Stability of dark screening solitons in photorefractive media

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Abstract

Biased photorefractive media are known to admit bright and dark solitons. Their bright solitons are always stable, nevertheless, their dark counterparts are unstable above certain background intensity and below a critical velocity. Here, we use the stability criterion and the Vakhitov-Kolokolov function to precisely determine the unstable parameter region. We also anticipate the strength of the instability by determining the unstable eigenvalues and eigenmodes using the Evans function method. The results are confirmed by numerical simulation of the full evolution equation.

1 Introduction

Photorefractive materials support the propagation of self-guided beams since the variations in refractive index produced by the beams may be sufficient to compensate for their diffraction. These beams are usually named photorefractive (PR) solitons. Whenever the self-guiding effect benefit from an external electric field, these solitons are called screening solitons. Both bright and dark screening solitons were predicted theoretically [2] and experimentally observed [11, 10, 3]. The bright solitons are parameterised by peak value or power and all of them are stable [4]. The dark solitons are parameterised by the intensity of the background and velocity or minimum of intensity [2, 5]. For relatively large background intensities, there is a critical velocity below which the dark solitons are unstable. This threshold of stability is easily determined by the stability criterion for dark solitons whenever gray solitons are considered [1, 7]. However, there are numerical difficulties associated with the application of the above criterion to black solitons (dark solitons with zero intensity).

Here, we investigate the stability of black solitons using the Vakhitov-Kolokolov function at zero [8]. Moreover, the strength of the instability is investigated using the Evans function method to compute the unstable eigenvalues and eigenmodes.

The article is structured as follows. In section 2, we introduce the ordinary differential equation for the dark soliton profile, its phase and general characteristics. In section 3, we determine the stability threshold for black and gray solitons and use the Evans function method to calculate the unstable eigenvalues and eigenmodes. Then, we present the results of direct numerical simulations in section 4 and the conclusions in section 5

2 Stationary solutions

The propagation of dark beams in biased photorefractive planar waveguides is modeled by [2]

$$iq_z + \frac{1}{2}q_{xx} + (1+\rho)\frac{q}{1+|q|^2} = 0. \tag{1}$$

The propagation is along z and diffraction is only allowed along x which coincides with the c axis of the crystal. The above equation is valid whenever the transport of charge is dominated by drift so that diffusion may be neglected. The above condition is fulfilled for strong bias fields and relatively large beams. The equation (1) admits localized stationary solutions corresponding to a hole over a background of constant intensity [2, 5], so that the nonzero boundary condition at infinity is given by

$$q \to \sqrt{\rho}\mathrm{e}^{i(\theta_0 \pm S/2)} \qquad x \to \pm\infty \tag{2}$$

where S is the phase difference across x. Let $q(z,x) = \sqrt{\rho}y(\eta)\mathrm{e}^{i\theta(z,\eta)}$ be a solution to (1) where $\eta = x - \omega z + \eta_0$, then the phase is given by

$$\theta(z,\eta) = z - \omega\int_0^\eta \frac{d\eta'}{y^2} + \omega\eta + \theta_0.$$

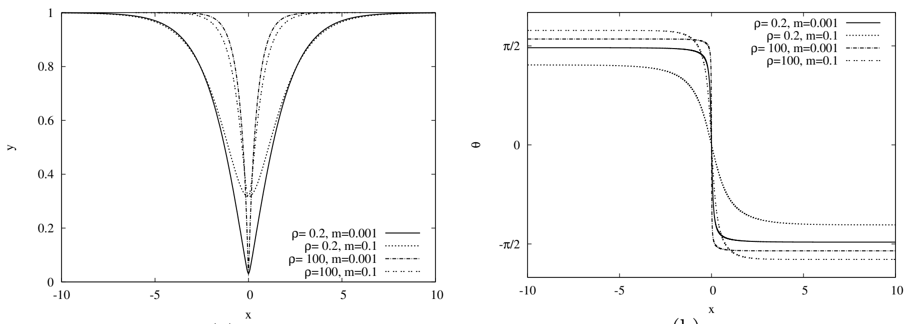
and $y(\eta)$ is a real non-negative function such that $y(\eta) \leq 1$, $y(\eta) \to 1$ as $\eta \to \pm\infty$ and satisfies the following ordinary differential equation

$$y'' + (\omega^2 - 2)y - \frac{\omega^2}{y^3} + (1+\rho)\frac{2y}{1+\rho y^2} = 0 \tag{3}$$

The profiles $y(\eta)$ (see Fig. 1(a)) are parameterized by ρ and ω such that $\omega^2 < \rho/(1+\rho)$. The minimum of $y(\eta)$ is $\sqrt{m}$, where m relates to the velocity ω by

$$\omega^2 = \frac{2m}{1-m}\left[\frac{1+\rho}{\rho(1-m)}\ln\left(\frac{1+\rho}{1+\rho m}\right) - 1\right].$$

Note that when $\omega = m = 0$ the above phase and ODE describe a black soliton. In this case and for all ρ, the phase difference between both tails is



(a) (b)

Figure 1: Normalised field profile (a) and phase profile (b) of gray solitons.

π which occurs as a discontinuity at the minimum η-position. Nevertheless, generally the phase difference of gray solitons is not π (Fig. 1(b)) . For small ρ all the gray solitons (every $m > 0$) have phase differences less than π, however, for larger ρ and small m the phase difference is greater than π (Fig. 2). These latter solitons are usually named *darker than black*.

3 Stability Analysis

The stability region for solutions to (1) can be determined using the stability criterion for dark solitons, which asserts that the beam is stable unless $\partial P/\partial\omega > 0$ [1, 7], where P is the renormalized momentum given by

$$\begin{aligned} P &= \frac{i}{2}\int_{-\infty}^{\infty}(q_x^* q - q_x q^*)dx - \rho \mathrm{Arg}\, q|_{-\infty}^{+\infty} \\ &= \omega\rho\int_{-\infty}^{\infty}(y^2-1)d\eta - \rho \mathrm{Arg}\, q|_{-\infty}^{+\infty}. \end{aligned} \tag{4}$$

The latter expression for P shows that for a dark soliton to be unstable it is necessary (not sufficient) that the soliton is black or darker than black. To prove it, let us consider $\omega > 0$ for which the phase difference is negative so that the second term is positive. In the limit $\omega = 0$, the first term is zero and $P = \rho\pi$. As ω moves away from zero, the first term is always negative in such a away that cannot contribute to increase P, therefore only solitons whose phase difference is, in modulus, greater than π may have $\partial P/\partial\omega > 0$.

It is important to realize that this criterion only tell us if a given soliton is stable or unstable, i.e., it is not possible to directly obtain the properties of unstable solitons, such as the stength of the instability, from that derivative. Therefore, one must resort to alternative methods in order to further study this subject.

We proceed by considering the linear stability equations, which are obtained by assuming a solution to equation (1) given by the above dark solution plus a

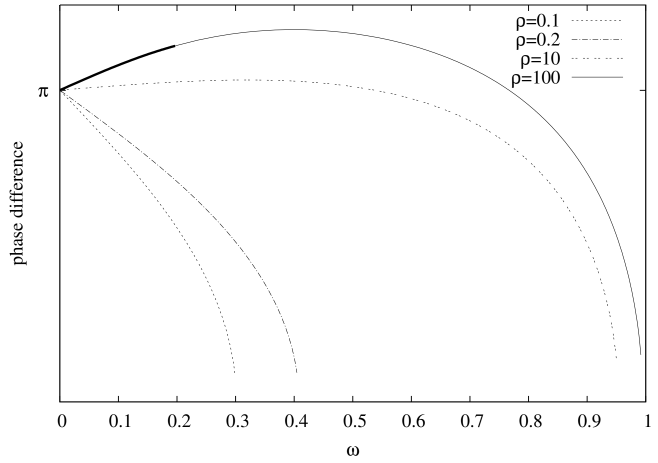


Figure 2: Modulus of the phase difference as function of ρ and ω. The thicker part of the line $\rho = 100$ corresponds to unstable solitons.

small perturbation term, namely

$$q(z,x) = \sqrt{\rho}\mathrm{e}^{i\theta(z,\eta)}\left[y(\eta) + \Delta(z,\eta)\right].$$

Introducing this solution into equation (1) we obtain equation (3) as the zero-order approximation and

$$i\Delta_z + \frac{1}{2}\Delta_{\eta\eta} - i\omega\frac{\Delta_\eta}{y^2} + i\omega\frac{\Delta y'}{y^3} + \left(\frac{\omega^2}{2} - 1 - \frac{\omega^2}{2y^4} + \frac{1+\rho}{1+\rho y^2}\right)\Delta \\ - \frac{\rho(1+\rho)y^2}{(1+\rho y^2)^2}(\Delta + \Delta^*) = 0 \quad (5)$$

as the first order approximation. To investigate spectral stability we assume that Δ has an exponential dependence on z, that is, $\Delta(z,\eta) = u(\eta)\mathrm{e}^{i\lambda z} + v^*(\eta)\mathrm{e}^{-i\lambda^* z}$ and $\Delta^*(z,\eta) = u^*(\eta)\mathrm{e}^{-i\lambda^* z} + v(\eta)\mathrm{e}^{i\lambda z}$. By substituting these expressions in the previous equation, we arrive to the following eigenvalue problem

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix}. \quad (6)$$

The operator L is given by

$$L = \sigma_3\left(\frac{1}{2}\partial_{\eta\eta} + F(\eta) - G(\eta)\right) - i\sigma_2 G(\eta) + iI_2\left(\frac{\omega y'}{y^3} - \frac{\omega}{y^2}\partial_\eta\right)$$

where I_2 is the 2×2 identity matrix and σ_2 and σ_3 are the Pauli matrices given by

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The functions $F(\eta)$ and $G(\eta)$ are given by

$$F(\eta) = \frac{\omega^2}{2} - 1 - \frac{\omega^2}{2y^4} + \frac{1+\rho}{1+\rho y^2}$$

$$G(\eta) = \frac{(1+\rho)\rho y^2}{(1+\rho y^2)^2}.$$

The stationary solution is spectrally stable if the spectrum of L has no strictly negative imaginary part.

The symmetry of L implies that if λ is an eigenvalue $-\lambda$, λ^* and $-\lambda^*$ are also eigenvalues. Following Henry [6], the continuous spectrum of L is on the regions defined by the curves defined by the continuous spectrum of the operator L_∞, where L_∞ stands for the form of L as $\eta \to \infty$, namely

$$L_\infty = \sigma_3 \left(\frac{1}{2}\partial_{\eta\eta} - \frac{\rho}{1+\rho} \right) - i\sigma_2 \frac{\rho}{1+\rho} - iI_2\omega\partial_\eta.$$

Since the continuous spectrum of L_∞ is $\mathbb{R}$, then $\mathbb{R}$ is also the continuous spectrum of L.

Due to the symmetry of (6) relatively to the real axis, the existence of any discrete eigenvalues implies instability. We search for discrete eigenvalues following the standard Evans function method. Let $Y = \begin{pmatrix} u & u_\eta & v & v_\eta \end{pmatrix}^T$, then Y satisfies the equation

$$\frac{dY}{d\eta} = A(\eta, \lambda) Y, \tag{7}$$

where

$$A(\eta,\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2(G - F - i\omega\frac{y'}{y^3} + \lambda) & 2i\frac{\omega}{y^2} & 2G & 0 \\ 0 & 0 & 0 & 1 \\ 2G & 0 & 2(G - F + i\omega\frac{y'}{y^3} - \lambda) & -2i\frac{\omega}{y^2} \end{pmatrix}. \tag{8}$$

For $\eta \to \pm\infty$ the matrix $A(\eta, \lambda)$ is independent of η and we shall denote it by $A_\infty(\lambda)$. Therefore, the system (7) transforms to a constant coefficient system of ordinary differential equations. It has solutions of the form $Y_r^\infty(\eta,\lambda) = y_r(\lambda)\exp[r(\lambda)\eta]$, where $r(\lambda)$ is one of the eigenvalues of $A_\infty(\lambda)$ and $y_r(\lambda)$ is the corresponding eigenvector. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, there are two r values with positive real part and other two with negative real part. Let us denote them as follows:

$$\mathrm{Re}(r_{1,2}) > 0 \qquad \mathrm{Re}(r_{3,4}) < 0.$$

The full system (7) has two bounded solutions as $\eta \to -\infty$ satisfying

$$Y_{r_{1,2}}^-(\eta,\lambda) \sim Y_{r_{1,2}}^\infty \qquad \text{as} \qquad \eta \to -\infty,$$

and two bounded solutions as $\eta \to +\infty$ satisfying

$$Y^+_{r_{3,4}}(\eta,\lambda) \sim Y^\infty_{r_{3,4}} \qquad \text{as} \qquad \eta \to +\infty.$$

The localized eigenfunction corresponding to discrete eigenvalues should be a linear combination of $Y^-_{r_1}(\eta,\lambda)$ and $Y^-_{r_2}(\eta,\lambda)$ (they span the unstable manifold) and also a linear combination of $Y^+_{r_3}(\eta,\lambda)$ and $Y^+_{r_4}(\eta,\lambda)$ (they span the stable manifold).

Following Alexander *et al* we work on the *exterior space* $\Lambda^2(\mathbb{C}^4)$ where the 2-vector $U^-(\eta,\lambda) = Y^-_{r_1}(\eta,\lambda) \wedge Y^-_{r_2}(\eta,\lambda)$ represents the unstable manifold and $U^+(\eta,\lambda) = Y^+_{r_3}(\eta,\lambda) \wedge Y^+_{r_4}(\eta,\lambda)$ represents the stable manifold. In $\Lambda^2(\mathbb{C}^4)$, λ is an eigenvalue if and only if $U^-(\lambda,\eta) \wedge U^+(\lambda,\eta) = 0$. The function $\tilde{D}(\lambda,\eta) = U^-(\lambda,\eta) \wedge U^+(\lambda,\eta)$ is independent of η and analytic on $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Here, we define

$$D(\lambda) = U^-(\lambda,0) \wedge U^+(\lambda,0)$$

as our Evans function, whose zeros are eigenvalues of L. Moreover, we define a normalized version of the Evans function as

$$E(\lambda) = \frac{D(\lambda)}{D_\infty}, \tag{9}$$

where D_∞ stands for $D(\lambda \to \infty)$. Therefore, we obtain $E(\lambda) \to 1$ as $|\lambda| \to \infty$.

The unstable eigenvalues can then be found using the Evans function. Since the Evans function is analytic, the existence of unstable eigenvalues may be investigated by calculating it on an infinite line parallel and very close to the real axis and applying the argument principle. For ρ and ω inside the region of instability, we found one pair of eigenvalues symmetrically located in the imaginary axis. For fixed ρ, they start at some $\pm bi$ (b real positive) and travel toward the origin as ω increases. Then, they reach the origin when ω attains the stability boundary. Fig. 3 represents the absolute value of the unstable eigenvalues using a grayscale plot. Once the eigenvalues are known, the associated eigenmodes may also be determined.

Note that the stability/instability boundary for photorefractive dark solitons, shown in Fig. 3, is consistent with the results obtained by directly applying the stability criterion, i. e., by using the condition $\partial P/\partial\omega = 0$. Finalizing some controversies about the definiteness of $\partial P/\partial\omega$ at $\omega = 0$ and the possibility of its application as stability criterion of black solitons, Pelinovsky and Kevrekidis [9] have recently demonstrated that both are true. Although this criterion could be used for all dark photorefractive solitons, its application to the black case presents some numerical difficulties, which are associated with the phase integral in definition (4). In order to find the stability boundary for black solitons we followed an alternative approach suggested by di Menza and Gallo [8]. We begin by assuming now that Δ and Δ^* are of the form $(\Delta+\Delta^*)(z,\eta) = U(\eta)\mathrm{e}^{i\lambda z} + U^*(\eta)\mathrm{e}^{-i\lambda^* z}$ and $(\Delta-\Delta^*)(z,\eta) = V(\eta)\mathrm{e}^{i\lambda z} - V^*(\eta)\mathrm{e}^{-i\lambda^* z}$ and obtain the following eigenvalue problem for the black solitons ($\omega = 0$)

$$L_0 V = 2\lambda U, \qquad L_1 U = 2\lambda V$$

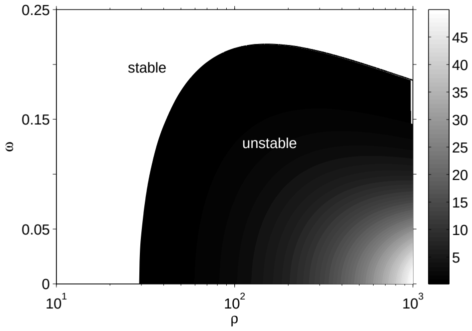


Figure 3: Stability and instability regions and absolute value of the unstable eigenvalue in the parameter space (ρ, ω).

where L_0 and L_1 are Sturm-Liouville operators given by:

$$L_0 = \partial_{\eta\eta} - 2 + \frac{2(1+\rho)}{1+\rho y^2},$$

$$L_1 = \partial_{\eta\eta} - 2 + \frac{2(1+\rho)}{1+\rho y^2} - \frac{4\rho y^2(1+\rho)}{(1+\rho y^2)^2}.$$

Then following a procedure identical to the one used by Vakhitov and Kolokolov [12], we define the function

$$g(\xi) = \left\langle y', (L_0 - \xi)^{-1} y' \right\rangle \tag{10}$$

The black solitons are unstable if $g(0) > 0$ and stable otherwise. In fact, it was demonstrated that $g(0)$ is equal to $-\frac{1}{2}\partial P/\partial\omega(0)$ [9]. Nevertheless, the numerical calculation of $g(0)$ is preferable to the numerical calculation of P close to $\omega = 0$.

Hence, to determine the sign of $g(0)$, we numerically find $\psi(\eta)$ such that $\psi(\eta) = L_0^{-1} y'$. This is done by solving

$$\psi'' - 2\psi + (1+\rho)\frac{2\psi}{1+\rho y^2} = y'$$

with ψ decaying as $\eta \to \pm\infty$. Then, we determine the sign of $g(0)$ as

$$g(0) = \int_{-\infty}^{\infty} y'(\eta)\psi(\eta)d\eta.$$

Following this procedure we determine that y_{black} is stable for $\rho \leq \rho_c$ where ρ_c is approximately 29.3 and unstable otherwise. These results are in complete agreement with the application of the stability condition $\partial P/\partial\omega = 0$ in the limit of very small ω.

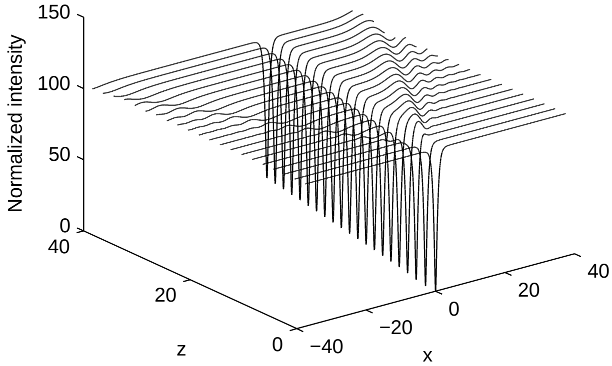


Figure 4: Evolution of the unstable soliton with $\rho = 100$ and $m = 0.001$ ($\omega = 0.0845$).

4 Numerical simulations

The evolution equation (1) was numerically integrated using a beam propagation method, with the solitary waves as the input beams. The numerical results obtained are in good agreement with our stability results. In effect, dark solitons evolve in a stable way for parameter values inside the stability region and destabilize otherwise, as it is shown in Fig. 4 for the soliton associated with $\rho = 100$ and $m = 0.001$. Moreover, for relatively small z, the perturbation of unstable solitons grows exponentially with propagation distance, with a growth rate that coincides with the absolute value of the unstable eigenvalue. This is illustrated in Fig. 5, which depicts the evolution of the maximum of the perturbation absolute value, for this soliton, as well as the expected growth rate associated with its unstable eigenvalue (approximately $-2.03i$). On the other hand, our numerical simulations also indicate that the growing perturbations of unstable solitons are quite similar to the associated eigenfunctions. This is clearly indicated in Fig. 6, which represents the unstable eigenfunction associated with the previously considered soliton and the perturbation obtained numerically for different values of z. For comparison purposes, the absolute values on this figure are normalized to their maxima. Furthermore, is is also interesting to mention that our simulations suggest that the unstable solitons evolve to stable ones while radiate part of their energy. This instability induced dynamics will be studied elsewhere.

5 Conclusions

We have determined parameter region for stability of PR screening dark solitons. For small ρ ($\rho < \rho_c \sim 29.3$), all the dark solitons are stable. Note that in the limit of small ρ the model resembles the defocusing NLS for which all the dark solitons are stable. For $\rho > \rho_c \sim 29.3$, there is a critical velocity (dependent

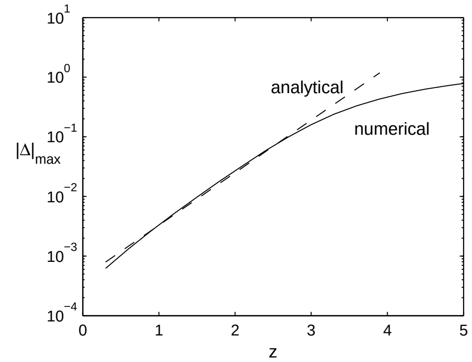


Figure 5: Growth rate of the perturbation for $\rho = 100$ and $m = 0.001$ whose unstable eigenvalue is $\lambda \sim -2.03i$.

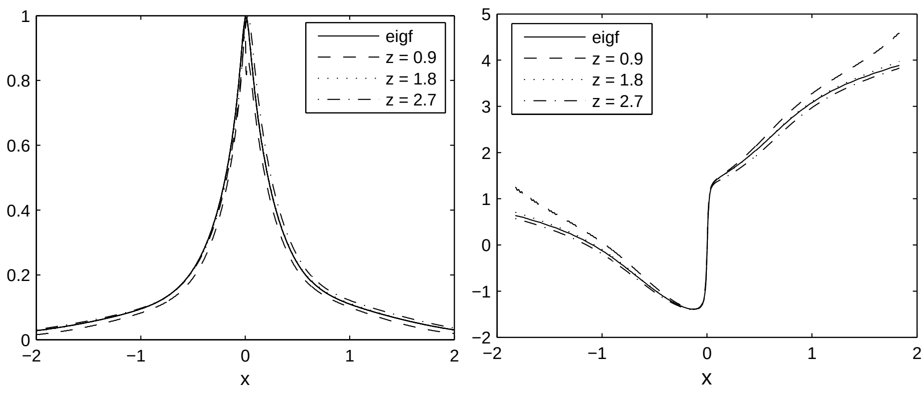


Figure 6: Comparison between the unstable eigenfunction and the perturbation that was obtained using direct integration of eq. (1) for several values of z. Results concern the gray soliton with $\rho = 100$ and $m = 0.001$: (a) absolute value and (b) phase.

on ρ) below which the solitons are unstable and stable solitons otherwise. Using the Evans function method, we have determined unstable eigenvalues and eigenmodes of the linear stability eigenvalue problem. The absolute value of the unstable eigenvalue (strength of the instability) decreases with ω for fixed ρ. Growth rates and eigenmodes agree reasonably with the initial instability evolution as observed by direct simulation of the full equation.

Acknowledgments

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