An Optimal Control Approach to Dynamic Nonlinear Infinite-Horizon Economic Models[1]

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Keep Calm

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

KEEP
CALM
AND
DO MORE
CALCULUS

About the Economists

"You and I come by road or rail, but economists travel on infrastructure"

– Margaret Thatcher

Outline

Outline

1. Preliminaries
2. Wrapping up
3. Limits to the analytical analysis
4. A numerical framework for infinite-horizon economic problems

An Optimal Growth Model

As an introductory note, we present the seminal Ramsey-Cass-Koopmans optimal growth model.

An Optimal Growth Model

Assumption 1

Families maximize their welfare by consuming goods, subject to a financial constraint.

Assumption 2

Firms produce an homogeneous good, subject to a resource constraint.

An Optimal Growth Model

Assumption 3

All markets clear: families face no unemployment (labour market) and all products are sold at given prices (goods market).

An Optimal Growth Model

Families L_t offer labour so that they can earn an income and pay for goods. Population grows at rate n.

$$L_t = L(0) \cdot e^{nt}, \quad L(0) = 1$$

which implies that

$$\dot{L} = nL$$

An Optimal Growth Model

Families work so that they can **consume** C. Their **welfare** $u(\cdot)$ is maximized by consumption.

$$max\ U = \int_0^\infty u(C)dt$$

Per capita $(c \equiv C/L)$ we have that for all families

$$max\ U = \int_0^\infty u(c)e^{nt}dt$$

An Optimal Growth Model

Let us assume a continuous discount rate $\rho > 0$ — families would rather not postpone their consumption.

This is the **objective function** of the households/families.

$$max\ U = \int_0^\infty u(c)e^{(n-\rho)t}dt$$

where the following properties hold: $u'(c) > 0$ and $u''(c) < 0$ — the utility function $u(\cdot)$ is concave and $\rho > n$ to avoid divergence.

An Optimal Growth Model

Firms produce a homogenenous good by employing labour L and capital K, according to a given level of technology A.

Goods are then produced according to the **production function**:

$$Y \equiv f(A, K, L) = A \cdot K^{\alpha} \cdot L^{1-\alpha} \quad (0 < \alpha < 1)$$

Per capita $(y \equiv Y/L,\ k \equiv K/L)$ we have

$$y \equiv f(k) = Ak^{\alpha}, \quad k(0) = k_0$$

An Optimal Growth Model

If we derive the production function in order to each one of the inputs, L and K, we obtain

$$r = \frac{\partial y}{\partial k} = f_k'$$

$$w = \frac{\partial y}{\partial l} = f_l'$$

where r is the **real interest rate** on capital and w the **wage of labour**.

An Optimal Growth Model

Finally, there is a **global resource constraint** that governs the whole economy and is obtained from the two separate optimisation problems — families maximize C and firms maximize π. We will omit how it is derived.

$$\dot{k} = f(k) - c - (\delta + n)k$$

where $0 \leq \delta < 1$ denotes the depreciation rate.

An Optimal Growth Model

Global Resource Constraint

$$\dot{k} = f(k) - c - (\delta + n)k$$

In economic terms, it reads like this: given the output of the economy $(y = f(k))$, if we discount resources to be consumed (c) and resources used to either replace end-of-life capital (δk) or to equip new borns (nk), what is left over is used to **invest in new capital** — lesson of the day: **saving is paramount** to produce more in the future.

An Optimal Growth Model

The following step is familiar. We formulate the Hamiltonian in order to apply the Maximum Principle and extract the Necessary Optimality Conditions along with the transversality condition.

$$H = u(c)e^{(n-\rho)t}dt + \lambda(f(k) - c - (\delta + n)k)$$

The first-order conditions are

$$\begin{aligned}\frac{\partial H}{\partial c} &= 0\\ \frac{\partial H}{\partial k} &= -\dot{\lambda}\\ \frac{\partial H}{\partial \lambda} &= \dot{k}\end{aligned}$$

and the transversality condition $\lim_{t\to\infty} k \cdot \lambda = 0$

An Optimal Growth Model

Assume that $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$

Let us focus on the first equation given by

$$\frac{\partial H}{\partial c} = 0 \Leftrightarrow e^{(n-\rho)t}c^{-\theta} = v$$

We take logs and differentiate to obtain

$$(n-\rho) - \theta\frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda}$$

If we replace $\dot{\lambda}/\lambda$ we get...

An Optimal Growth Model

The **Euler equation**

$$\dot{c} = \frac{c}{\theta}(f'(k) - \delta - \rho)$$

It describes the **optimal trajectory** of the control variable c.

An Optimal Growth Model

The Euler equation

$$\dot{c} = \frac{c}{\theta}(f'(k) - \delta - \rho)$$

along with the resource constraint

$$\dot{k} = f'(k) - c - (\delta + n)k$$

the transversality condition

$$\lim_{t \to \infty} k \cdot \lambda = 0$$

and the initial value for k

$$k(0) = k_0$$

is the analytical description of our (buoyant) economy.

An Optimal Growth Model

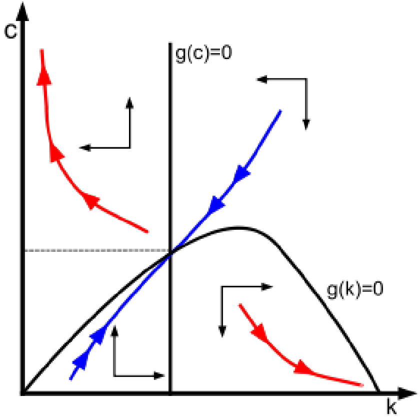

Figure: Phase diagram in the (k-c) space.

An Optimal Growth Model

Economists focus mainly in two properties of the dynamic system:

- The long run or **steady state** — when $\dot{c} = 0$ and $\dot{k} = 0$
- The short run or **transition dynamics** — when the system is subject to a shock. A change in the value of a parameter, for instance.
    - How the system *actually* moves from one state to another is also of extreme importance — consider a policy that would cause a decrease in consumption in the short term but exhibit a boost of its new steady state value. Is it worth it?
        - This is called *dynamic inefficiency*

Outline

In a nutshell

- Devise an economic growth model
- Solve the underlying optimal control problem
- Find the dynamic system that describes the economy
- (Log-)linearize to study the stability of the steady state
- (Log-)linearize again or run a numerical simulation to find the trajectories of a given transition

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Outline

Overview

- Economic models are becoming **increasingly complex**
- Most of these models are **nonlinear**
  - An analytical solution may not exist or can not be obtained in a timely manner
  - We could **linearize** (in our case, log-linearize)
    - It does come with a cost
  - Transition trajectories are extremely difficult to fiddle with
- Most **numerical tools** are not precise

Did you say “in a timely manner?”

- The model we have seen (Ramsey-Cass-Koopmans) dates back to 1926[5]
- 80 years later, 2006, we still have contributions to the literature[6] for closed-form solutions to the underlying analytical problem of this model
  - But they require **odd and very specific** combinations of parameters that allow for such closed-form solution

[5] [Ramsey(1928), Cass(1965), Koopmans(1963)]

[6] [Smith(2006)]

80 years

- Yes, your math is correct. **80 years** for a *particular* closed-form solution to be derived
  - Can we wait that long?

- How misleading can linearization be?
  - A lot

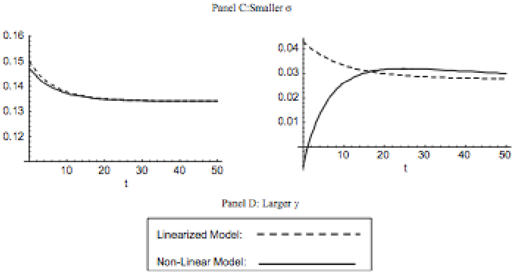

Figure: [Atolia et al.(2010)Atolia, Chatterjee, and Turnovsky] show how misleading linearization can be.

Other methods

- What about other methods?
  - Better — but should we settle with these results?

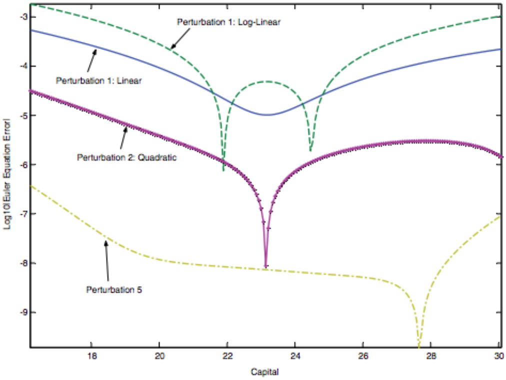

Fig. 8. Euler equation errors at $z = 0$, $\tau = 2/\sigma = 0.007$.

Figure: [Aruoba et al.(2006)Aruoba, Fernández-Villaverde, and Rubio-Ramírez] compare several known procedures, analytical and numerical.

The transitional dynamics

- [Futagami et al.(2008)Futagami, Iwaisako, and Ohdoi] devise an endogenous growth model with public debt
  - They define the target debt as a ratio to the stock of private capital ($b \equiv B/K$)
  - That gives origin to **two** balanced growth paths (one of high growth, another of low growth) and a possible indeterminacy of the transition path
- [Minea and Villieu(2012)] make a very small change to that model
  - They define the target debt as a ratio to the output of the economy ($b \equiv B/Y$)
    - Everything else remains the same
  - That gives origin to **one** balanced growth path and to a unique adjustment path to equilibrium

Outline

Our contribution

- We follow [Fontes(2001)] and we suggest a general framework to tackle the underlying optimal control problem of economic growth models
    - No need for linearization
    - No need to determine the analytical solution of the Euler equation
    - The transitional dynamics can be studied along the way and not only at the steady state
    - It allows for something not yet much studied: ***anticipated*** **exogenous shocks**.

Fontes (2001)

Consider the following generic optimal control problem.

$$\text{maximize}\left\{J = \int_{t}^{t+T} L(s, x(s), u(s))ds + W(t+T, x(t+T))\right\}, \qquad (1)$$

subject to:

$$\dot{x}(s) = f(s, x(s), u(s)) \quad a.e.\ s \in [t, t+T],$$

$$x(t) = x_t,$$

$$u(s) \in U(s) \quad a.e.\ s \in [t, t+T],$$

$$x(t+T) \in S.$$

Fontes (2001)

H1-H4

Assume that the hypothesis H1-H4 hold.

Fontes (2001)

SC1-SC5

To guarantee the stability of the finite horizon problem, the stability conditions SC1-SC5 must be verified.

Fontes (2001)

SC4

The time horizon T is such that, the set S is reachable in time T from any initial state and from any point in the generated trajectories: that is, there exists a set X containing X_0 such that for each pair $(t_0, x_0) \in \mathbb{R} \times X$ there exists a control $u : [t_0, t_0 + T] \to \mathbb{R}^m$ satisfying

$$x(t_0 + T; t_0, x_0, u) \in S.$$

Also, for all control functions u in the conditions above $x(t; t_0; x_0; u) \in X$ for all $t \in [t_0, t_0 + T]$.

Fontes (2001)

SC5

There exists a scalar $\varepsilon > 0$ such that for each time $t \in [T, \infty)$ and each $x_t \in S$, we can choose a control function $\widetilde{u} : [t, t+\varepsilon] \to \mathbb{R}^m$, with $\widetilde{u}(s) \in U(s)$ for all $s \in [t, t+\varepsilon]$, satisfying

$$W_t(t, x_t) + W_x(t, x_t) \cdot f(t, x_t, \widetilde{u}(t)) \leq -L(t, x_t, \widetilde{u}(t))$$

and

$$x(t+r; t, x_t, \widetilde{u}) \in S$$

for all $r \in [0, \varepsilon]$.

Fontes (2001)

Theorem 3

Assume the system satisfies hypothesis H1-H4. Choose the design parameters to satisfy SC. Then, for a sufficiently small inter-sample time δ, the closed-loop system resulting from the application of the MPC strategy is asymptotically stable in the sense that $\|x^*(t)\| \to 0$ as $t \to \infty$.

The stability conditions for the RCK model

These are general results for the stability of Model Predictive Control. For our case we just require SC4-5 to hold.

For $k \in S$ then $\exists c^*$ s.t.

$$\dot{W} \leq -L$$

$$k(t+\delta) \in S$$

This implies that we need to impose a **boundary cost** W and a **boundary condition**.

The stability conditions for the RCK model

We now apply these results to our problem. Recall the objective function and note that $(L \equiv -U)$.

$$U = \int_0^\infty u(c)e^{(n-\rho)t}dt$$

In our problem, the cost at the end time (*boundary cost*) W will be given by

$$\dot{W} = u(c^*)e^{(n-\rho)t}dt$$

Integrating we get the **boundary cost** W (note that $u(c^*)$ is constant since c^* is constant)

$$W = \frac{e^{(n-\rho)t}}{\rho - n} \cdot u(c^*)$$

The stability conditions for the RCK model

The **boundary condition** is implied by the definition of the region S.

$$S = \{k \in \mathbb{R} : \ \dot{k} = 0 \Leftrightarrow y - c - (\delta + n + x)k = 0\} \quad k(T) \in S$$

The boundary condition along with the boundary cost guarantee that

Infinite Horizon Problem $\equiv$ Finite Horizon Problem

i.e.,

$$\int_0^\infty U(t)dt \equiv \int_0^T U(t)dt + \int_T^\infty U(t)dt, \quad (W \equiv \int_T^\infty U(t)dt)$$

The tools

- Matlab runs the show
  - ICLOCS for numerically solving the optimal control problem
    - IPOPT is used to solve the ODE equations as an alternative to *fmincon()*
  - We implement the five stability conditions as defined by [Fontes(2001)]
    - Otherwise the numerical estimates will show a finite-horizon behavior

The new approach

- Concoct the economic growth model, assume that markets clear and get the constraints of the economy
- Implement the model in ICLOCS and add the stability conditions SC
- Profit!

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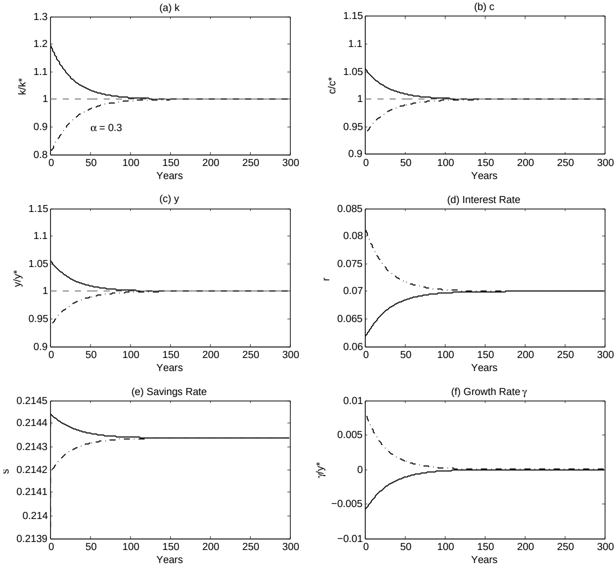
An application: The Ramsey-Cass-Koopmans



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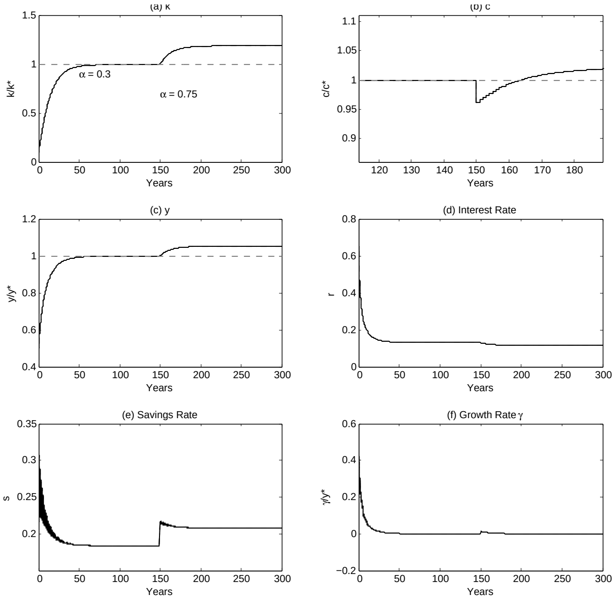

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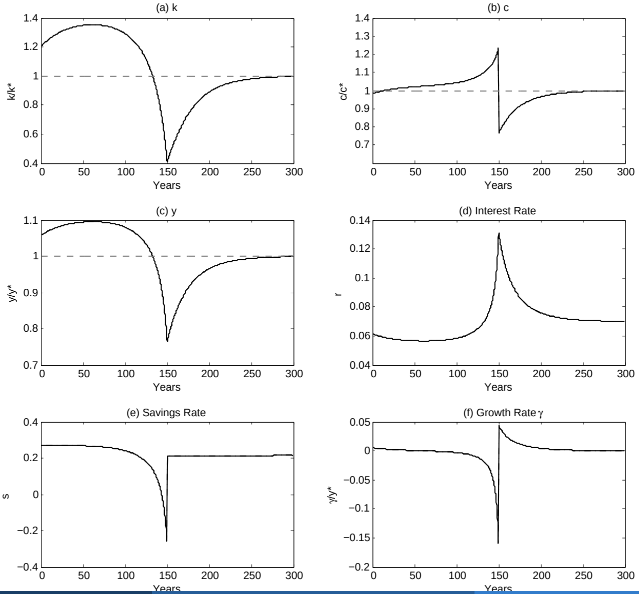

Evaluation of the procedure

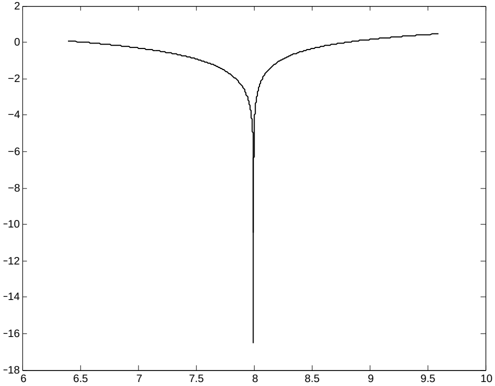

Figure: Euler equation error function for our numerical procedure.

Another quote

"Principles have no real force except when one is well-fed."
– Mark Twain

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