Optimal Control Methods for Infinite-Horizon Economic Growth Models

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Abstract

We propose a framework for solving nonlinear infinite-horizon economic growth models using an NLP direct method. This approach has several benefits in comparison to the indirect and analytical methods found in the literature:

- It can solve a model in its nonlinear form
- It allows for the study of anticipated and multiple, sequential shocks
- It does no require the dynamic system to start from a steady-state equilibrium
- Opens the way for more complex models that are analytically intractable

Introduction

Optimal control theory has been extensively applied to the solution of economic problems since the pioneering work of [1]. The study of dynamic growth models has been following a standard procedure, which consists in applying the Maximum Principle and obtaining the NCO for a linearized version of the infinite-horizon problem. This approach has served the economics profession well but lacks the flexibility and robustness of state-of-the-art numerical tools.

Building on the work of [2], we propose a framework that transforms the original infinite-horizon problem into a finite-horizon equivalent form and solves it using direct methods. This new procedure is capable of dealing with complex models, once deemed intractable when using analytical tools.

We exemplify the usage of this framework by numerically solving the Uzawa-Lucas endogenous growth model and analyzing how it copes with anticipated shocks, but more recent models like [3] are also eligible. This framework opens a whole new realm of possibilities and we believe it will be an important asset in the toolkit of a macro-growth researcher.

Procedure

The procedure consists in first discretizing the problem and then optimizing, and works as follows:

1. Transform the original infinite-horizon optimal control problem P_∞ into an equivalent finite-horizon problem P_T by using Theorem 1
2. Numerically solve P_T using a direct method optimal control solver (e.g. ICLOCS).

Theorem 1

Given a generic optimal control problem

$$P_\infty: \ \min \int_0^\infty L(x(t), u(t)) \cdot dt$$

subject to:

$$\begin{aligned} &\dot{x} = f(x,u) \quad \text{a.e.} \\ &x(0) = x_0, \\ &x(t) \in \Gamma(t), \\ &u(t) \in \Omega(t) \end{aligned}$$

for which we assume there is a finite solution. Assume additionally that after some time T, the state is within some invariant set S (that is, $x(t) \in S$, $S \subset \Gamma(t)$, for all $t \geq T$) for which the problem still has a finite solution. Then, there exists a terminal cost function W, such that the problem is equivalent to the finite horizon problem

$$P_T: \ \min \int_0^T L(x(t), u(t)) \cdot dt + W(x(T))$$

subject to:

$$\begin{aligned} &\dot{x} = f(x,u) \quad \text{a.e.} \\ &x(0) = x_0, \\ &x(t) \in \Gamma(t), \\ &u(t) \in \Omega(t), \\ &x(T) \in S. \end{aligned}$$

Numerical Results for an Anticipated Shock

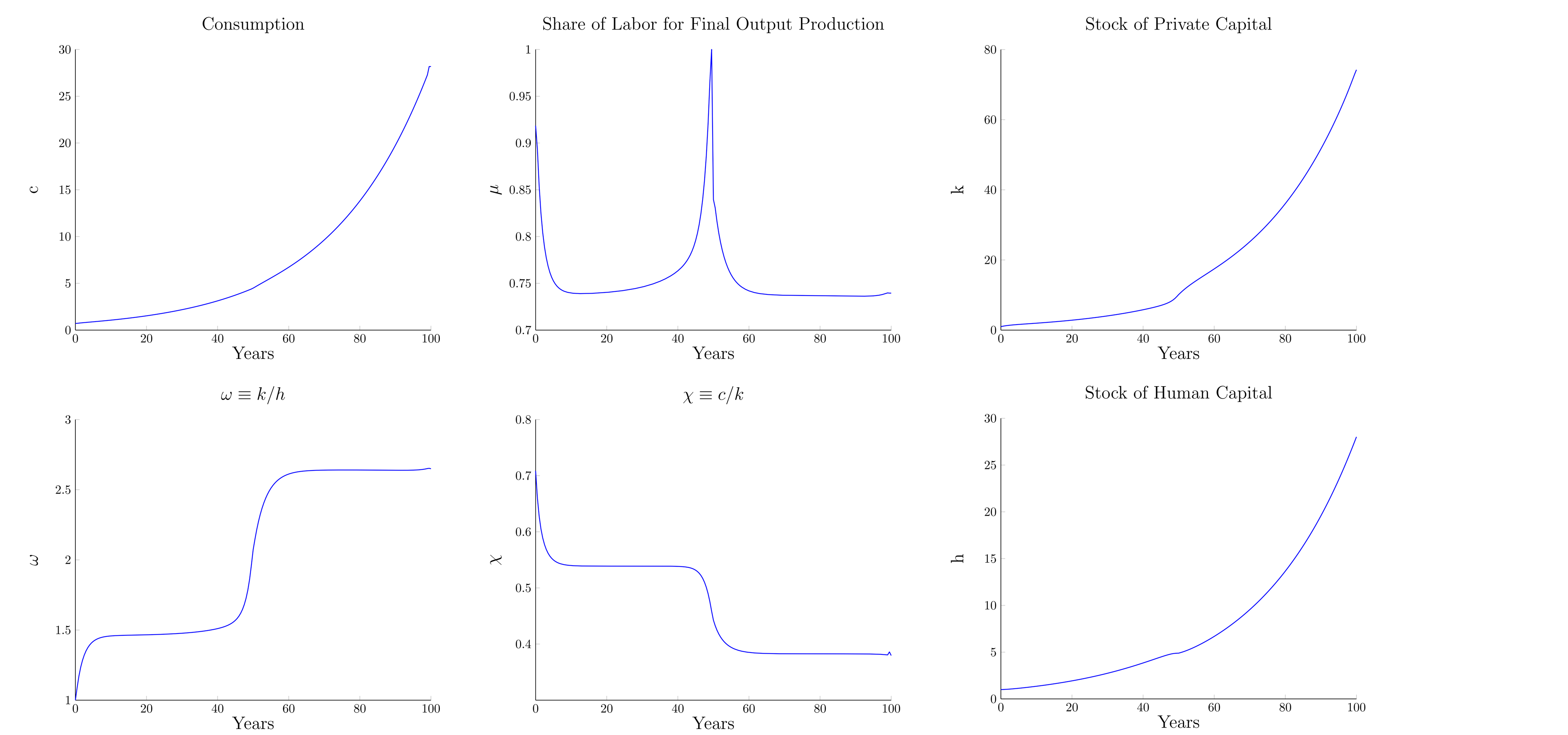

Figure 1: Numerical optimization of the Uzawa-Lucas endogenous growth model when subject to an anticipated capital elasticity increase from $\alpha = 0.3$ to $\alpha = 0.4$ at time $t = 50$.

Application

We exemplify the usage of the framework solving the Uzawa-Lucas endogenous growth model.

$$\max U = \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{-\rho t} dt, \quad s.t.$$

$$\begin{aligned} &c > 0, && 0 \leq \mu \leq 1 \\ &\dot{k} = Ak^\alpha (\mu h)^{1-\alpha} - c - \delta_k k && \\ &\dot{h} = B(1-\mu)h && \\ &k(0) = k_0 && k \geq 0, \ \forall t > 0 \\ &h(0) = h_0 && h \geq 0, \ \forall t > 0 \end{aligned}$$

with the social planner or household choosing an allocation $(c, \mu)_{t=0}^\infty$ that maximizes U. In the long-run, the economy enters a BGP with $\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{b}}{b} = \gamma$.

Further Work

We are extending this framework to run unanticipated shocks by running a similar problem with a moving-horizon (MPC).

Moreover, we are using the framework to study optimal debt adjustment using time-variant tax rates, something deemed intractable using analytical methods.

References

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