Class Notes
MAD – Decision Aid Methodologies – FEUP 2005

Multiobjective problems

Manuel Matos

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Multiobjective problems

$$\min \quad \mathbf{f}(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \dots \\ f_m(\mathbf{x}) \end{cases}$$

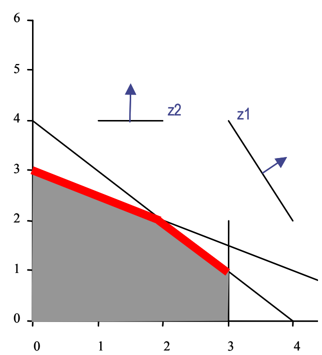
$$st: \quad \mathbf{g}(\mathbf{x}) = \mathbf{0}$$
$$\mathbf{h}(\mathbf{x}) \leq \mathbf{0}$$
$$\mathbf{x} \geq \mathbf{0}$$

$\mathbf{x}$ vector of decision variables *(may include integer or binary variables)*
$\mathbf{f}(\mathbf{x})$ vector of objective functions
$\mathbf{g}(\mathbf{x})$ set of equality constraints
$\mathbf{h}(\mathbf{x})$ set of inequality constraints

- Main characteristics
  - Alternatives are not known in advance
  - Optimization procedures are always needed
  - May have a big number of constraints and decision variables
  - May not be completely described by the mathematical formulation
  - Sometimes interpreted as optimization problems with more than one objective function (vector optimization)

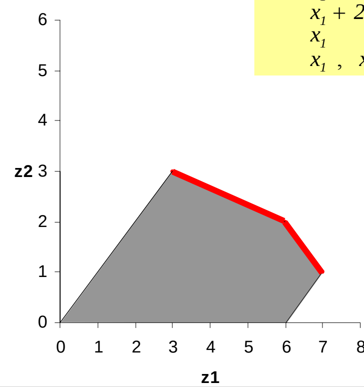
Decision space vs attribute's space

- Decision space



- Attribute's space

$$\max\ z_1 = 2x_1 + x_2$$
$$\max\ z_2 = x_2$$
$$\begin{aligned} suj:\ & x_1 + x_2 \le 4 \\ & x_1 + 2x_2 \le 6 \\ & x_1 \le 3 \\ & x_1\ ,\ x_2 \ge 0 \end{aligned}$$



Multicriteria analysis - main approaches

- Ensure that the DM follows a "rational" behavior (Normative option)
- Give some advice based on reasonable (but not indisputable) rules
- Find the preferred solution from partial decisions about decision hypothesis
- Prepare decision sets

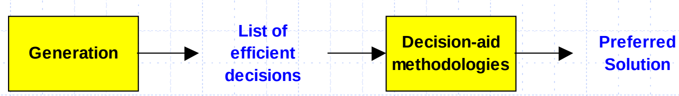
- *Value functions, Utility theory, distance to the Ideal*
- *The French School*
- *Interactive methods*
- *Generation methods*
  Filtering of efficient solutions

MO problems – basic strategies

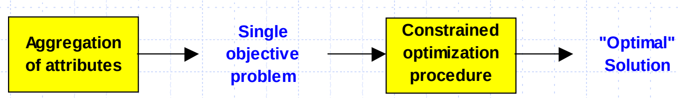
- Generation methods
- Aggregation of criteria (use of a value function)
  - Transforms the problem into an optimization one
- Interactive methods
  - Based on an implicit value function (never explicitly known!)
    - Geoffrion-Dyer-Feinberg, Surrogate Worth Trade-off, Zionts-Wallenius
  - Without special conditions
    - STEM, Trimap
- Goal programming

Multiobjective approaches

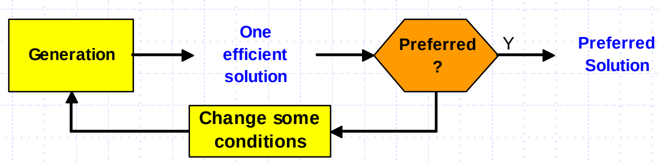
- Two phase (or generation)



- Aggregation



- Interactive



Some arguments

Strategy	Pro	Con
Generation	Doesn't have parameters Gives the global picture Doesn't require the DM's presence	Doesn't produce a solution or an order Risk of generating to many solutions Heavy calculations
Aggregation	Leads to optimization Induces a total order No further intervention of the DM	Difficulties in building the Value Function Some arbitrariness Tendency to predefinitions and confusion between Obj. Func. and Value Func.
Interactive	Reduces information overload Easier calculations (in general) Induces learning	Loss of holistic vision Produces only a final solution May need many judgments
Goal Prog.	Well established in OR Easy to apply Adequate to large dimension problems	Only linear problems Needs goal definition Requires a lexicographic order of the criteria (no compensation)

Aggregation (use of a value function)

$$\min \ \mathbf{f}(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \cdots \\ f_m(\mathbf{x}) \end{cases}$$

$$\text{st:} \quad \mathbf{g}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{h}(\mathbf{x}) \le \mathbf{0}, \quad \mathbf{x} \ge \mathbf{0}$$

→

$$v(\mathbf{x}) = v[f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})]$$

↓

$$\max \ v(\mathbf{x})$$

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Interactive approaches

(typically, only for MO linear problems)

- General procedure
  1. Find an initial solution (efficient)
  2. Ask the DM if he is satisfied → if he is, this is the preferred solution. STOP
  3. Ask the DM which criteria he wants to improve and which criteria he accepts to worsen
  4. Use the precedent information to find a new solution
  5. Return to 2
- Some classics
  - STEM
    - STRANGE
  - Zionts-Wallenius
  - Interval Criterion Weights
  - Surrogate Worth Trade-off
  - Geoffrion-Dyer-Feinberg
  - Pareto Race
  - Trimap

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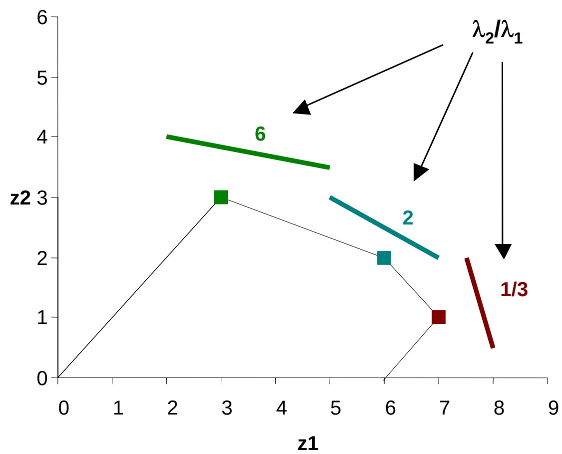
Generation methods

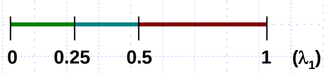
- Parametric variation of $\lambda>0$ in $min\ f(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x})$
  - The optimal solution of this auxiliary problem is an efficient solution of the original multiobjective problem
  - The parameters λ are only instrumental (not judgments of the DM)
- Constrained optimization
  - Define additional constraints in n-1 objective functions
  - Optimize the remaining objective function
  - Repeat for different RHS values of the additional constraints
- Multiobjective simplex
- Multiobjective metaheuristics

Parametric variation

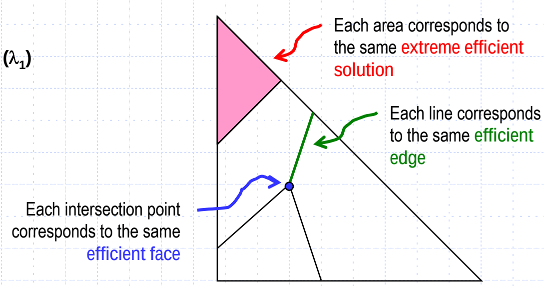


Parametric variation

- In MO linear problems, post-optimization (parametric analysis) can be used to find all the efficient solutions
  - e.g. (previous problem)

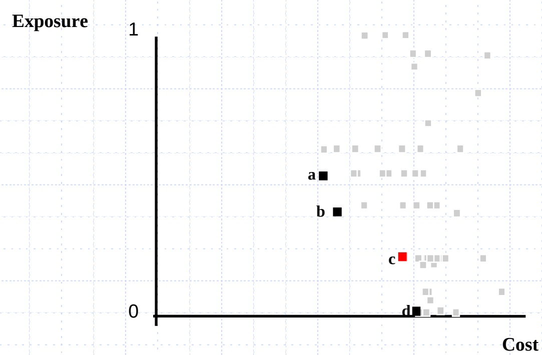


e.g. (tricriteria problem)



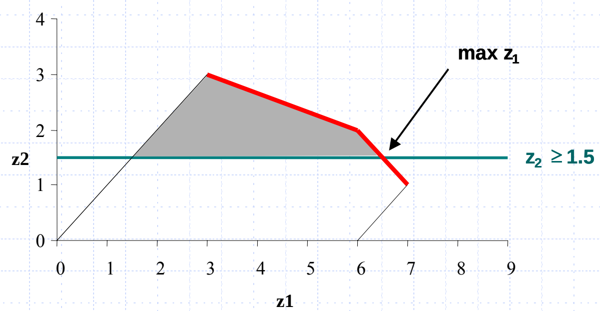
Parametric variation

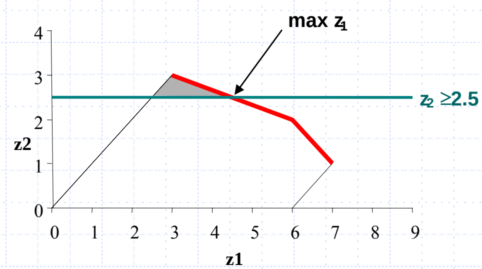
- Difficulties in combinatory problems
  - Some efficient solutions are never selected



a,b,c,d - Efficient solutions
c - "Convex dominated" **but not dominated**

Constrained optimization (ε - constraint)

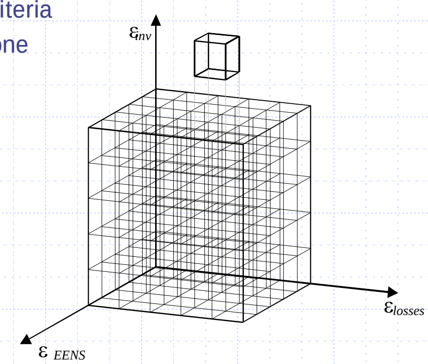




ε - constraint method

- Repeat
  - Fix lower limits for n-1 criteria
  - Optimize the remaining one
- Example
  - max Robustness

  s.t.:
  - Inv ≤ ε_{INV}
  - Loss ≤ ε_{LOSS}
  - EENS ≤ ε_{EENS}



  - Variation of the ε leads to the generation of efficient solutions

Compound and emergent strategies

- Generation > Filtering
  - Use aspiration levels and elimination rules
  - Reduces the number of alternatives to consider
  - Still doesn't produces a solution or order
- Generation > Multiattribute method
  - Constitutes a complete approach
  - Opens the way to the use of less prescriptive methodologies
- Meta-heuristics and multiobjective genetic algorithms
  - Adequate for MO problems with integer or binary variables
  - Explore the efficient zone (or part of it)
  - May include interactivity