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A RBRDO APPROACH BASED ON FEASIBILITY ROBUSTNESS AND IMPOSED RELIABILITY LEVEL

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ABSTRACT

An approach for the optimization of composite structures, including uncertainties in mechanical properties and structural parameters, is proposed. The purpose is to estimate a response that is also a function of those uncertainties and seek robust and reliable structures. The principles of Reliability-based Robust Design Optimization (RBRDO) are applied. A PMA-based approach that allows avoiding the computational expensive second cycle of the reliability analysis is proposed.

Keywords: uncertainty, robustness, reliability, RBRDO, optimization, composites.

INTRODUCTION

The introduction of uncertainty in a system has the purpose of estimating a response that is also a function of that uncertainty. The uncertainty is considered to be of the random-kind. This way, probability distributions may be assigned. A common and efficient way of doing so is to consider a new set of variables that may contain increment variables and/or totally new variables. Theses variables are called random variables and the original ones are denoted as design variables. Design variables may be random, too (Beyer, 2007).

An optimal solution of f is denoted robust if its variability tends to a minimum. Robustness is introduced in the optimization problems by manipulation of objective (performance robustness) and/or constraints (feasibility robustness). It is common to consider, for each objective, two functions, such as expected value and variance, such that the original single-optimization problem becomes a two-objective problem. The resultant multi-objective optimization problem is called Robust Design Optimization (RDO) (Beyer, 2007).

Optimal solutions might be close to the limits of integrity of the systems, because the sought objective might oppose strength. So, under uncertainty, safety limits can be surpassed and the system, fail. Reliability is measured in terms of a probability of failure, p_f, which is defined as the probability of a limit-state function, G, being negative. Cumulative probabilities are defined as integrals and require evaluations of physical models numerous times, in each cycle, to be estimated. The resultant optimization is a double-cycle problem and it is named Reliability-based Design Optimization (RBDO) (Valdebenito, 2010).

DETERMINISTIC OPTIMIZATION PROBLEM

In the field of structural optimization, the base problem, from which additional developments are introduced, is deterministic and may be written as follows

$$\begin{aligned} &\underset{\mathbf{x}}{\text{Optimize}} && \mathbf{F}(\mathbf{x}) = \{f_1(\mathbf{x}), \ldots, f_N(\mathbf{x})\} \\ &\text{subject to} && g_j(\mathbf{x}) \leq 0\,, \qquad j = 1,2,\ldots,m \\ &&& h_k(\mathbf{x}) = 0\,, \qquad k = 1,2,\ldots,p \\ &&& x_i^l \leq x_i \leq x_i^u, \quad i = 1,2,\ldots,n \end{aligned} \tag{1}$$

where $f_i \in \mathbb{R}$ is one of the functions to optimize, named objective-functions, which compose the solution space $\mathbf{F}(\mathbf{x})$, $D = \{\mathbf{x} \in \mathbb{R}^n \mid g_j(\mathbf{x}) \leq 0 \wedge h_j(\mathbf{x}) = 0 \wedge x_i^l \leq x_i \leq x_i^u\}$ is the design space, defined by the intersection of inequality, equality, and size constraints, respectively, and $\mathbf{x}$ is the vector of design variables. Problem (1) is said to be a multi-objective optimization problem, since more than one objectives are expected to be optimized. Concepts of a multi-objective optimization will be addressed in a later section. In the present problem, only two inequality constraints will be considered: one constraint for the displacements and another constraint for the stress state of the structure. Generically, each constraint may be written as

$$\begin{aligned} g_1(\mathbf{x}) &= \frac{\bar{u}(\mathbf{x})}{u_a} - 1 \leq 0 \\ g_2(\mathbf{x}) &= \frac{\bar{R}(\mathbf{x})}{R_a} - 1 \leq 0 \end{aligned} \tag{2}$$

where $\bar{u}(\mathbf{x})$ and $\bar{R}(\mathbf{x})$ are two functionals, one related to the maximum displacement of the structure and the other with the most critical Tsai number,

$$\begin{aligned} \bar{u}(\mathbf{x}) &= \max(u_1, \ldots, u_q) \quad, q = 1, \ldots, N_{dis} \\ \bar{R}(\mathbf{x}) &= \max(R_1, \ldots, R_r) \quad, r = 1, \ldots, N_{str} \end{aligned} \tag{3}$$

being N_{dis} the total number of calculated displacements and N_{str} the total number of points of the structure where the stress vector is evaluated. These critical values are compared maximum allowable values u_a and R_a. The stress analysis is performed using the strength parameter R, known as *Tsai number*, and calculated as the ratio between the maximum allowable stress and the actual stress at the r-th point of the structure where the stress vector is evaluated (Tsai,1987). The response vector is represented as $\boldsymbol{\varphi}(\mathbf{x}) = (\bar{u}, \bar{R})$.

PROPAGATION OF UNCERTAINTY

In this work, uncertainty is considered to be of the random kind. Random uncertainty is related to the inherent variability of the physical system under consideration, due to the somewhat random nature of the variables themselves and/or the way they interact with each other. In the *theory of propagation of uncertainty* (Cacuci, 2003), the goal is to determine the moments of a given response function. Let $\varphi(\mathbf{x}) \in \mathbb{R}$ be a generic system response and $\mathbf{x} \in \mathbb{R}^N$ a generic vector of N random variables of the system, about which only the *nominal* *values* $\mathbf{x^0}$ and their *uncertainties* $\delta\mathbf{x}$ are estimated. Then, the true value of the variables may be given by

$$\mathbf{x} = \mathbf{x^0} + \delta\mathbf{x} \tag{4}$$

From (4), let the deviation from the mean value of the variables be $\delta x_i = (x_i - x_i^0)$. The expansion in Taylor's series of $\varphi(\mathbf{x})$, around the nominal value $\mathbf{x^0}$, considering only up to the first-order terms, is the following

$$\varphi(x_1, \dots, x_n) \cong \varphi^0 + \sum_{i=1}^{N} S_i \delta_i \tag{5}$$

being $\varphi^0 = \varphi(\mathbf{x^0})$ and $S_i = (\partial\varphi/\partial x_i)_{\mathbf{x^0}}$ the sensitivity of the response to the variable x_i. From (5), the first two moments of φ are defined as follows

$$\mathrm{E}(\varphi) = \varphi^0 \tag{6}$$

$$\mathrm{Var}(\varphi) = \sum_{i=1}^{N} S_i^2 \, \mathrm{Var}(x_i) + 2 \sum_{i \neq j=1}^{N} S_i \, S_j \, \mathrm{Cov}(x_i, x_j) \tag{7}$$

or

$$\mathrm{Var}(\varphi) = \mathbf{sVs}^{\mathrm{T}} \tag{8}$$

where $\mathbf{s}$ is the column-vector of the sensitivities and $\mathbf{V}$ is the covariance matrix of the random variables, defined as follows

$$V_{i,j} = \begin{cases} \mathrm{Cov}(x_i, x_j) = \sigma_i \sigma_j \rho_{ij} & , \quad i \neq j \\ \mathrm{Var}(x_i) = \sigma_i^2 & , \quad i = j \end{cases} \tag{9}$$

where σ_i is the standard deviation of x_i and ρ_{ij} the correlation coefficient of x_i and x_j. The previous concepts may be extended to the case of multiple response functions, as follows

$$\mathrm{E}(\boldsymbol{\varphi}) = \boldsymbol{\varphi}^{\mathbf{0}} \tag{10}$$

$$\mathbf{C}_{\boldsymbol{\varphi}} \equiv \mathrm{Var}(\boldsymbol{\varphi}) = \mathbf{SVS}^{\mathbf{T}} \tag{11}$$

where $\mathbf{S}$ a $(m \times N)$ matrix whose components are the sensitivities of the j-*th* response to the i-*th* variable, $S_{ij} = \left(\partial\varphi_j/\partial x_i\right)_{\mathbf{x^0}}$, and $\mathbf{C}_{\boldsymbol{\varphi}}$ is named *variance-covariance matrix*.

THE ADJOINT VARIABLE METHOD

The goal of sensitivities analysis is to analyze the behavior of the response of the system and to evaluate it sensitivity to variations in the input variables, around their nominal values. The methodology presented here is based on the Adjoint Variable Method. The method was developed in connection with structural analysis of composite structures (António, 1995). The structural analysis of laminate composite structures is based on a displacement formulation of the Finite Element Method (FEM), in particular using the shell finite element model developed by Ahmad (1969) and further improvements (Figueiras, 1989). A short description of the Ahmad element can be found in the following reference (António and Hoffbauer, 2008). In this work, it is considered the linear behavior of structural systems, with the equilibrium matrix equation established as

$$\mathbf{Ku} - \mathbf{f} = \mathbf{0} \tag{12}$$

where $\mathbf{K} \equiv \mathbf{K}(\mathbf{x})$ is the stiffness matrix, $\mathbf{u} \in \mathbb{R}^{N_u}$ is the vector of state variables (displacements), and solution of (12), $\mathbf{x} \in \mathbb{R}^{N_x}$ is the vector of design variables and $\mathbf{f}$ is the vector of external loads. Now, let $\varphi(\mathbf{u}, \mathbf{x}) \in \mathbb{R}$ be a functional, with explicit dependence on $\mathbf{u}$ and $\mathbf{x}$ and with implicit dependence on $\mathbf{x}$, also, and for which (12) holds. Then, the following Lagrangian can be defined as

$$\mathcal{L}(\mathbf{u}, \mathbf{x}, \boldsymbol{\lambda}) = \varphi(\mathbf{u}, \mathbf{x}) - \boldsymbol{\lambda}^T[\mathbf{Ku} - \mathbf{f}] \tag{13}$$

where $\lambda \in \mathbb{R}^{N_\lambda}$ is vector of Lagrange multipliers, also known as adjoint variables (states). The associated optimality conditions, in the state space, written as

$$\frac{\partial \mathcal{L}}{\partial \mathbf{u}} = \frac{\partial \varphi}{\partial \mathbf{u}} - \boldsymbol{\lambda}^T \frac{\partial}{\partial \mathbf{u}}[\mathbf{Ku} - \mathbf{f}] = \mathbf{0} \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}} = \mathbf{0} \tag{15}$$

Also, as shown by Arora and Cardoso (1982), the following equality is written

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \frac{\mathrm{d}\varphi}{\mathrm{d}\mathbf{x}} \tag{16}$$

In the Adjoint Variable Method, the augmented Lagrangian of (15) is defined in terms of adjoint states in order to calculate the implicit components of the total derivatives of φ, with respect to the design variables $\mathbf{x}$. Considering the independence of $\mathbf{f}$ to the state variables $\mathbf{u}$, the adjoint set of equations is obtained from (14) and is written as

$$\mathbf{K}(\mathbf{x})\boldsymbol{\lambda} - \frac{\partial \varphi}{\partial \mathbf{u}} = \mathbf{0} \tag{17}$$

where $\mathbf{K}(\mathbf{x})$ is given by (15), which is equivalent to the equilibrium equation (12). Considering the independence of $\mathbf{f}$ to te design variables $\mathbf{x}$, from equality (16), it follows that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\mathbf{x}} = \frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \frac{\partial \varphi}{\partial \mathbf{x}} - \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial \mathbf{x}} \mathbf{u} \tag{18}$$

The developed methodology for sensitivity analysis is twofold (António, 1995; Arora and Cardoso, 1992):

1. Solve the adjoint set of equations, defined by (17);
2. Get the sensitivities from (18).

Using equation (18), the components of matrix $\mathbf{S}$, in equation (11), can be calculated to obtain the variance-covariance matrix $\mathbf{C}_{\boldsymbol{\varphi}}$, associated with the variability of the structural response.

STRUCTURAL FEASIBILITY ROBUSTNESS AND RDO

In this work, Robustness is considered as a measure of variability of the critical structural responses, included in the vector-function $\boldsymbol{\varphi}(\mathbf{x}) = (\bar{u}, \bar{R})$ and defined in equation (3). For that purpose, the evaluation of the effects of the uncertainty in the structural response is done in a simple and systematic way by calculating the determinant of the variance-covariance matrix $\mathbf{C}_{\boldsymbol{\varphi}}$, defined in equation (11). This definition of Robustness is seen as a *structural feasibility robustness*, since the final optimization problem will also include reliability constraints and only the deterministic constraints in (2) are evaluated in terms of robustness. The RDO process is executed over two sets of variables: (1) deterministic design variables $\mathbf{d} \in \mathbb{R}^{N_\mathbf{d}}$ and (2) and the mean values $\boldsymbol{\mu}_\mathbf{z}$ of the random design variables $\mathbf{z} \in \mathbb{R}^{N_\mathbf{z}}$. A third set of random structural parameters $\boldsymbol{\pi} \in \mathbb{R}^{N_{\boldsymbol{\pi}}}$ is also considered, albeit its mean values $\boldsymbol{\mu}_{\boldsymbol{\pi}}$ are kept constant and are an input of the system, in the RDO. The standard deviations of $\mathbf{z}$ are also kept constant and an input of the system. The proposed approach for RDO of composite structures is defined as a bi-objective optimization problem, with one objective for (a) optimality and (b) another for robustness, namely: (a) the minimization of the weight of the composite structure $W(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z})$ and (b) the minimization of the determinant of the variance-covariance matrix $\det \mathbf{C}_{\boldsymbol{\varphi}}\,(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_{\boldsymbol{\pi}})$, which is formally written as follows

$$\begin{aligned} &\min_{\mathbf{d},\, \boldsymbol{\mu}_\mathbf{z}} && \mathbf{F}(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) = \left\{ W(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}), \det \mathbf{C}_{\boldsymbol{\varphi}}\,(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) \right\} \\ &\text{subject to} && \mathbf{x} = (\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}) \in D \end{aligned} \tag{19}$$

with $\boldsymbol{\varphi}(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) = (\bar{u}, \bar{R})$ given by (3). Formulation (19) represents a general RDO problem, meaning that the set of variables considered in each functional depends on the problem under study. According to António and Hoffbauer (2015), uncertainties in different groups of random design variables and parameters show distinct behaviors and importance upon the response variability during the RDO search. At the end of the search process the Pareto front, representing the trade-off between optimality and robustness is obtained.

STRUCTURAL RELIABILITY ASSESSMENT AND RBDO

In this work, the authors propose a process for the optimization of composite structures, which combines structural optimization under the concept of Robustness with the imposition of a minimum reliability level, measured as the probability of failure of the system. In this particular case, structural reliability is understood as the ability of the composite structure to support a certain level of external loads. Such evaluation is based on the stress state of the system. Here, the stress state is given by $\bar{R}(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_{\boldsymbol{\pi}})$, in (3), and the corresponding constraint function is given by $g_2(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_{\boldsymbol{\pi}})$, in (2), also called *limit-state function.* However, the existence of uncertainty, in both variables and parameters of the system, may introduce relevant fluctuations in the deterministic realization of the value of g_2 and, thus, the structure fail. Therefore, it is of vital importance to evaluate the stress limit-state in a probabilistic fashion and impose an upper bound for such probability, to guarantee a minimum level of reliability. Formally, the RBDO problem with probabilistic evaluation of the stress limit-state is written as follows, in conformity with the deterministic problem (1) and with constraints in equation (2),

$$\begin{aligned}
&\underset{\mathbf{d},\,\boldsymbol{\mu}_\mathbf{z}}{\text{Optimize}} && \mathbf{F}(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_\boldsymbol{\pi}) = \{f_1, \dots, f_N\} \\
&\text{subject to} && p_{f_{\bar{R}}}(\mathbf{d}) \le p_{f_{\bar{R}}}^{a} \\
& && g_1(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_\boldsymbol{\pi}) \le 0 \\
& && d_i^l \le d_i \le d_i^u, \quad i = 1,2, \dots, \mathrm{N}_\mathbf{d} \\
& && \mu_{z_i}^{\,l} \le \mu_{z_i} \le \mu_{z_i}^{\,u}, \quad i = 1,2, \dots, \mathrm{N}_\mathbf{z}
\end{aligned} \tag{20}$$

where $p_{f_{\bar{R}}}(\mathbf{d})$ represents the calculated probability of failure and $p_{f_{\bar{R}}}^{a}$ the imposed one. Mathematically, $p_{f_{\bar{R}}}$ is defined as the probability of violation of the limit-state of the structure. While from a deterministic perspective, feasibility is considered in the negative side of the limit-state functions, in Reliability assessment a solution is feasible for positive values of the constraint functions, by convention, meaning it is located on the safe side. Therefore, the stress limit state shall be rewritten as

$$g_2(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_\boldsymbol{\pi}) = 1 - \frac{\bar{R}(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_\boldsymbol{\pi})}{R_a} \tag{21}$$

and the probability of failure defined by (Melchers, 1999)

$$p_{f_{\bar{R}}}(\mathbf{d}) = \Pr(g_2(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_\boldsymbol{\pi}) \le 0) = \int_{g_2(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_\boldsymbol{\pi}) \le 0} p_{\mathbf{z},\boldsymbol{\pi}}(\mathbf{z}, \boldsymbol{\pi} \,|\mathbf{d})\, d\mathbf{z} d\boldsymbol{\pi} \tag{22}$$

where $p_{\mathrm{z},\pi}(\cdot\,|\,\cdot)$ is the conditioned joint probability density function. The solution of the canonic RBDO problem (20) is numerically expensive and its application to the design of real structures is limited, due to the need to evaluate a very low probability, defined as the integral in (22). Also, the RBDO is a double-cycle problem: the outer cycle corresponding to the evaluation of the physical model and the inner cycle for Reliability assessment. This means that a massive number of model evaluations are needed, particularly when using genetic algorithms (GA) as optimization tool. For these reason, in this work, the chosen method to measure the reliability design solutions is the Performance Measure Approach (PMA). The goal is to make the execution of the inner reliability cycle more efficient and estimate the probability of failure by the concept of the *reliability index* β, without the need to explicitly calculate p_f. Formally, β is defined, in the space of standard normal random variables $\mathbf{u}$, as the Euclidean distance from a given point to the origin. In the PMA, such point is defined as the point for which the limit-state function is minimized, given an imposed distance β^a (allowable), which translated in the following optimization problem (Valdebenito, 2010)

$$\begin{aligned}
&\underset{\mathbf{u}_{\mathbf{z},\boldsymbol{\pi}}}{\min} && g_2(\mathbf{d}, \mathbf{u}_{\mathbf{z},\boldsymbol{\pi}}) \\
&\text{subject to} && \beta(\mathbf{u}_{\mathbf{z},\boldsymbol{\pi}}) \equiv (\mathbf{u}_{\mathbf{z},\boldsymbol{\pi}}{}^{\mathbf{T}} \cdot \mathbf{u}_{\mathbf{z},\boldsymbol{\pi}})^{\frac{1}{2}} = \beta^a
\end{aligned} \tag{23}$$

where $\mathbf{u}_{\mathbf{z},\boldsymbol{\pi}} \sim N(0,1)$ is the vector containing independent standard normal random design variables and/or standard normal random parameters. Since p_f is not calculated explicitly, an equivalent reliability constraint, based on the PMA, is defined through the concept of the quantile-function (Chiralaksnakul and Mahadevan, 2007) and is given by

$$g_2\left(\mathbf{d}, \mathbf{u}^*_{\mathbf{z},\boldsymbol{\pi}}\right) \geq 0 \qquad (24)$$

where $\mathbf{u}^*_{\mathbf{z},\boldsymbol{\pi}}$ is the optimal solution of (23).

RBRDO OF COMPOSITE SHELLS STRUCUTURES

Considering the definitions in the previous sections, it is now possible to write the proposed RBRDO problem, for the optimization of composite structures under the effects of uncertainty, as

$$\begin{aligned}
&\min_{\mathbf{d},\, \boldsymbol{\mu}_\mathbf{z}} && \mathbf{F}(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) = \left\{W(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}), \det \mathbf{C}_{\boldsymbol{\varphi}}\left(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_{\boldsymbol{\pi}}\right)\right\} \\
&\text{subject to} && g_1(\mathbf{d}, \boldsymbol{\mu}_\mathbf{z}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) \leq 0 \\
& && g_2\left(\mathbf{d}, \mathbf{u}^*_{\mathbf{z},\boldsymbol{\pi}}\right) \geq 0 \\
& && d_i^l \leq d_i \leq d_i^u, \quad i = 1,2,\dots,\mathrm{N}_\mathbf{d} \\
& && {\mu_{z}}_i^l \leq \mu_{z_i} \leq {\mu_{z}}_i^u, \quad i = 1,2,\dots,\mathrm{N}_\mathbf{z}
\end{aligned} \qquad (25)$$

The optimization process of the RBRDO problem comprises two nested minimization cycles: an exterior cycle of RDO optimization complemented with an inner cycle for the reliability assessment of each design solution. Because of the expected highly nonlinear nature of limit-state functions both cycles of the RBRDO are solved by suitable evolutionary algorithms.

THE MOGA-2D EVOLUTIONARY ALGORITHM

The proposed algorithm to solve the exterior bi-objective RDO problem is the Bi-level Dominance Multi-Objective Genetic Algorithm (MOGA-2D), developed by António and Hoffbauer (2016). The algorithm searches the design space to find multiple Pareto-optimal solutions in parallel, using two simultaneous populations, namely, small population (SP) and enlarged population (EP). It performs using the concept of local dominance at SP and storing the new generated non-dominated solutions (rank 1), from SP, into the EP. This enlarged population is continuously updated based on global dominance and has two main functionalities: to build the global Pareto front and to transmit its best solutions' genetic properties to the next small populations of the evolutionary process.

Dominance definition: Let $\mathbf{Q} \subseteq \mathbb{R}^{\mathrm{N_Q}}$ be the population set of individuals being sorted and ranked according to the concept of non-constrained dominance. Following the definition by Deb (2001), an individual $\mathbf{v}_i \in \mathbf{Q}$ is said to constrain-dominate an individual $\mathbf{v}_j \in \mathbf{Q}$, if any of the following conditions is verified:

(1) $\mathbf{v}_i$ and $\mathbf{v}_j$ are both feasible, with

a. $\mathbf{v}_i$ is no worse than $\mathbf{v}_j$ for all objectives, and
b. $\mathbf{v}_i$ is strictly better than $\mathbf{v}_j$ at least in one objective,

(2) $\mathbf{v}_i$ is feasible, while $\mathbf{v}_j$ is not,
(3) $\mathbf{v}_i$ and $\mathbf{v}_j$ are both infeasible, but $\mathbf{v}_i$ has a smaller constraint violation.

The total constraint violation of an individual $\mathbf{v}$ is defined as the sum of the absolute values of the violated constraint functions, in the multi-objective optimization, and is defined by

$$\xi(\mathbf{v}) = \sum_{i=1}^{2} \Gamma_{\mathrm{i}}\big(g_i(\mathbf{v})\big) \tag{26}$$

with

$$\Gamma_{\mathrm{i}}\big(g_i(\mathbf{v})\big) = \begin{cases} 0 & , if\ \mathbf{v}\ is\ feasible \\ \mathrm{abs}(g_i) & , if\ \mathbf{v}\ is\ unfeasible \end{cases} \tag{27}$$

The concept of constrained-domination enables to compare two solutions in problems with multiple objectives and constraints, since if $\mathbf{v}_i$ constrain-dominates $\mathbf{v}_j$, then $\mathbf{v}_i$ is better than $\mathbf{v}_j$, or vice-versa. If none of the three conditions above is verified, then no solution dominates the other.

Fitness assignment based on local dominance: In the SP, solutions are ranked according to their fitness, which no longer depends on an absolute value related to a certain fitness function, but rather on the concept of dominance. The individual fitness is calculated according to the niche occupied by the solution an also depending on the number of individuals with the same level of dominance in its neighborhood. It is called *shared fitness*. A *sharing function* is used to improve the distribution of rank 1 solutions along the local Pareto front, at SP level, during the evolutionary process. Though the Elitist strategy adopted in SP is based on fitness, it is also based in dominance, implicitly. The procedure to assign a fitness value to each solution is described in a previous work by Conceição António (2013). A detailed description of these mechanisms is found in works by Conceição António (2002, 2013).

THE mGA EVOLUTIONARY ALGORITHM

The proposed algorithm to solve the inner single-objective RBDO problem is a single-objective GA developed by Conceição António (2002). Here it shall be referred as *micro genetic algorithm* (mGA), due to the fact that its application is limited to local problems and with small populations. Observing the PMA problem in equation (23) it is seen that all the search process is taken in the standard uncertainty domain. However, since the stress limit-state function is derived from a discretized model of the structure, it is not defined explicitly in terms of the variables of the system. So, predicting its value in such a space may require extra computational cost, as transformation methods are usually hard to apply. Therefore, the search procedure is taken in the actual non-standard uncertainty domain, of variables $\mathbf{z}$ e $\boldsymbol{\pi}$, and only the equality constraint is verified in the standard space, since the concept of β is defined in such conditions.

In order to apply the mGA, a search domain has to be defined, with lower and upper bounds for the search variables. To make sure the equality constraint is always verified, size constraints should be defined as a function of β^a. In the case of non-standard normal random variables, size constraints are given by

$$\begin{aligned} &\mu_{z_i} \pm \beta^a \sigma_{z_i} \quad , i = 1, \dots, \mathrm{N}_{\mathbf{z}} \\ &\mu_{\pi_i} \pm \beta^a \sigma_{\pi_i} \quad , i = 1, \dots, \mathrm{N}_{\boldsymbol{\pi}} \end{aligned} \tag{28}$$

For practical reasons, related to the applicability of the mGA, the equality constraint shall be relaxed and equated to a positive small valued slack variable, Δ. Thus, a small feasible domain is defined and unfeasible solutions are given a penalty on their fitness. Given the nature of GA's, the PMA problem in (23) is rewritten as follows

$$\begin{aligned} &\max_{\mathbf{z}, \boldsymbol{\pi}} \qquad \bar{f} = C - [\alpha_1 g_2(\mathbf{d}, \mathbf{z}, \boldsymbol{\pi})^2 + \alpha_2 \Gamma(\beta)] \\ &\text{subject to} \quad \phi(\mathbf{u}_{\mathbf{z},\boldsymbol{\pi}}) \equiv \beta(\mathbf{u}_{\mathbf{z},\boldsymbol{\pi}}) - \beta^a = \Delta \end{aligned} \tag{29}$$

where C is a high valued constant, α_1 and α_2 are control parameters and

$$\Gamma(\beta) = \begin{cases} 0 & , if\ \phi(\mathbf{u}_{\mathbf{z},\boldsymbol{\pi}}) \in [-\Delta, \Delta] \\ K|\phi(\mathbf{u}_{\mathbf{z},\boldsymbol{\pi}})|^q & , if\ \phi(\mathbf{u}_{\mathbf{z},\boldsymbol{\pi}}) \notin [-\Delta, \Delta] \end{cases} \tag{30}$$

where K and q are penalty parameters. As seen in (30), reliability assessment for each design solution still requires successive evaluations of g_2 for different realizations of the random variables. In order to avoid multiple runs of the FE model, and to make use of the already available information from the RDO, it was chosen to approximate the limit-state function by a first-order Taylor's series around the *mean value of random variables* $(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}})$. A detailed description of the operators of the mGA is found in works by Conceição António (2002).

APPLICATION TO COMPOSITE SHELL STRUCTURES

Problem definition

To study the ability of the proposed approach for bi-objective optimization of RBRDO, a clamped cylindrical shell laminated structure is considered as show in Fig 1. Nine vertical loads of mean value $P_k = 7\ KN$ are applied along the free linear side (AB) of the structure. This side is constrained in the y-axis direction. The structure is divided into four macro-elements, grouping all elements, and there is one laminate for each macro-element. The laminate distribution is as shown in Fig. 1. The balanced angle-ply laminates with five layers and the stacking sequence $[+a/-a/\,0/-a/+a]$ are considered in a symmetric construction. Ply angle a is a design variable and is referenced to the x-axis of the reference axis, as in Fig. 1. The design variable h_i denotes the laminate thickness and four laminates are considered. A composite material built with the carbon/epoxy system denoted T300/N5208 (Tsai, 1987) is used in the presented analysis. The macro mechanics' mean values of the elastic and strength properties of the ply material used in the laminates of the structure is presented in Table 1. In the studied case, the uncertainty of the variables and parameters is organized in the following four groups:

Group 1: mechanical properties $\mathbf{m} \subseteq \boldsymbol{\pi}$, defined as random parameters;
Group 2: ply angles of the laminates $\mathbf{a} \subseteq \mathbf{z}$, defined as random design variables;
Group 3: laminate thicknesses $\mathbf{h} \subseteq \mathbf{z}$, defined as random design variables;
Group 4: point loads $\mathbf{p} \subseteq \boldsymbol{\pi}$, defined as random parameters.

Table 1 - Mean values of mechanical properties of composite layers

Material	E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	ν_{12}
T300/N5208	181.00	10.30	7.17	0.28
	$X; X'$	$Y; Y'$	S (MPa)	ρ (kg/m^3)
T300/N5208	1500; 1500	40; 246	68	1600

The group of mechanical properties $\mathbf{m}$ includes the following random parameters: E_{1j}, E_{2j}, Y_j e S_j, where the subscript j denotes the laminate number. Being four the number of laminates, there are sixteen mechanical properties aggregated in vector $\boldsymbol{\pi}$. Five random design variables are considered in vector $\mathbf{z}$: one for ply-angle α, for all symmetric laminates, and four laminates thicknesses. Depending on the cycle of optimization (RDO or PMA) the sources of uncertainty differ. In the RDO outer-cycle, the uncertainty of the system is considered through vectors $\mathbf{z}$ and $\boldsymbol{\pi}$, for which the nominal values are the mean values $\boldsymbol{\mu}_{\mathbf{z}}$ and $\boldsymbol{\mu}_{\boldsymbol{\pi}}$. The design variables of the RDO problem are the mean values $\boldsymbol{\mu}_{\mathbf{z}}$. Standard deviations are kept constant during the RDO. The design variables are encoded using binary code format with different numbers of digits. The MOGA-2D parameters used at SP evolution and the size constraints are defined in Table 2.

In the PMA inner-cycle of reliability assessment, the uncertainty of the system is considered only through the mechanical parameters $\mathbf{m}$, for which nominal values are the mean values $\boldsymbol{\mu}_{\mathbf{m}}$. The PMA is applied to each design solution found by the corresponding RDO cycle, and thus the evaluation of the limit-state function is done for fixed values of $\boldsymbol{\mu}_{\mathbf{z}}$, corresponding to the different realizations of the design variables, of each design solution. During the PMA, the standard deviations of the random mechanical properties are kept constant and are necessary to define the local search space and to transform the referred variables to standard normal random variables. Assuming the original random variables follow $\mathbf{m}\sim N(\boldsymbol{\mu}_{\mathbf{m}}, \boldsymbol{\sigma}_{\mathbf{m}})$, then the following transformation is applied

$$\mathbf{u}_{\mathbf{m}} = \frac{\mathbf{m} - \boldsymbol{\mu}_{\mathbf{m}}}{\boldsymbol{\sigma}_{\mathbf{m}}} \qquad (31)$$

to obtain $\mathbf{u}_{\mathbf{m}}\sim N(\mathbf{0}, \mathbf{1})$. Again, the design variables are encoded using binary code format with different numbers of digits. The mGA parameters and the size constraints (see 28) are defined in Table 3. In the RDO procedure, the allowable value u_a of the constraint of displacement is $u_a = -1.0 \times 10^{-2}\, m$. In the PMA procedure the allowable distance in the standard uncertainty space is $\beta_a = 3$. The standard deviations corresponding to each group of variables are presented in Table 4.

Table 2 - MOGA-2D parameters and size constraints

Population size	30
Elite group size (%)	33.33
Mutation group size (%)	20
Number of generations	300
Code format (digit nr.)/ /size constraint for ply angle	4/[0°,90°]
Code format (digit nr.)/ /size constraint for laminate thicknesses	5/[0.005m, 0.040 m]
Population size	30
Elite group size (%)	33.33

Table 3 - mGA parameters

Population size	15
Elite group size (%)	33.33
Mutation group size (%)	20
Number of generations	100
Code format (digit nr.)/ /size constraint for $\mathbf{m}$	4/ /eq.(28)
Population size	15

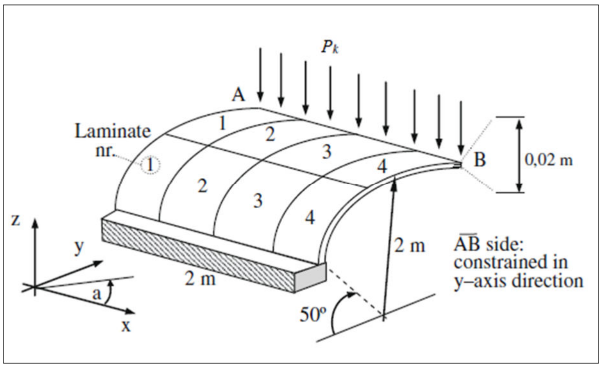

Fig. 1 - Geometric definition of the cylindrical shell structure and composite laminates distribution.

Table 4 - Standard deviations of the of variables

Group 1	$\sigma_{m_i} = 6\%\ \mu_{m_i}, i = 1, \dots, 16$
Group 2	$\sigma(a) = 5°$
Group 3	$\sigma\left(h_j\right) = 5 \times 10^{-4}, j = 1, \dots, 4$
Group 4	$\sigma_{P_k} = 6\%\ P_k, , k = 1, \dots, 9$

The RBRDO problem is rewritten for this case study as follows

$$\begin{aligned}
\min_{\boldsymbol{\mu}_{\mathbf{z}}} \quad & \mathbf{F}(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) = \left\{W(\boldsymbol{\mu}_{\mathbf{z}}), \det \mathbf{C}_{\boldsymbol{\varphi}}\,(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}})\right\} \\
\text{subject to} \quad & g_1(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) \le 0 \\
& g_2(\boldsymbol{\mu}_{\mathbf{z}}, \mathbf{u}_{\mathbf{m}}^{*}) \ge 0 \\
& \mu_{z_i}^{\,l} \le \mu_{z_i} \le \mu_{z_i}^{\,u}, \quad i = 1,2, \dots, \mathrm{N}_{\mathbf{z}}
\end{aligned} \tag{32}$$

and the PMA problem as

$$\begin{aligned}
\max_{\mathbf{m}} \quad & \bar{f} = C - \left[\alpha_1 g_2(\boldsymbol{\mu}_{\mathbf{z}}, \mathbf{m})^2 + \alpha_2 \Gamma(\beta)\right] \\
\text{subject to} \quad & \phi(\mathbf{u}_{\mathbf{m}}) \equiv \beta(\mathbf{u}_{\mathbf{m}}) - \beta^{a} = \Delta
\end{aligned} \tag{33}$$

where

$$\Gamma(\beta) = \begin{cases} 0 & , if\ \phi(\mathbf{u}_{\mathbf{m}}) \in [-\Delta, \Delta] \\ K|\phi(\mathbf{u}_{\mathbf{m}})|^{q} & , if\ \phi(\mathbf{u}_{\mathbf{m}}) \notin [-\Delta, \Delta] \end{cases} \tag{34}$$

Fig. 2 shows the evolution of the construction of the Pareto Optimal front, screening solutions of rank 1 of the last generation of the MOGA-2D. The bi-objective RBRDO problem based on minimization of weight and variability of the structural response appears to have contradictory objectives, as the minimization of one implies an augmentation in the other. The Pareto optimal front has a good number of solutions, indicating a good convergence of the algorithm. The influence of the sharing function is evident, as the solutions are well spread along the Pareto front. Since the determinant of the variance-covariance matrix do not allow to distinguish between the partial contributions of each critical response, $\bar{R}$ and $\bar{u}$, to the total variability, the respective coefficients of variation $CV(\bar{R}) = \sqrt{var(\bar{R})}/\bar{R}_a$ and $CV(\bar{u}) = \sqrt{var(\bar{u})}/\bar{u}_a$ were calculated and it is concluded that the variability of the critical stress response is low and fairly stable, while the variability of the critical displacements is larger and incrementing, in agreement with $\det \mathbf{C}_{\boldsymbol{\varphi}}$. Then, if needed, it is possible to establish a preference function for the desired optimal solution of the bi-objective problem, based on the obtained values of $CV(\bar{u})$.

When it comes to the reliability-based optimization, Fig. 3 shows that the values of $g_2(\boldsymbol{\mu}_\mathbf{z}, \mathbf{u}_\mathbf{m}^*)$ vary for different Pareto-optimal solutions. It is interesting to notice that solutions with high variability have high values of $g_2(\boldsymbol{\mu}_\mathbf{z}, \mathbf{u}_\mathbf{m}^*)$ suggesting that, while in terms of stress analysis they might be reliable, since stresses do not induce relevant variability on the system, when it comes to displacements, reliability may not be granted. For further conclusions, reliability-based optimization must be applied to the displacements' response, as well. Also, from Fig. 3, it is possible to conclude that the minimum reliability level $\beta^a = 3$ is well imposed, numerically.

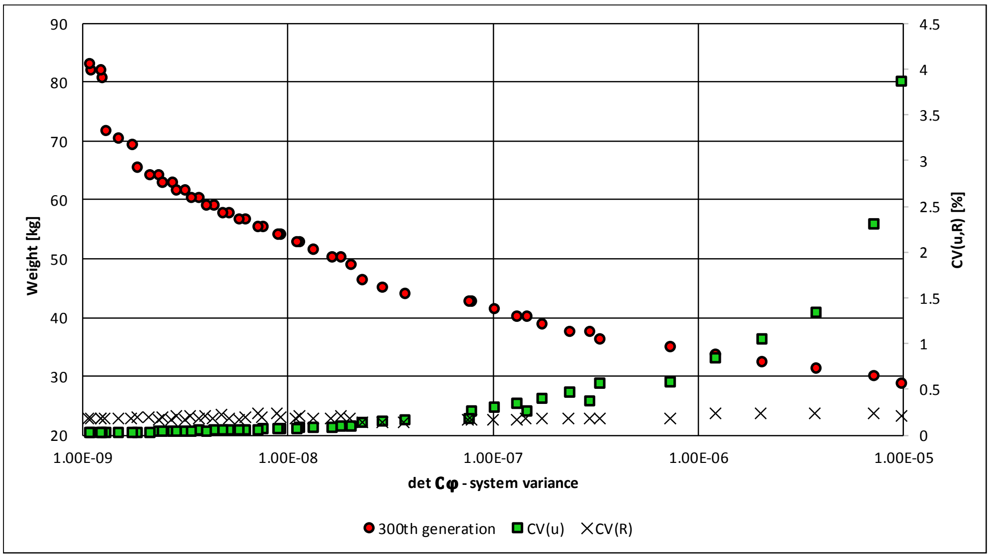

Fig. 2 - Pareto front evolution, at generation 300, and response of both critical displacements and Tsai number.

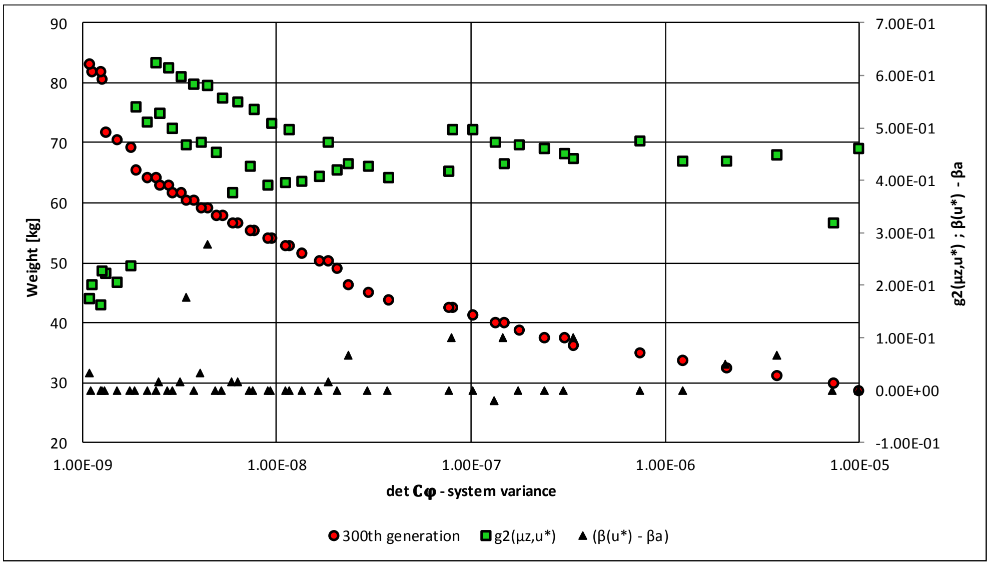

Fig. 3 - Pareto-optimal front, stress limit-state value after PMA and imposed reliability level validation.

CONCLUSIONS

In this work an efficient formulation of the RBRDO for composite shell structures is proposed. Robustness is defined as a measure of variability of the structural responses of displacement and stress. Reliability of the structural stress response is evaluated through the PMA. The resulting two-cycle bi-objective optimization is solved with the application of two GA's, one for each cycle. For the RDO outer-cycle, the search in the design space is based on the concept of constrained-dominance, performed by the bi-level genetic algorithm MOGA-2D. The PMA internal cycle, is solved performing a local search in the uncertainty space formed by stress related variables, using a micro-GA. Even though the structural system is considered to be linear elastic, the limit-state function of stress is estimated by a first-order expansion in Taylor's series, to avoid numerous evaluations of the FE model. The presented example shows that the MOGA-2D converges to a Pareto-optimal front with two distinct zones: one with well distributed solutions, in the weight varying branch; another with sparse solutions, in the variability varying branch. The calculation of individual coefficients of variation, show that the critical displacement response is responsible for the most of the variability of the system, while for critical stress it is low and stable. Reliability results show that the Pareto-optimal solutions respect the desired reliability level also with reliability-feasible values of the stress limit-state function.

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