricts the design space to a local region and risks the convergence to a local optimum (Blum, 2003). The heuristic algorithms start typically either with guess solutions randomly generated or with a designer preferred trial solution. The diversification is gradually reduced as the algorithm proceeds; simultaneously, the intensification is increased. One of the roles of injected randomness in stochastic search is

to allow for "spontaneous" movements to unexplored areas of the search space that may contain an unexpectedly good design. This is especially relevant when the search is stalled near a local solution. Injected randomness may also be used for the creation of simple random quantities that act like their deterministic counterparts, but which are much easier to obtain and more efficient to compute. Metaheuristics algorithms are either population-based or trajectory-based. Examples of the former ones are the genetic algorithms (Holland, 1975) or the particle swarm optimization (Cagnina, 2008; Kayham, 2010); the last ones include the simulated annealing optimization (Kirkpatrick, 1983) or the harmonic search Geem, 2001).

If a sufficient numerous and diversified group of people is asked to decide on subjects of general interest, the decisions of the group are better than the decisions that an isolated individual would take (Surowiecki, 2005). Well informed and sophisticated an expert is, his or her advice and predictions should be pooled with those of others to get the most out of him or her. Practical examples, simple and complex, are described in (Surowiecki, 2005), remarking the principle of group think, and the concept that the masses are better problem solvers, forecasters, and decision makers than any one individual. A classic example of group intelligence is the jelly-beans-in-the-jar experiment, in which invariably the group's average estimate for the number of the jelly beans in the jar is superior to the vast majority of the individual guesses. The idea that large groups of people are smarter than an elite few may be a surprise to many involved in decision making and problem solving, and can be a concept that is difficult to accept. The theory that groups are remarkably intelligent and often smarter than the smartest people in them, demonstrates the significant impact on how businesses operate, how knowledge is increased, how economies are structured, and how people live their daily lives (Williams, 2006). The necessary conditions for a crowd to be wise include diversity, independence, and a specific type of decentralization. These conditions are essential to making good decisions which are the result of disagreement and contest rather than consensus or compromise. Diversity means individuals have some private information or their own interpretation of known facts. Diversity helps because it actually adds perspectives that would otherwise be absent and because it takes away, or at least weakens, some of the destructive characteristics of group decision-making. Homogeneous groups are great at doing what they do well, but they become progressively less able to investigate alternatives. Independence means freedom from the influence of others. It keeps the individual mistakes from becoming correlated. Diversity is essential to preserving this independence. In the jelly-beans-in-the-jar experiment, most group members are not talking to each other or solving problems together. Decentralization means people draw on local knowledge. It encourages individuals to make important decisions, not just in one location based only on one specific type of information, but dispersed through a variety of locations from where local knowledge is drawn and shared. The information coming out of a decentralized group must be aggregated throughout the system, to maintain a balance between local and global counterparts. Aggregation needs a mechanism that turns individual judgments into a collective decision. For instance, in a free market, the aggregating mechanism is price; in the jelly-beans-in-the-jar experiment, individual guesses were aggregated and then averaged, i.e., the aggregating mechanism is the average guess.

The aim of the present article is the proposal of an algorithm based on the concept just described. The algorithm considered is stochastic in the sense that it relies on random numbers and that different results may be obtained upon running the algorithm repeatedly. The algorithm is population-based, where the population individual designs are randomly generated. These individuals are diversified and independent since their design variables

values are chosen stochastically without any correlation. The individual designs are also decentralized since the design variables are chosen all over the entire design space. Finally, the different values of the design variables are aggregated as the plain or weighted average of those values. However, generating a diverse set of possible solutions isn't enough. The designer, as the body of people, also has to be able to distinguish the good solutions for the bad. So, at each of iterations, the present algorithm selects two designs: the best design, the one with the best objective so far; and the averaged design, the one which design variable values are the mean of those variable values for the iteration. The best design represents the intensification component of the algorithm; the averaged design represents the diversity part. In the current article, the optimization problem is understood as a minimization problem, where the function of merit to be assessed is an extended cost function that takes in account penalization due to violated constraints. A reference design is considered as a linear combination of both the best and the averaged design variables. A simple recurrence formula, centered on the reference design, is used to actualize the design for the next randomly generated population. The population can be normally or uniformly generated.

In optimization, there is traditionally a concern with developing a good stopping criterion. Unfortunately, the quest for an automatic means of stopping an algorithm with a guaranteed level of accuracy seems doomed to failure in general stochastic search problems. The fundamental reason is that in nontrivial problems, there will always be a significant region within the design space that will remain unexplored in any finite number of iterations, and there is always the possibility the optimum could lie in this unexplored region. A danger arises in making broad claims about the performance of an algorithm based on the results of numerical studies. Performance can vary tremendously under even small changes in the form of the functions involved or the coefficient settings within the algorithms themselves. Outstanding performance on some types of functions is consistent with poor performance on other types of functions. This is a manifestation of no free lunch theorems (Spall, 2003; Wolpert, 1997).

The present algorithm is applied to several typical engineering design problems profusely used as test problems.

## THE OPTIMAL DESIGN PROBLEM

The optimal design problem may be formulated in a generalized fashion as

$$\begin{aligned} &\min_{\boldsymbol{b}} \Psi_0(\boldsymbol{b}) \\ &\text{s.t.} \\ &\Psi_j(\boldsymbol{b}) \le 0; \qquad j = 1,2,\ldots,m \\ &b_{il} \le b_i \le b_{iu}; \qquad i = 1,2,\ldots,n \end{aligned} \tag{1}$$

where $\Psi_0$ is the cost or objective function, $\Psi_j$ are the constraints in a dimensionless form, $\boldsymbol{b} \equiv (b_1, b_2, \ldots, b_n)$ is the design vector, $b_{il}$ and $b_{iu}$ are respectively the lower and upper bounds of the design variable $b_i$.

In order of applying the present algorithm, the formulation of the Eq. (1) is particularized as the following one:

$$\begin{aligned} &\min_{\boldsymbol{b}} \ \bar{\Psi}_0 \equiv \Psi_0 + |\Psi_0| P \\ &s.t. \\ &b_{il} \le b_i \le b_{iu}; \qquad i = 1,2,\ldots,n \end{aligned} \tag{2}$$

In Eq. (2), $P$ stands for the penalty factor which value depends on the violation of the constraints as

$$P = \begin{cases} 0, & \text{for satisfied constraints} \\ \sum_{v=1}^{m'} \dfrac{\alpha \Psi_v}{m'}, & \text{for violated constraints} \end{cases} \tag{3}$$

where $\alpha > 0$ and $m'$ is the number of violated constraints. In practical terms, the constraint $\Psi_j$ can be considered violated if $\Psi_j > \varepsilon$, with $\varepsilon$ a very small number. Note that the absolute value of the objective is considered in the Eq. (2) in order to accommodate problems with negative objective functions.

## THE AVERAGE-BASED CONCEPT ALGORITHM

The present algorithm may be described in the following manner:

1. Start with a preference design guess and an imposed standard deviation $s_i = b_{iu} - b_{il}$, $(i = 1,2,\ldots,n)$. Compute the starting values of $\Psi_0, \Psi_j, (j = 1,2,\ldots,m)$ and $\bar{\Psi}_0$, and establish the starting reference design $\boldsymbol{b}^R$ as the preference guess, i.e. $\boldsymbol{b}^R \equiv \boldsymbol{b}$ .

Start iterates as

2. Launch a normally or uniformly random population of $N$ designs as

$$b_i^k = b_i^R + s_i r_i^k; \ i = 1,2,\ldots,n; \ k = 1,2,\ldots,N \qquad \text{or} \qquad \boldsymbol{b}^k = \boldsymbol{b}^R + \boldsymbol{s} \square \boldsymbol{r}^k \tag{4}$$

   where $r_i^k$ is the $k$-th random number related to the design variable $b_i$. Do $b_i = b_{il}$ if $b_i < b_{il}$, $b_i = b_{iu}$ if $b_i > b_{iu}$.

3. Evaluate $\Psi_0, \Psi_j, (j = 1,2,\ldots,m)$ and $\bar{\Psi}_0$ for the entire population of $N$ designs.

4. Evaluate the best design $\boldsymbol{b}^B$ corresponding to the minimum value $\bar{\Psi}_0^{min}$ of the extended function. Evaluate the averaged design $\boldsymbol{b}^A$ for the distribution of designs as

$$\boldsymbol{b}^A = \sum_{k=1}^{N} \boldsymbol{b}^k p_k \Big/ \sum_{k=1}^{N} p_k \tag{5}$$

   where $\boldsymbol{b}^k$ is an arbitrary design vector in the population and $p_k$ is the weight accounted for design $\boldsymbol{b}^k$ on the average. The weights are selected by the designer.

If a plain average is chosen, then $p_k = 1$ for every design.

Another choice used in this work is to compare the values of $\boldsymbol{b}^k$ corresponding extended function $\overline{\Psi}_0$ and of the best extended function $\overline{\Psi}_0^{min}$ in the previous iteration; then assign a weight $p_k = 2$ if the first value is smaller than the second one, a weight $p_k = 1$ if it is equal or larger.

5. If there are no constraint violations and there are no improvements of the objective function within a prescribed number of iterations, go to 7.

6. Evaluate a reference design as the linear combination

$$\boldsymbol{b}^R = \theta \boldsymbol{b}^B + (1-\theta)\boldsymbol{b}^A \tag{6}$$

with $0 \le \theta \le 1$ and assume the distribution of designs $\boldsymbol{b}^k$ centered at $\boldsymbol{b}^R$ with a standard deviation vector $\boldsymbol{s} = 2\left|\boldsymbol{b}^A - \boldsymbol{b}^B\right|$.

Go to 2.

End the iterates

7. Stop.

In order of handling tabular discrete value design variables, the Eq. (4) is rewritten as

$$b_i^k = int\left(\frac{b_i^R + s_i r_i^k}{\Delta}\right)\ \Delta; \qquad i = 1,2,\ldots,n; \quad k = 1,2,\ldots,N$$

where $\Delta$ is the difference value between two consecutive design variables.

## NUMERICAL APPLICATIONS

In this Section, one is going to solve three well-known engineering design optimization test problems by applying the formulation and algorithm presented on the previous sections: the welded beam design, the pressure vessel design and the tension-compression spring design. For all the applications, normally distributed populations were used. A total of 30 independent runs were performed per problem. In the first three sections the runs of the algorithm were performed with a particular initial seed for different population sizes and different weights on the calculation of the average and reference designs. The results are compared with analytical solutions and/or heuristic and nonlinear programming algorithms. In the fourth section are presented the mean and standard deviation for each problem as well as its best solution over the 30 independent runs referred above.

### Welded Beam Design

The welded beam design problem is well studied in the context of single-objective optimization. A beam *A* needs to be welded on another beam *B* and must carry a certain load *P* as shown in Fig. 1. The welded beam is designed for minimum fabrication cost subject to

constraints on shear stress $\tau$, bending stress in the beam $\sigma$, buckling load on the bar $P_c$, end deflection of the beam $\delta$, cost of the weld and beam $A$ materials, and side constraints (Ragsdell, 1975; Rao, 1996). One wants to find four design parameters: thickness of the beam *b*, width of the beam *t*, length of the weld *l*, and thickness of the weld *h*.

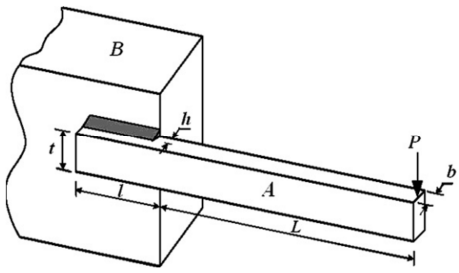


Fig. 1 - Welded beam structure

In order of formulating the problem in a standard form, let $\boldsymbol{x} \equiv (x_1, x_2, x_3, x_4) = (h, l, t, b)$ be the design vector. Then, our problem may be described as

$$
\begin{aligned}
& min\ \Psi_0 \equiv \left(C_1 + C_2\right) x_1^2 x_2 + C_3\, x_3 x_4 \left(L\ +\ x_2\right) \\
& s.t. \\
& \Psi_1 \equiv \tau / \tau_{max} - 1 \le 0 \\
& \Psi_2 \equiv \sigma / \sigma_{max} - 1 \le 0 \\
& \Psi_3 \equiv x_1 / x_4 - 1 \le 0 \\
& \Psi_4 \equiv (C_1\, x_1^2 + C_3\, x_3 x_4 \left(L\ +\ x_2\right)) / 5 - 1 \le 0 \\
& \Psi_5 \equiv \delta / \delta_{max} - 1 \le 0 \\
& \Psi_6 \equiv 1 - P_c / P \le\ 0 \\
& 0.125 \le x_1 \le 2\ ; \quad 0.1 \le x_4 \le 2\ ; \quad 0.1 \le x_2, x_3 \le 10
\end{aligned}
$$

where $C_1 = 0.10471(\$/\text{in}^3)$ is the cost per unit volume of the weld material, $C_2 = 1(\$/\text{in}^3)$ is the labor cost per unit weld volume, $C_3 = 0.04811(\$/\text{in}^3)$ is the cost per unit volume of the beam *B*, $\tau_{max} = 13{,}600\,\text{psi}$, $\sigma_{max} = 30{,}000\,\text{psi}$, $\delta_{max} = 0.25$ in, and

$$\tau = \sqrt{\tau_1^2 + \tau_1 \tau_2\, x_2 / R + \tau_2^2}, \qquad \tau_1 = P / (\sqrt{2} x_1 x_2), \qquad \tau_2 = MR / J, \qquad M = P(L + x_2/2)$$

$$R = (1/2)\sqrt{x_2^2 + (x_1 + x_3)^2}, \qquad J = (\sqrt{2}/6)\, x_1 x_2 [x_2^2 + 3(x_1 + x_3)^2]$$

$$\sigma = 6PL / (x_3^2 x_4), \qquad \delta = 4PL^3 / (E\, x_3^3 x_4)$$

$$P_c = 4.013 (E / L^2)(x_3 x_4^3 / 6)[1 - (\ 0.5 x_3 / L)\sqrt{0.25\, E / G}]$$

$$P = 6{,}000\,\text{lb}, \qquad L = 14\,\text{in}, \qquad E = (30)(10^6)\,\text{psi}, \qquad G = (12)(10^6)\,\text{psi}$$

By using mathematical programming, (Rao, 1996) presents the optimum cost function $\Psi_0^* \equiv 2.3810$ corresponding to the design point $\boldsymbol{x}^* \equiv (0.2444, 6.2177, 8.2915, 0.2444)$. A better nonlinear programming solution has been achieved in (Andrei, 2013) by adding GAMS created nonlinear model: $\Psi_0^* \equiv 1.72485$, $\boldsymbol{x}^* \equiv (0.206, 3.470, 9.037, 0.206)$. The lowest

optimal solution known so far by the authors is given in (Kayham, 2010) as $\Psi_0^* \equiv 1.724717$, $\boldsymbol{x}^* \equiv (0.205830,\ 3.468338,\ 9.036624,\ 0.205730)$.

The optimization results achieved with the present algorithm are shown in Table 1 for different population sizes *N*, $\theta = 0.85$, and different average choices on Eq. (5). In Table 1, *IP* = 0 stands for plain average and *IP* = 1 means weighted average as written on step 4 of the algorithm described on Section 3. The first row for each choice combination of *N* and *IP* represents the optimum at the convergence of the algorithm. The following rows are the results at intermediary iterations. For all these design points all the constraints are satisfied.

Table 1 - Welded beam optimization solutions

| | Iterate | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\Psi_0$ |
|---|---|---|---|---|---|---|
| N = 5000, IP = 0 | 16645 | 0.205729 | 3.470519 | 9.036630 | 0.205730 | 1.724858 |
| | 268 | 0.205468 | 3.476687 | 9.038588 | 0.205723 | 1.725573 |
| | 32 | 0.205436 | 3.477702 | 9.034963 | 0.205964 | 1.726863 |
| | 16 | 0.204191 | 3.530001 | 9.042473 | 0.205769 | 1.731815 |
| N = 5000, IP = 1 | 11144 | 0.205726 | 3.470586 | 9.036639 | 0.205730 | 1.724864 |
| | 303 | 0.204957 | 3.488440 | 9.035478 | 0.205797 | 1.726386 |
| | 57 | 0.205409 | 3.475931 | 9.042488 | 0.205808 | 1.726697 |
| | 19 | 0.204431 | 3.523721 | 9.041501 | 0.205707 | 1.730704 |
| N = 1000, IP = 0 | 9434 | 0.205727 | 3.470567 | 9.036656 | 0.205730 | 1.724866 |
| | 27 | 0.204332 | 3.495317 | 9.051999 | 0.205779 | 1.729056 |
| | 19 | 0.204359 | 3.503654 | 9.084193 | 0.205596 | 1.734414 |
| N = 1000, IP = 1 | 10419 | 0.205730 | 3.470530 | 9.036634 | 0.205730 | 1.724866 |
| | 1335 | 0.205707 | 3.470647 | 9.038382 | 0.205760 | 1.725373 |
| | 522 | 0.205451 | 3.477778 | 9.038218 | 0.205742 | 1.725777 |
| | 37 | 0.205635 | 3.479632 | 9.039730 | 0.205804 | 1.727052 |
| | 21 | 0.206227 | 3.492924 | 9.013144 | 0.206816 | 1.732874 |
| N = 100, IP = 0 | 15080 | 0.205727 | 3.470617 | 9.036695 | 0.205729 | 1.724877 |
| | 1596 | 0.205693 | 3.470650 | 9.039433 | 0.205731 | 1.725315 |
| | 205 | 0.205585 | 3.477111 | 9.031481 | 0.206261 | 1.728674 |
| | 177 | 0.206150 | 3.503281 | 9.041299 | 0.206170 | 1.734152 |
| N = 100, IP = 1 | 27689 | 0.205724 | 3.470646 | 9.036589 | 0.205731 | 1.724869 |
| | 672 | 0.204555 | 3.495600 | 9.037673 | 0.205735 | 1.726630 |
| | 300 | 0.205584 | 3.488843 | 9.039643 | 0.205969 | 1.729467 |
| | 141 | 0.202780 | 3.547630 | 9.027128 | 0.206246 | 1.732925 |
| N = 20, IP = 0 | 11462 | 0.205731 | 3.470480 | 9.036694 | 0.205735 | 1.724909 |
| | 2265 | 0.205546 | 3.475157 | 9.036914 | 0.205765 | 1.725515 |
| | 714 | 0.207129 | 3.459350 | 8.988936 | 0.208278 | 1.736547 |
| N = 20, IP = 1 | 22011 | 0.205726 | 3.470606 | 9.036623 | 0.205730 | 1.724868 |
| | 2788 | 0.205607 | 3.474374 | 9.036784 | 0.205731 | 1.725223 |
| | 115 | 0.202686 | 3.587231 | 9.037897 | 0.205726 | 1.736020 |

The best minimum cost value obtained in the present article is $\Psi_0^* \equiv 1.724858$, corresponding to the point $\boldsymbol{x}^* \equiv (0.205729,\ 3.470519,\ 9.036630,\ 0.205730)$. One may observe that the algorithm solutions compare very well with the best solution presented above, even for earlier iterates of the algorithm. We may also observe that convergence is faster when the plain average is used in Eq. (5).

**Pressure Vessel Design**

The pressure vessel design problem has been proposed in (Kannan, 1994). It is one of the most used test problems for validating optimization algorithms. The problem is to find the optimal design of a compressed air storage tank (Fig. 2) with a working pressure of 1000 psi and a minimum capacity volume of $V_{min} = 1,296,000\,\text{in}^3$. The pressure vessel is composed of a cylindrical shell capped at both ends by hemispherical heads.

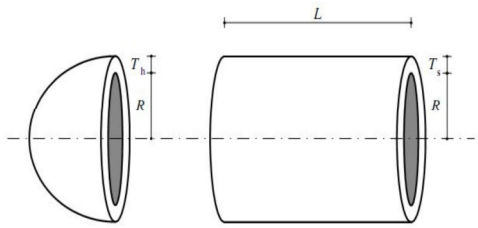


Fig. 2 - Pressure vessel

Let the design variables be $x_1 \equiv T_s$ the thickness of the shell, $x_2 \equiv T_h$ the thickness of the heads, $x_3 \equiv R$ the inner radius and $x_4 \equiv L$ the length of the cylindrical shell. The variables $x_1$ and $x_2$ should be integer multiples of 0.0625 in. The objective is to minimize the manufacturing cost (material, welding and forming costs) of the pressure vessel (Sandgren, 1990), subjected to constraints on volume capacity and in accordance with respective ASME codes. The mathematical model of the problem is:

$$
\begin{aligned}
& min\ \Psi_0 \equiv\ 0.6224\,x_1x_3x_4 + 1.7781\,x_2x_3^2 + 3.1661\,x_1^2x_4 + 19.84\,x_1^2x_3 \\
& s.t. \\
& \Psi_1 \equiv 0.0193x_3/x_1 - 1 \le 0 \\
& \Psi_2 \equiv 0.00954x_3/x_2 - 1 \le 0 \\
& \Psi_3 \equiv V_{min}/V - 1 \le 0; \qquad V = \pi(x_3^2x_4 + 4x_3^3/3) \\
& 0.0625 \le x_1, x_2 \le (99)(0.0625)\ ; \quad 10 \le x_3, x_4 \le 200
\end{aligned}
$$

The analytical optimum for this problem is calculated in the Annex A as $\Psi_0^* \equiv 6059.714335$ at the point $\boldsymbol{x}^* \equiv (0.8125,\ 0.4375,\ 42.0984456,\ 176.6365958)$ with the first and third constraints being active.

The optimal results achieved with the present algorithm are shown in Table 2 for different population sizes $N$, $\theta = 0.85$, and different average choices on Eq. (5). For all these solutions there is no violation of the constraints. The first constraint is nearly active at the optimal point for all the different population sizes; it takes values in the interval $-(0.3666)(10^{-7}) \le \Psi_1^* \le -(0.1215)(10^{-5})$.

Table 2 - Pressure vessel optimal solutions

| | Iterate | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\Psi_0$ |
|---|---|---|---|---|---|---|
| N = 1000, IP = 0 | 4143 | 0.81250 | 0.43750 | 42.09843 | 176.63712 | 6059.7231 |
| N = 1000, IP = 1 | 13441 | 0.81250 | 0.43750 | 42.09843 | 176.63678 | 6059.7158 |
| N = 500, IP = 0 | 8285 | 0.81250 | 0.43750 | 42.09843 | 176.63701 | 6059.7212 |
| N = 500, IP = 1 | 11909 | 0.81250 | 0.43750 | 42.09844 | 176.63673 | 6059.7168 |
| N = 200, IP = 0 | 4173 | 0.81250 | 0.43750 | 42.09838 | 176.63742 | 6059.7227 |
| N = 200, IP = 1 | 5077 | 0.81250 | 0.43750 | 42.09843 | 176.63704 | 6059.7222 |

The results for the present algorithm compare well with the analytical solution. Again, the plain average of the Eq. (5) gives origin to faster convergence.

**Tension/compression Spring Design**

The tension/compression spring design optimization problem is described in (Arora, 1989). The goal is to minimize the weight of a tension/compression spring (Fig. 3) subject to constraints on minimum deflection, shear stress, surge frequency, limits on outside diameter and side constraints. The design variables to be considered are the wire diameter *d*, the mean coil diameter *D* and the number *n* of active coils. Let us set up the vector of design variables as $\boldsymbol{x} \equiv (x_1, x_2, x_3) \equiv (d, D, n)$.

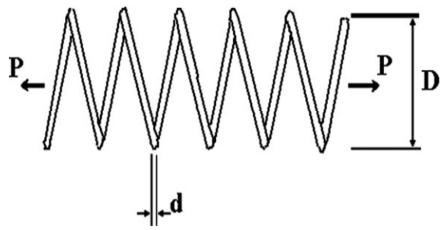


Fig. 3 - Tension-compression spring

The problem may be formulated as

$$
\begin{aligned}
& min\,\Psi_0 \equiv (x_3+2)\,x_1^2 x_2 \\
& \text{s.t.} \\
& \Psi_1 \equiv 1 - x_2^3 x_3 \big/ (71785\, x_1^4) \le 0 \\
& \Psi_2 \equiv (4x_2^2 - x_1 x_2) \big/ [12566\,(x_2 x_1^3 - x_1^4)] + 1 \big/ (5108\, x_1^2) - 1 \le 0 \\
& \Psi_3 \equiv 1 - 140.45\, x_1 \big/ x_2^2 x_3 \le 0 \\
& \Psi_4 \equiv (x_1 + x_2) \big/ 1.5 - 1 \le 0 \\
& 0.05 \le x_1 \le 2, \qquad 0.25 \le x_2 \le 1.3, \qquad 2 \le x_3 \le 15
\end{aligned}
$$

The analytical solution for this problem is presented in the Annex B. The minimum weight of the spring is achieved as $\Psi_0^* = 0.012665232$, $\boldsymbol{x}^* \equiv (0.051690,\ 0.356740,\ 11.287642)$. At the optimum, the constraints $\Psi_1$ and $\Psi_2$ are active, and $\Psi_3^* = -4.054$, $\Psi_4^* = -0.7277$.

The minimum objective function value obtained in (Arora, 1989) by nonlinear programming is $\Psi_0^* \equiv 0.012679$ corresponding to the optimum point $\boldsymbol{x}^* \equiv (0.051699,\ 0.35695,\ 11.289)$. The best result known so far by the authors is given in (Cagnina, 2008) at the design point $\boldsymbol{x}^* \equiv (0.051583,\ 0.354190,\ 11.438675)$, at which corresponds $\Psi_0^* \equiv 0.012665$.

The optimum results achieved with the present algorithm are shown in Table 3 for different sizes *N* of the population, $\theta = 0.85$, and plain average selection on Eq. (5). For all these solutions there is no violation of the constraints, being practically active the first two constraints $\Psi_1$ and $\Psi_2$. The last two constraints have values within the intervals $-4.08 \le \Psi_3^* \le -4.04$ and $-0.734 \le \Psi_4^* \le -0.720$.

Table 3 - Tension-compression spring optimal solutions for IP = 0

| | Iterate | $x_1$ | $x_2$ | $x_3$ | $\Psi_0$ |
|---|---|---|---|---|---|
| N = 1000 | 10063 | 0.051718 | 0.357409 | 11.248693 | 0.012665 |
| N = 500 | 11690 | 0.051775 | 0.358778 | 11.169669 | 0.012666 |
| N = 200 | 24818 | 0.051308 | 0.347629 | 11.842695 | 0.012668 |
| N = 100 | 13899 | 0.052127 | 0.367321 | 10.694988 | 0.012671 |

Again, the present algorithm results compare well with the analytical solution.

### Statistical Tests

Thirty independent runs were performed for each of the problems described above. Table 4 presents the mean and the standard deviation as well as the best solution obtained for each problem over these 30 runs. In the table are also indicated the population sizes used for the various problems. Other parameters used in these runs are $IP = 0$ and $\theta = 0.85$.

Table 4 - Mean, standard deviation and best values for the results

| Problem | Mean | Standard Deviation | Best Objective | Design Variables for the Best Objective | | | |
|---|---|---|---|---|---|---|---|
| | | | | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
| Welded Beam N = 1000 | 1.724866 | 7.61E-06 | 1.724852 | 0.205730 | 3.470484 | 9.036630 | 0.205730 |
| Press. Vessel N = 1000 | 6,060.391 | 6.55E-01 | 6,059.763 | 0.81250 | 0.43750 | 42.09809 | 176.64127 |
| Spring N = 1000 | 0.012666 | 7.65E-07 | 0.012665 | 0.051718 | 0.357409 | 11.248693 | |

## CONCLUDING REMARKS

This article presents an average concept algorithm to solve various optimization problems which include structural engineering test design constrained problems. To evaluate the performance of the present algorithm, numerical applications are conducted and the results are compared to the results obtained analytically and/or to the best ones achieved by other optimization methods. The analytical solutions for two of the constrained problems, namely the pressure vessel design and the tension-compression spring design problems, are determined in Annexes A and B of the present article. The solutions found by the proposed algorithm compare well with those results. We may conclude that the algorithm finds the global solution or a near-global solution in each problem tested. Characterizing the principal advantage of the algorithm one should emphasize the good balance between the accuracy of the solutions it achieves and its rare simplicity.

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## ANNEX A: PRESSURE VESSEL CLASSICAL OPTIMUM DESIGN

Let us assume initially the variables $x_1$ and $x_2$ are continuous and later on make the convenient correction to table (multiple of 0. 0675) values.
The cost function $\Psi_0$ decreases monotonically with $x_1$ or $x_2$ decrease. The constraint $\Psi_1$ is the only constraint that increases as $x_1$ decreases, and the constraint $\Psi_2$ is the only constraint that increases as $x_2$ decreases. Then, $\Psi_1$ and $\Psi_2$ provide respectively lower bounds for $x_1$ and $x_2$, and these variables may be minimized out as

$$x_1 = 0.0193\,x_3, \qquad x_2 = 0.00954\,x_3 \tag{A.1}$$

Substituting these relationships into the original problem, we have

$$\begin{aligned} &min\,\bar{\Psi}_0 \equiv 0.01319166\,x_3^2 x_4 + 0.024353275\,x_3^3 \\ &s.t. \\ &\bar{\Psi}_3 \equiv 1296000 - \pi x_3^2 x_4 - (4/3)\pi x_3^3 \le 0, \qquad 10 \le x_3, x_4 \le 200 \end{aligned} \tag{A.2}$$

where the upper bars on the cost and constraint symbols mean we are determining by now the solution for all variables continuous. The Lagrangian function for this problem is

$$L = \bar{\Psi}_0 + \lambda_3 \bar{\Psi}_3 - \mu_3^-(x_3 - 10) + \mu_3^+(x_3 - 200) - \mu_4^-(x_4 - 10) + \mu_4^+(x_4 - 200) \tag{A.3}$$

where $\lambda_3, \mu_3^-, \mu_3^+, \mu_4^-, \mu_4^+$ are the Lagrange multipliers for the constraint $\bar{\Psi}_3$ and side constraints. The necessary (Karush-Kuhn-Tucker) conditions for the problem (A.2) may now be set as

$$\begin{aligned} &\partial L/\partial x_3 \equiv (2)(0.01319166)x_3 x_4 + (3)(0.024353275)x_3^2 - \lambda_3 2\pi(x_3 x_4 + 2x_3^2) - \mu_3^- + \mu_3^+ = 0 \\ &\partial L/\partial x_4 \equiv 0.01319166\,x_3^2 - \lambda_3 \pi x_3^2 - \mu_4^- + \mu_4^+ = 0 \\ &\lambda_3 \bar{\Psi}_3 \equiv \lambda_3(1296000 - \pi x_3^2 x_4 - (4/3)\pi x_3^3) = 0 \\ &\mu_k^-(x_k - 10) = 0, \qquad \mu_k^+(x_k - 200) = 0; \quad k = 3,4 \\ &\lambda_3, \mu_3^-, \mu_3^+, \mu_4^-, \mu_4^+ \ge 0 \\ &\bar{\Psi}_3 \equiv 1296000 - \pi x_3^2 x_4 - (4/3)\pi x_3^3 \le 0 \\ &10 \le x_3, x_4 \le 200 \end{aligned} \tag{A.4}$$

Let us now search the different combinations of Lagrange multipliers:

1. $\mu_3^- > 0, \quad \mu_4^- > 0 \quad \Rightarrow \quad x_3 = x_4 = 10, \quad \Psi_3 = 1288669.6 > 0$ (violated)

   $\mu_3^- > 0, \quad \mu_4^+ > 0 \quad \Rightarrow \quad x_3 = 10,\ x_4 = 200, \quad \Psi_3 = 1228979.4 > 0$ (violated)

   Then, whatever the value of $\lambda_3$, must have $\mu_3^- = 0$
2. $\lambda_3 = 0$

   2.1 $\mu_3^- = 0 \quad \Rightarrow \quad \mu_3^+ < 0$ (from the 1st condition, then violating the 5th one)

   2.2 $\mu_4^- = 0 \quad \Rightarrow \quad \mu_4^+ < 0$ (from the 2nd condition, then violating the 5th one)
3. $\lambda_3 > 0 \quad \Rightarrow \quad \Psi_3 = 0 \quad \therefore \quad x_4 = 1296000/(\pi x_3^2) - (4/3)x_3$

3.1 $\mu_3^+ = \mu_4^- = \mu_4^+ = 0 \;\Rightarrow\; \lambda_3 = 0.01319166/\pi$ (from the 2nd condition)

$\Rightarrow\; x_3 = 0$ (after substituting $x_4$ and $\lambda_3$ into the 1st one)

3.2 $\mu_3^+ = 0,\ \mu_4^- = 0,\ \mu_4^+ > 0 \;\Rightarrow\; x_4 = 200$ (from the 4th conditions)

$$1296000 - \pi x_3^2 x_4 - (4/3)\pi x_3^3 = 0 \;\Rightarrow\; x_3 = 40.3196187244$$

$$\lambda_3 = [(2)(0.01319166)x_3x_4 + (3)(0.024353275)x_3^2]/[2\pi(x_3x_4 + 2x_3^2)]$$
$$= 0.004663057579 > 0$$

$$\mu_4^+ = \pi x_3^2 \lambda_3 - 0.01319166\, x_3^2 = 2.369851195 > 0$$

All the optimality necessary conditions are satisfied, then

$\bar{x}_1 = 0.0193x_3 = 0.778168646,\quad \bar{x}_2 = 0.00954x_3 = 0.384649165,$

$\bar{x}_3 = 40.3196187244,\quad \bar{x}_4 = 200$

is candidate local optimum point for the assumed continuous variables.

3.3 $\mu_3^+ = 0,\ \mu_4^- > 0,\ \mu_4^+ = 0 \;\Rightarrow\; x_4 = 10$ from the 4th conditions,

$$1296000 - \pi x_3^2 x_4 - (4/3)\pi x_3^3 = 0 \;\Rightarrow\; x_3 = 65.2252326139$$

$$\lambda_3 = [(2)(0.01319166)x_3x_4 + (3)(0.024353275)x_3^2]/[2\pi(x_3x_4 + 2x_3^2)]$$
$$= 0.005698937505 > 0$$

$\mu_4^- = 0.01319166\, x_3^2 - \pi x_3^2 \lambda_3 = -20.04674872 < 0$ (violates 5th condition)

3.4 $\mu_3^+ > 0,\ \mu_4^- = \mu_4^+ = 0 \;\Rightarrow\; x_3 = 200 \;\Rightarrow\; x_4 = -256.3534264 < 0$

3.5 $\mu_3^+ > 0,\ \mu_4^- = 0,\ \mu_4^+ > 0 \;\Rightarrow\; x_3 = 200,\ x_4 = 200$

$\Rightarrow\; \Psi_3 = -57347062.87 \neq 0$ (contradicts point 3: $\Psi_3 = 0$)

3.6 $\mu_3^+ > 0,\ \mu_4^- > 0,\ \mu_4^+ = 0 \;\Rightarrow\; x_3 = 200,\ x_4 = 10$

$\Rightarrow\; \Psi_3 = -33470958.7 \neq 0$ (contradicts point 3: $\Psi_3 = 0$)

Testing now the second-order sufficient conditions for the only point $\bar{\boldsymbol{x}} \equiv (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$, determined in 3.3, satisfying the necessary conditions, one may use the so-called bordered Hessian (Luenberger, 1984) calculated at that point:

$$\boldsymbol{B}(\bar{x}_3, \bar{x}_4) \equiv \begin{bmatrix} 0 & \partial\Psi_3/\partial x_3 & \partial\Psi_3/\partial x_4 \\ \partial\Psi_3/\partial x_3 & \partial^2 L/\partial x_3{}^2 & \partial^2 L/\partial x_3 \partial x_4 \\ \partial\Psi_3/\partial x_4 & \partial^2 L/\partial x_3 \partial x_4 & \partial^2 L/\partial x_4{}^2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -71095.91969 & -5107.198124 \\ -71095.91969 & 232.2341915 & -0.117553254 \\ -5107.198124 & -0.117553254 & 0 \end{bmatrix}$$

As $n - m$ for the problem (A.2) is $2 - 1 = 1$ one has to calculate the last principal minor $det\,\boldsymbol{B}$. Since its value is negative, its sign is coincident with $sign(-1)^m = sign(-1)$. Hence, the Hessian of $L$ is positive-definite and the point $\bar{\boldsymbol{x}}$ is a minimum point.

Now, rounding up the values of $\bar{x}_1$ and $\bar{x}_2$ to the table values, we have

$$x_1^* = 0.8125, \qquad x_2^* = 0.4375$$

then determine the other two variables as

$$x_3^* = min\{0.8125/0.0193, 0.4375/0.00954\} = 42.0984456$$

from the two first constraints $x_3 \le 0.8125/0.0193$, $x_3 \le 0.4375/0.00954$, i.e., the constraint $\Psi_1$ is active at the optimum, and

$$x_4^* = 176.6365958$$

from the condition of $\Psi_3 = 0 \quad \therefore \quad x_4 = 1296000/(\pi x_3^2) - (4/3)x_3$.
Therefore, the analytical optimum point for the pressure vessel design problem is

$$\boldsymbol{x}^* = (0.8125,\ 0.4375,\ 42.0984456,\ 176.6365958)$$

giving the minimum optimum cost $\Psi_0^* = 6059.714335$. At the optimum, the constraints have the values $\Psi_1^* = \Psi_3^* = 0$ and $\Psi_2^* = -0.082013323$.

## ANNEX B: TENSION-COMPRESSION SPRING CLASSICAL OPTIMUM DESIGN

Again, let us firstly to analyze the monotonicity of the problem (Papalambros, 1988). One should observe the constraint $\Psi_1$ is critical with respect to the design variable $x_3$. The cost function $\Psi_0$ increases monotonically in the variable $x_3$, and there is exactly one constraint, the constraint $\Psi_1$, whose monotonicity with respect to $x_3$ is opposite from that of the objective. Then, $\Psi_1$ is active and bounds $x_3$ from below:

$$x_3 = 71785\, x_1^4 / x_2^3 \tag{B.1}$$

Substituting the relationship (B.1) into the objective function and into the constraint $\Psi_3$ we get

$$\begin{aligned} \Psi_0 &= 2x_1^2 x_2 + 71785\, x_1^6 / x_2^2 \\ \Psi_3 &\equiv 1 - 0.001956536881\, x_2 / x_1^3 \le 0 \end{aligned} \tag{B.2}$$

Substituting the lower and upper bounds of $x_2$ into the constraint $\Psi_3$ we have that $x_1(0.25) \le 0.078790891$, $x_1(1.3) \le 0.136503503$; then the range of the design variable $x_1$ can be set up as

$$0.05 \le x_1 \le 0.136503503$$

If one uses now the upper bounds of $x_1$ and $x_2$ in the constraint $\Psi_4$, $0.136503503 + 1.3 \le 1.5$, it is obvious that this constraint is always inactive, not playing any role into the optimization problem.

Studying now the monotonicity of the objective expressed in (B.2) with respect to the design variable $x_2$, the minimum of $\Psi_0$ is given as

$$\partial \Psi_0 / \partial x_2 \equiv 2x_1^2 - (2)(71785)\, x_1^6 / x_2^5 = 0 \qquad \therefore \qquad x_2 = \sqrt[3]{71785\, x_1^4}$$

since $\partial^2 \Psi_0 / \partial x_2^{\,2} \equiv (6)(71785)\, x_1^6 / x_2^4 > 0$. Thus, $\Psi_0$ decreases monotonically in $x_2$ increase for $0.25 \le x_2 \le \sqrt[3]{71785\, x_1^4}$ and increases monotonically in $x_2$ increase for $\sqrt[3]{71785\, x_1^4} \le x_2 \le 1.3$. For example, within the range $0.25 \le x_2 \le \sqrt[3]{71785\, x_1^4}$, the function $\Psi_0$ decreases in the variable $x_2$ increase, achieving the minimum at $x_2 = 0.765545910$ for a prescribed $x_1 = 0.05$, and increases the value of the minimum point at $x_2$ as $x_1$ increases. For $x_1 \ge 0.074378786$ the function $\Psi_0$ is a decreasing function all along the feasible domain of $x_2$, $0.25 \le x_2 \le 1.3$.

The constraint $\Psi_2$ may be expressed as

$$4x_2^2 + (C-1)x_1 x_2 - C x_1 \le 0, \qquad C = 2.460062647 - 12566 x_1^2$$

This constraint increases monotonically in $x_2$ increase; then its monotonicity with respect to $x_2$ is opposite from that of $\Psi_0$ for $0.25 \le x_2 \le \sqrt[3]{71785\, x_1^4}$ and the constraint $\Psi_2$ is critical providing an upper bound for $x_2$:

$$x_2 = (x_1/8)(1 - C + \sqrt{C^2 + 14C + 1})$$

The reduced problem may now be expressed as

$$\begin{aligned} & min\, \Psi_0 \equiv 2x_1^2 x_2 + 71785\, x_1^6 / x_2^2 \\ & s.t. \\ & x_2 = (x_1/8)(1 - C + \sqrt{C^2 + 14C + 1}) \\ & 0.05 \le x_1 \le 0.136503503, \quad 0.25 \le x_2 \le 1.3 \end{aligned} \tag{B.3}$$

The optimum point can be determined easily by using a unidimensional search in $x_1$, with the active constraint $\Psi_2$ determining $x_2$. The variable $x_3$ is calculated by using the relationship (B.1) after solving the problem (B.3).

The optimum value of the objective function is obtained as $\Psi_0^* = 0.012665232$ at the point $\boldsymbol{x}^* \equiv (0.051690, 0.356740328, 11.28764160)$. At the optimum, the constraints have the values $\Psi_1^* = \Psi_2^* = 0,\ \Psi_3^* = -4.05383024,\ \Psi_4^* = -0.727713114$.