Solving Combinatorial Optimization Problems: exact techniques

- Explicit enumeration — building all the *admissible solutions* it is possible to obtain the *optimal solution*.
- Implicit enumeration — all the admissible solutions are considered and *implicitly evaluated* but are not explicitly built.

  Examples: Tree serach with “Branch and Bound”; lower and upper limits for the value of the optimal solution.
- Modeling the problems with integer programming models (decision variables take integer values), or even binary (variables with only two possible values: 0 or 1), and their resolution with adequate algorithms.

  Note: These formulations can also be used to obtain limits for the value of the optimal solution through *relaxations*.

Solving Combinatorial Optimization Problems: exact techniques

Relaxation — If we do not consider one or more constraints of the original problem PO, we can transform the problem in another problem that can be simpler to solve PR. Considering that the problem is a minimization problem, the optimal values of the objective function are related in the following way:

$$f^{\star}_{PR} \leq f^{\star}_{PO}$$

(with less constraints the solution can only improve or stay the same).

Linear relaxation – transform an integer problem in a problem with continuous variables by "dropping" the constraint that the variables must be integer (or Binary). This allows us to use the Simplex Algorithm instead of the more complex, and far more time consuming, tree search.

“Branch and bound” method

The “branch and bound” method is based on an intelligent enumeration of the solutions that are candidates to be to integer optimal by *successively partitioning* the solution space and *cutting the search tree* by considering limits that are calculated during the enumaration.

Graphical representation of the resolution of a problem with B&B

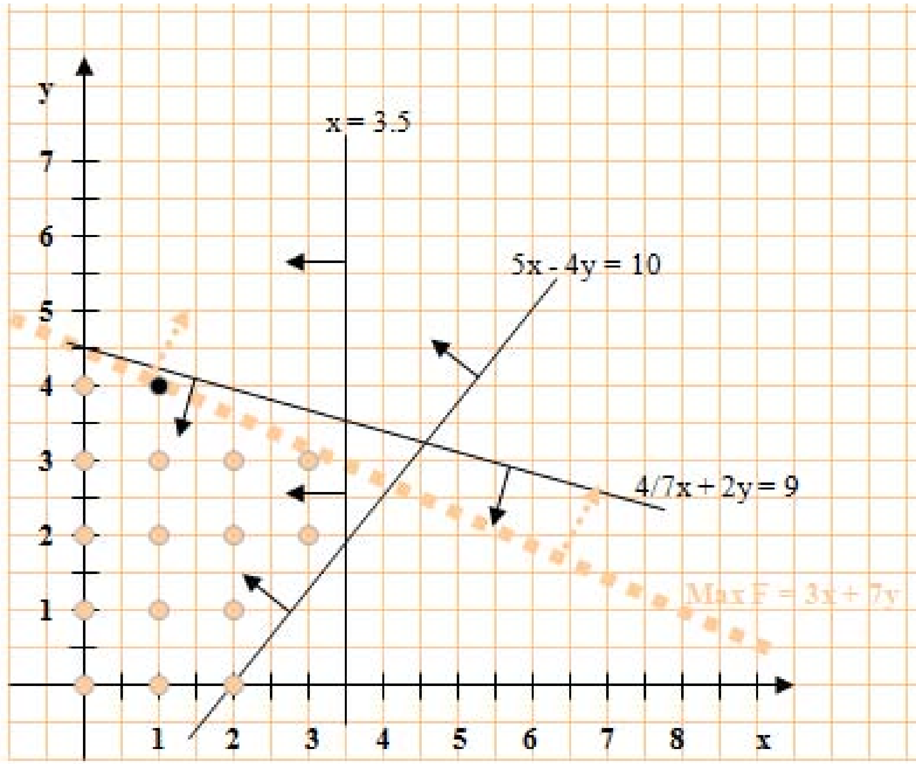
Consider the following integer programming problem:
Maximize:

$$F = 3x + 7y$$

subject to:

$$\begin{array}{ccccc} x & & & \leq & 3.5 \\ 5x & - & 4y & \leq & 10 \\ \frac{4}{7}x & + & 2y & \leq & 9 \\ x & , & y & \geq & 0 \textit{ and integer} \end{array}$$

and its graphical representation:



The integer optimal solution is: $x = 1$ and $y = 4$.

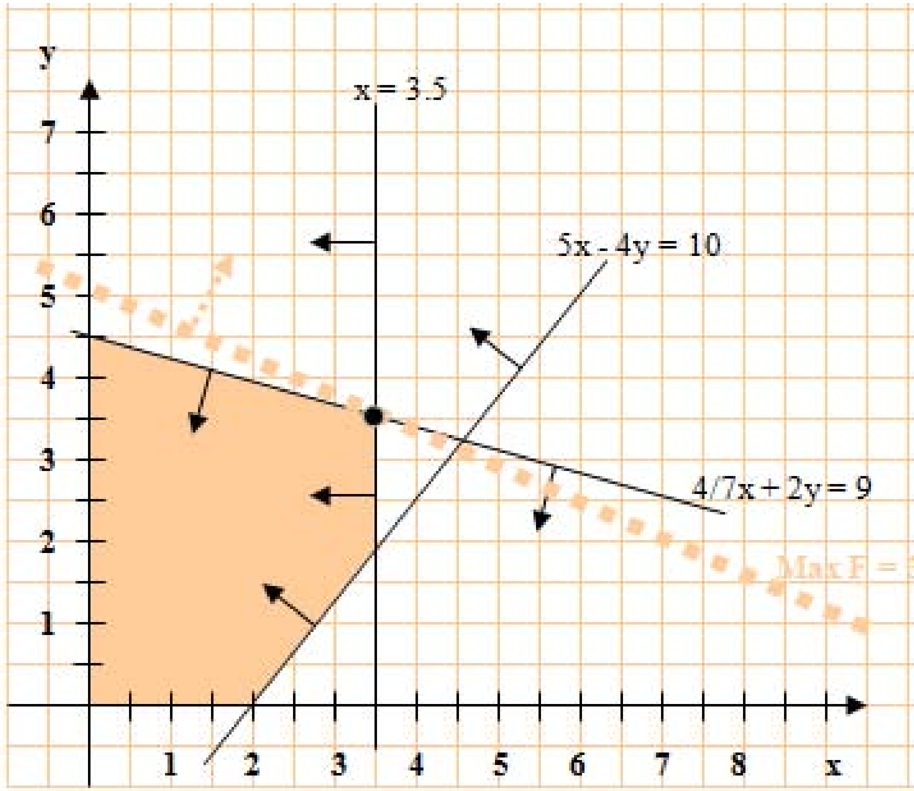
Solving the linear relaxation

Problem $\mathcal{PL}0$:

$$\max F = 3x + 7y$$

subject to:

$$\begin{array}{rcrcr} x & & & \leq & 3.5 \\ 5x & - & 4y & \leq & 10 \\ \frac{4}{7}x & + & 2y & \leq & 9 \\ x & , & y & \geq & 0 \end{array}$$



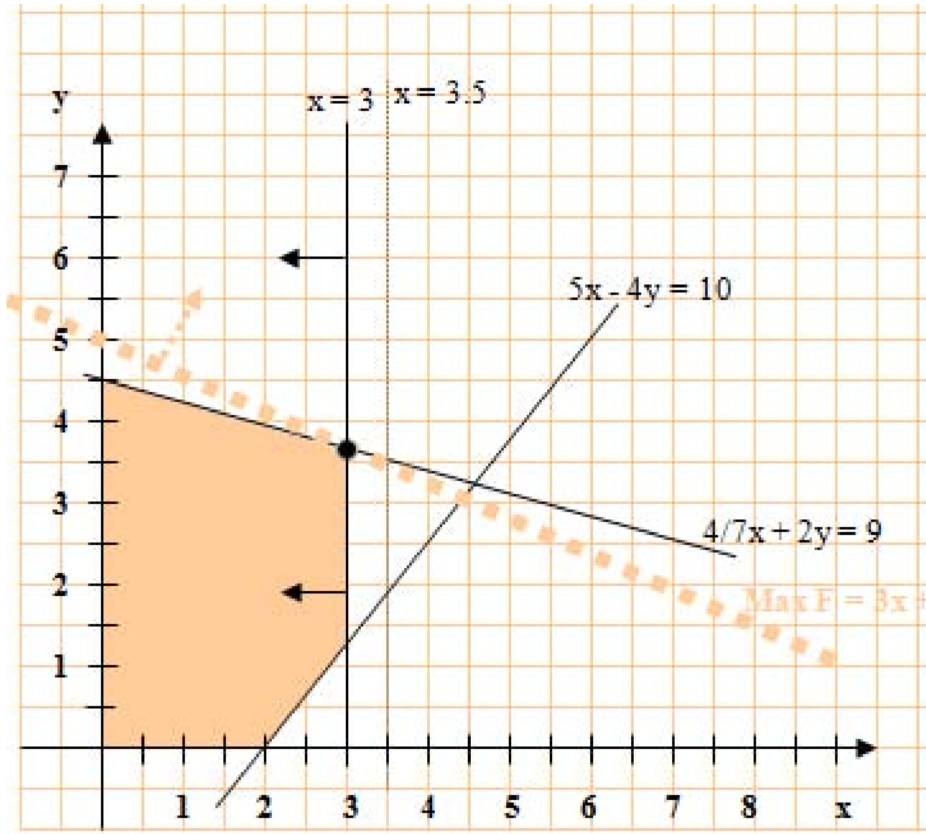
Optimal, non integer, solution:
$x = 3.5$ and $y = 3.5$; $F = 35$

Branching in x: $x \leq 3$

$$\max F = 3x + 7y$$

subject to:

$$\begin{array}{rcrcr} x & & & \leq & 3.5 \\ 5x & - & 4y & \leq & 10 \\ \frac{4}{7}x & + & 2y & \leq & 9 \\ x & , & y & \geq & 0 \\ x & & & \leq & 3 \end{array}$$



Solution (non integer):

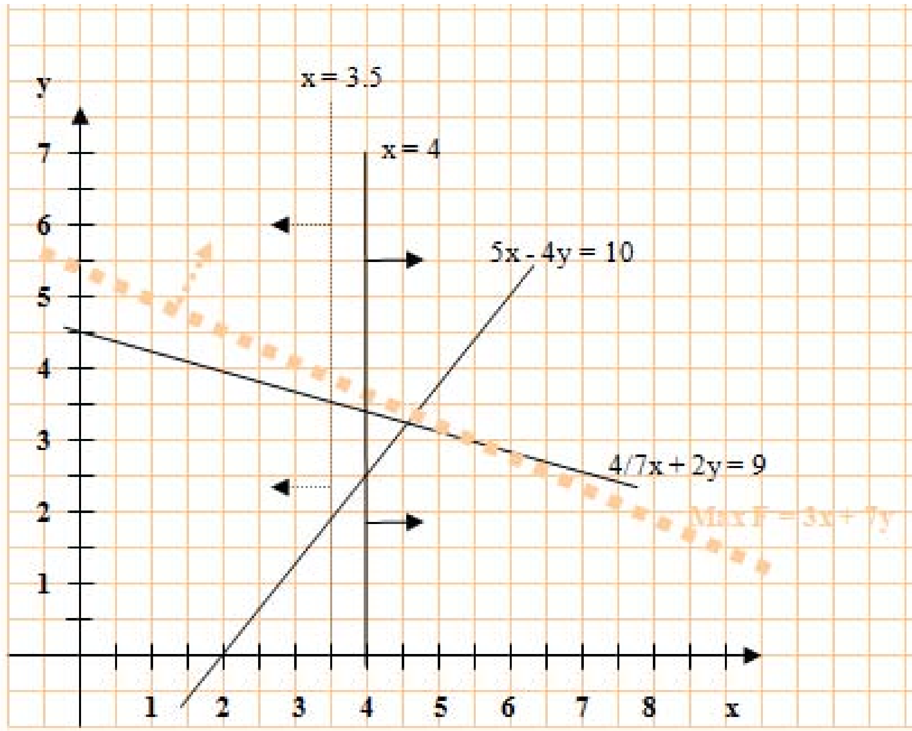
$x = 3$ and $y = 3.6$; $F = 34.5$

Branching in x: $x \geq 4$

$$\max F = 3x + 7y$$

subject to:

$$\begin{array}{rcrcr} x & & & \leq & 3.5 \\ 5x & - & 4y & \leq & 10 \\ \frac{4}{7}x & + & 2y & \leq & 9 \\ x & , & y & \geq & 0 \\ x & & & \geq & 4 \end{array}$$



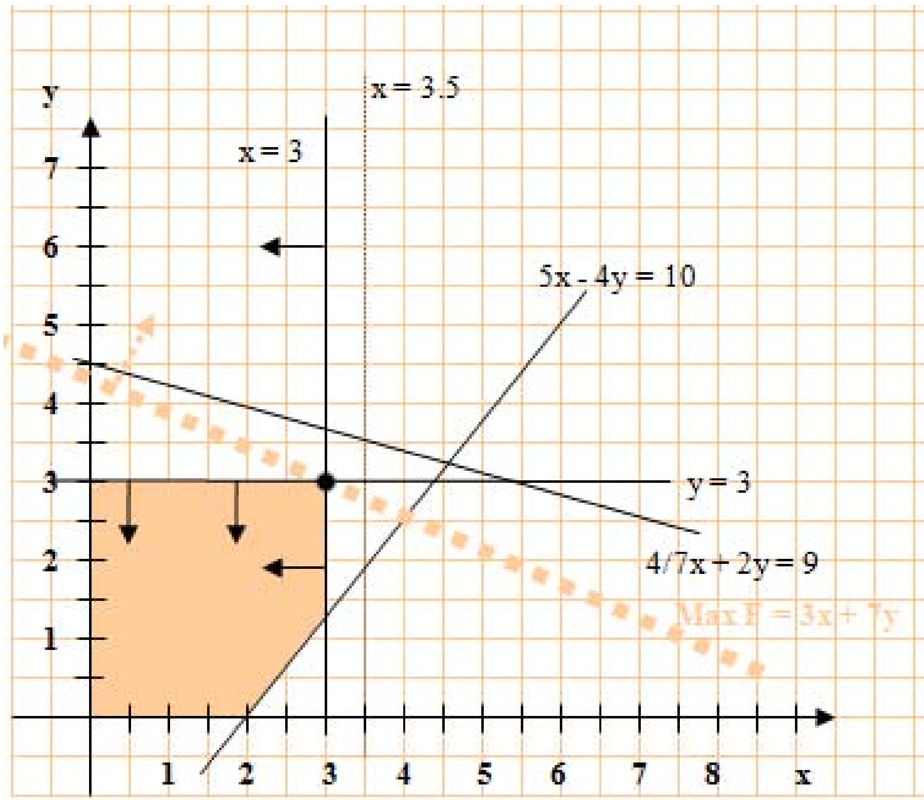
Without admissible solutions.

Branching in y: $y \leq 3$

$$\max F = 3x + 7y$$

subject to:

$$\begin{array}{rcrcr} x & & & \leq & 3.5 \\ 5x & - & 4y & \leq & 10 \\ \frac{4}{7}x & + & 2y & \leq & 9 \\ x & , & y & \geq & 0 \\ x & & & \leq & 3 \\ & & y & \leq & 3 \end{array}$$



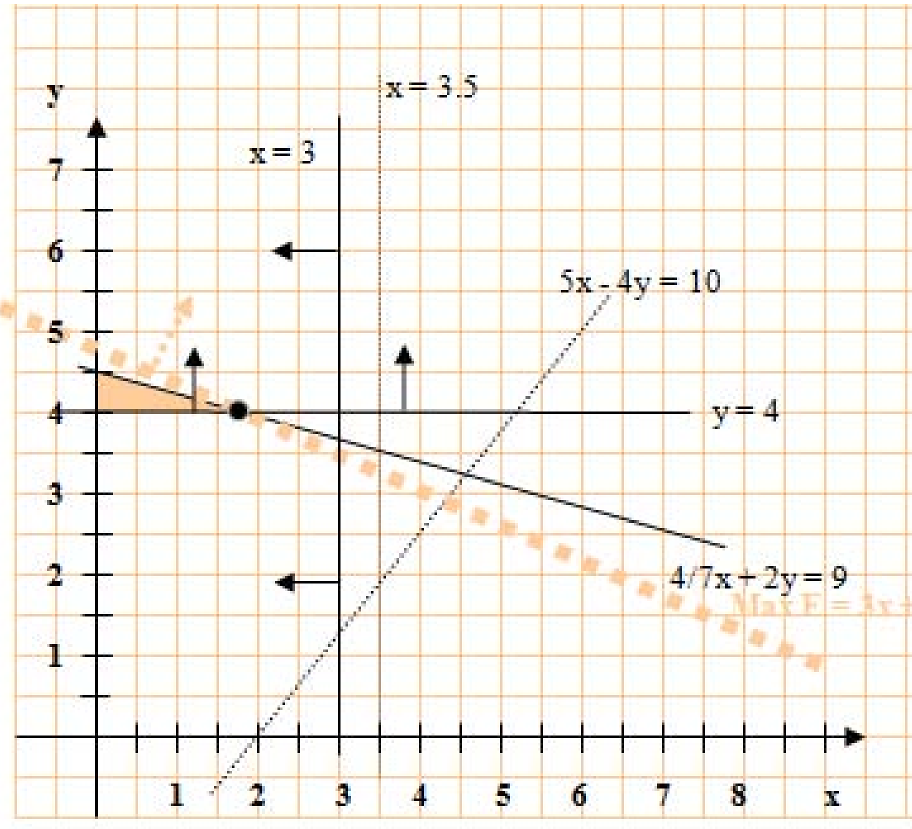
Solution (integer):
$x = 3$ and $y = 3$; $F = 30$
Lower limit $\Rightarrow$ non-integer solutions with an objective value F smaller than or equal to 30 do not need to be further explored!

Branching in y: $y \geq 4$

$$\max F = 3x + 7y$$

subject to:

$$\begin{array}{rcrcr} x & & & \leq & 3.5 \\ 5x & - & 4y & \leq & 10 \\ \frac{4}{7}x & + & 2y & \leq & 9 \\ x & , & y & \geq & 0 \\ x & & & \leq & 3 \\ & & y & \geq & 4 \end{array}$$



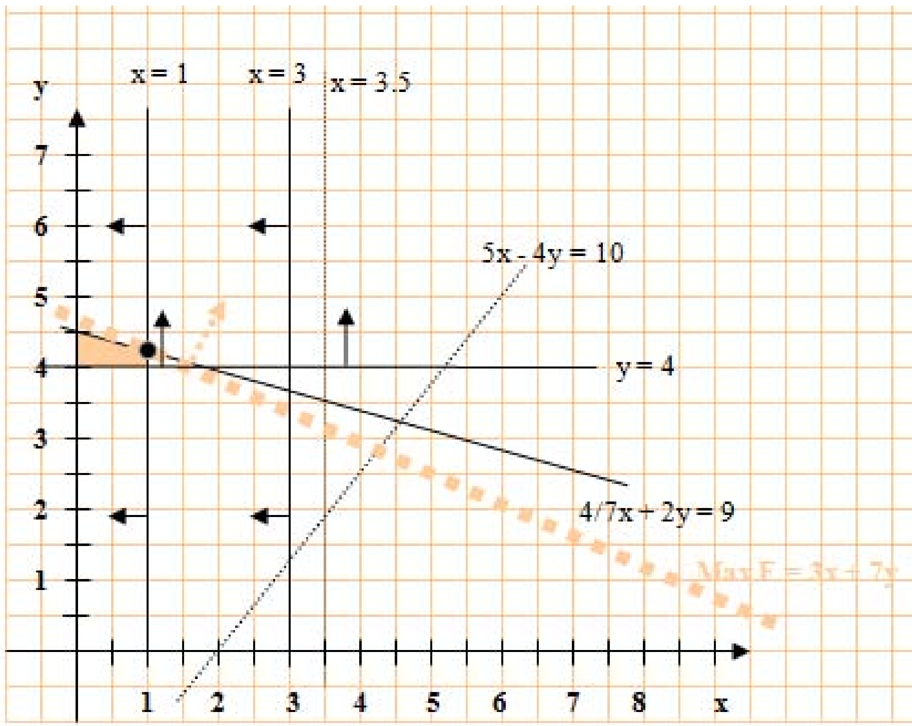
Solution (non-integer):
$x = 1.7$ and $y = 4$; $F = 33.2$

Branching in x: $x \leq 1$

$$\max F = 3x + 7y$$

subject to:

$$\begin{array}{rcrcr} x & & & \leq & 3.5 \\ 5x & - & 4y & \leq & 10 \\ \frac{4}{7}x & + & 2y & \leq & 9 \\ x & , & y & \geq & 0 \\ x & & & \leq & 3 \\ & & y & \geq & 4 \\ x & & & \leq & 1 \end{array}$$



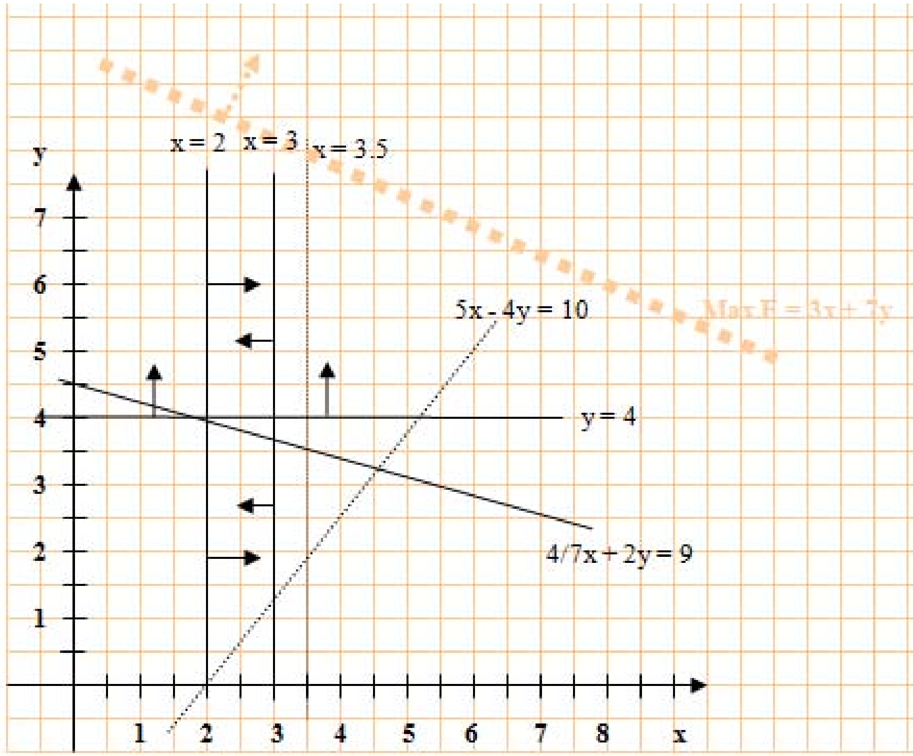
Solution (non-integer):
$x = 1$ and $y = 4.2$; $F = 32.5$

Branching in x: $x \geq 2$

$$\max F = 3x + 7y$$

subject to:

$$\begin{array}{rcrcr} x & & & \leq & 3.5 \\ 5x & - & 4y & \leq & 10 \\ \frac{4}{7}x & + & 2y & \leq & 9 \\ x & , & y & \geq & 0 \\ x & & & \leq & 3 \\ & & y & \geq & 4 \\ x & & & \geq & 2 \end{array}$$



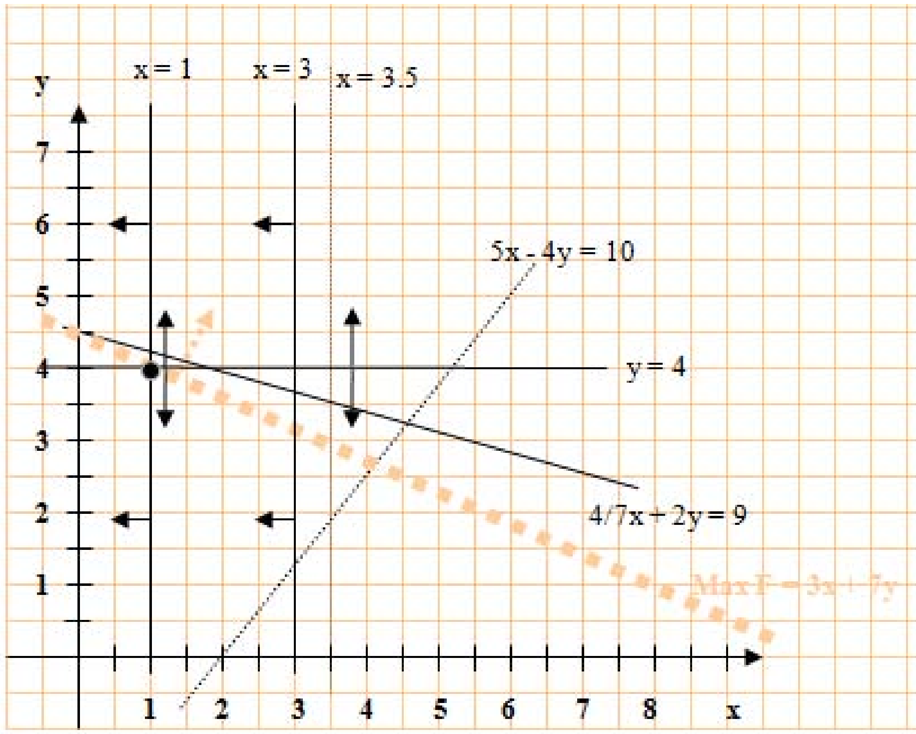
Without feasible solutions.

Branching in y: $y \leq 4$

$$\max F = 3x + 7y$$

subject to:

$$\begin{array}{rcrcr}
x & & & \leq & 3.5 \\
5x & - & 4y & \leq & 10 \\
\frac{4}{7}x & + & 2y & \leq & 9 \\
x & , & y & \geq & 0 \\
x & & & \leq & 3 \\
 & & y & \geq & 4 \\
x & & & \leq & 1 \\
 & & y & \leq & 4
\end{array}$$



Solution (integer):

$x = 1$ and $y = 4$; $F = 31$

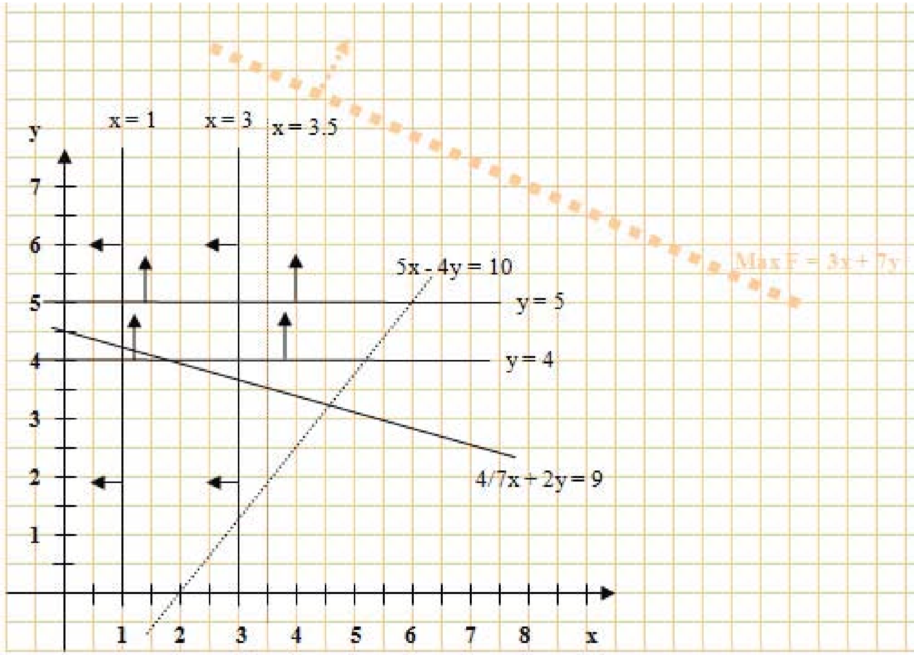
Best integer solution obtained so far!

Branching in y: $y \geq 5$

$$\max F = 3x + 7y$$

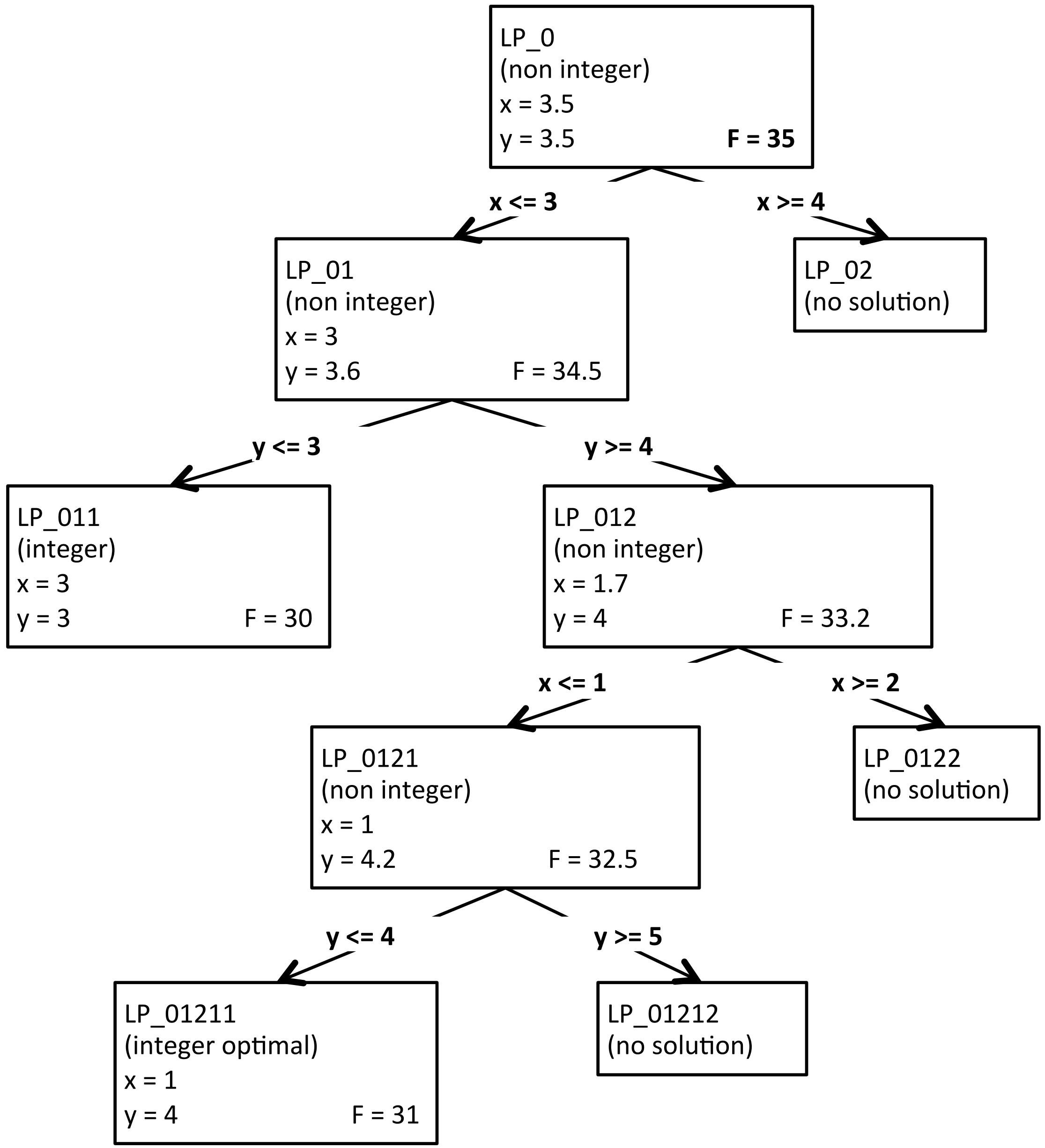
subject to:

$$\begin{array}{rcrcr}
x & & & \leq & 3.5 \\
5x & - & 4y & \leq & 10 \\
\frac{4}{7}x & + & 2y & \leq & 9 \\
x & , & y & \geq & 0 \\
x & & & \leq & 3 \\
 & & y & \geq & 4 \\
x & & & \leq 1 & \\
 & & y & \geq & 5
\end{array}$$



Without admissible solutions.

"Branch-and-Bound" tree

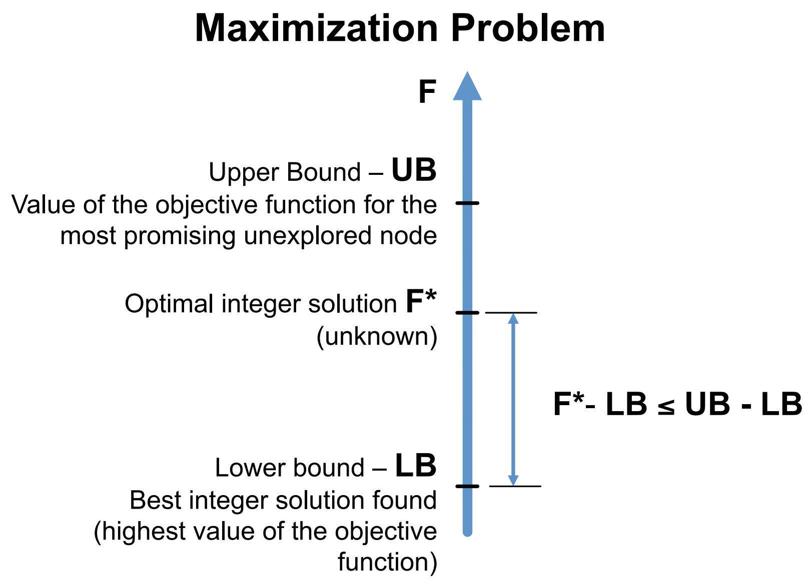


Upper and lower Bounds

- the "branch & bound" is more efficient because it is possible to cut nodes of the search tree without exploring them, because we are sure not to obtain better solutions than the ones we already have;
- allow us to "measure the distance", (considering the value of the objective function) to the optimal solution.

Bounds in a maximization problem:

- a **lower** limit LI is given by an integer solution that has been previously obtained – the optimal solution $F^\star$ can never be worse (lower) than the integer solution that we already have;
- an **upper** limit LS is given by the highest value of the objective function within all the nodes that are not yet completely explored (we hope to find there an integer solution that is better than the one we got).



Bounds – example

Consider a maximization problem where all the variables are integer. The tree in figure 1 is obtained along the resolution process.

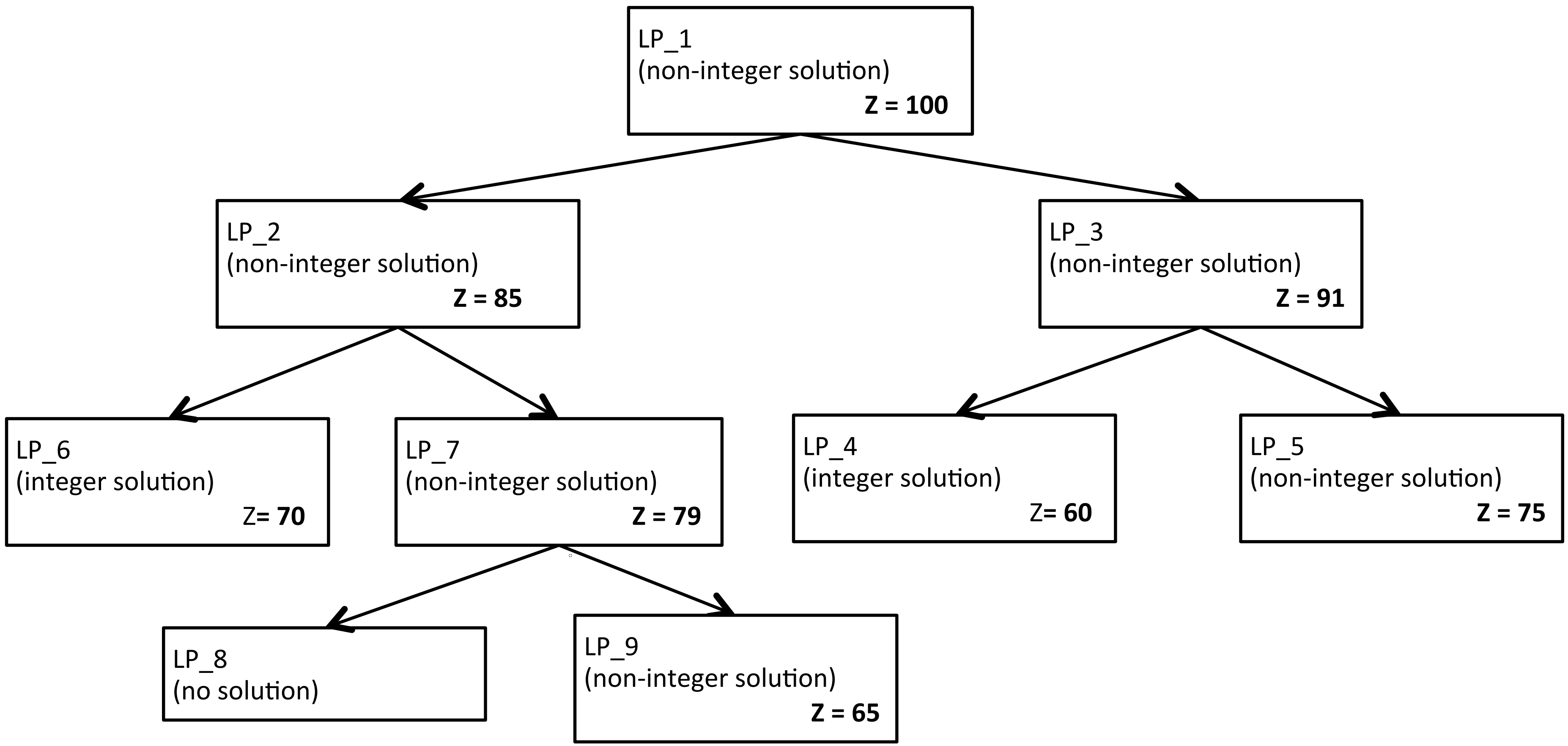


Figura 1: "Branch-and-Bound" tree.

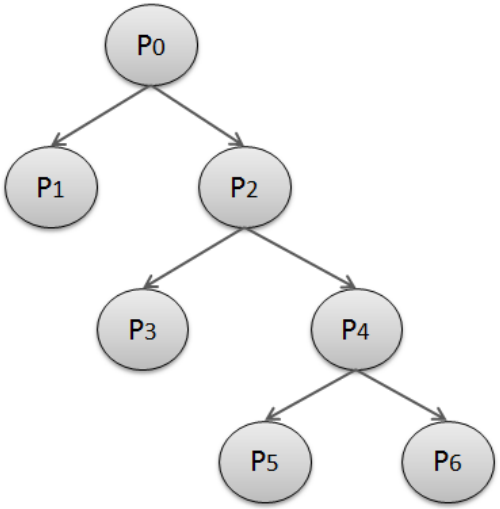
1. Which is now the best **upper bound** on the optimal (integer) solution?
2. Which is now the best **lower bound** on the optimal (integer) solution?
3. Which nodes have already been explored? Explain why.
4. Which nodes have not been explored yet? Explain why.
5. Do we already know the optimal solution? Explain why.
6. Which is the maximum gap if the algorithm finishes at this point?

Open questions – Branching strategies

Given a set of unexplored nodes how can we choose the next node to explore?

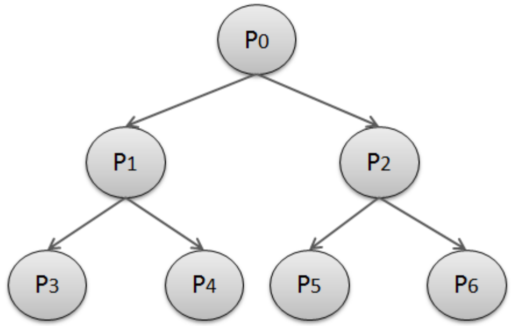
Depth-first search

Select the deepest node in the tree.



Breadth-first search

Select the unexplored node that is highest in the tree.



Most promising node

Select the node with the best objective function. The node that can potentially lead to the best integer solution.

Open questions

- **Selection of the variable to branch on** – After selecting the node to explore, which variable should be chosen to branch on within the variables that are not integer in the solution in the node?

  Some strategies have been described in literature, but their results depend heavily on the problem at hand.

  $$\Downarrow$$

  The strategies depend on the application and on the physical meaning of the variables.

Bibliography

- Alves, José Carlos (1989). *Provas de Aptidão Científica e Capacidade Pedagógica.* FEUP.
- Goldbarg, Marco Cesar e Luna, Henrique Pacca (2000). *Otimização Combinatória e Programação Linear*, Editora CAMPUS.