ELECTROTECNIA TEÓRICA

Transparências das aulas teóricas

Maria Inês Barbosa de Carvalho

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ELECTROTECNIA TEÓRICA

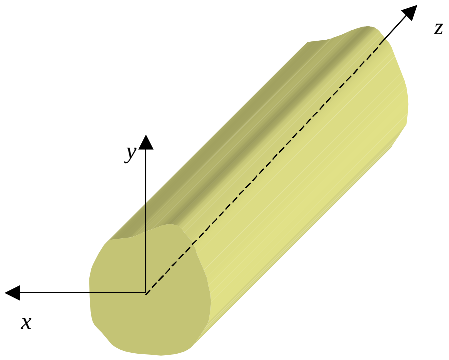
- Ondas electromagnéticas
- Linhas de transmissão
- Guias de onda cilíndricos
    - Guias metálicos
        - Placas paralelas
        - Rectangulares
        - Circulares
    - Guias dieléctricos
        - Planares
        - Fibras Ópticas

GUIAS DE ONDA CILÍNDRICOS

Guias de onda cilíndricos

- Formalismo teórico
  - Ondas guiadas
  - Método para obtenção de $\vec{E}$ e $\vec{H}$
  - Tipo de ondas
  - Frequência de corte
  - Impedância de onda
  - Potência média propagada
  - Energia média armazenada
  - Velocidade de transporte de energia
  - Condições fronteira

Guias de onda cilíndricos



- secção transversal não varia com distância longitudinal
- guias preenchidos com material (e , m) sem perdas
- podem estar limitados por condutor perfeito ($s = \infty$)
- comprimento infinito ⇨ propagação segundo +*z*

⇩

$$\vec{E}(x,y,z) = \vec{E}^{\,0}(x,y)e^{-g\,z}$$

$$\vec{H}(x,y,z) = \vec{H}^{\,0}(x,y)e^{-g\,z}$$

$$\nabla^2\vec{E} + w^2 me\,\vec{E} = 0$$

$$\nabla^2\vec{H} + w^2 me\,\vec{H} = 0$$

$$\nabla^2_{xy}\vec{E}^{\,0} + h^2\vec{E}^{\,0} = 0$$

$$\nabla^2_{xy}\vec{H}^{\,0} + h^2\vec{H}^{\,0} = 0$$

$$\nabla_{xy}^2 \vec{E}^0 + h^2 \vec{E}^0 = 0$$

$$\nabla_{xy}^2 \vec{H}^0 + h^2 \vec{H}^0 = 0$$

$$\Updownarrow$$

$$\frac{\partial^2 E_x^0}{\partial x^2} + \frac{\partial^2 E_x^0}{\partial y^2} + h^2 E_x^0 = 0$$

$$\frac{\partial^2 E_y^0}{\partial x^2} + \frac{\partial^2 E_y^0}{\partial y^2} + h^2 E_y^0 = 0$$

$$\frac{\partial^2 E_z^0}{\partial x^2} + \frac{\partial^2 E_z^0}{\partial y^2} + h^2 E_z^0 = 0$$

$$\frac{\partial^2 H_x^0}{\partial x^2} + \frac{\partial^2 H_x^0}{\partial y^2} + h^2 H_x^0 = 0$$

$$\frac{\partial^2 H_y^0}{\partial x^2} + \frac{\partial^2 H_y^0}{\partial y^2} + h^2 H_y^0 = 0$$

$$\frac{\partial^2 H_z^0}{\partial x^2} + \frac{\partial^2 H_z^0}{\partial y^2} + h^2 H_z^0 = 0$$

$$\nabla \times \vec{E} = -jwm\, \vec{H}$$
$$\nabla \times \vec{H} = jwe\, \vec{E}$$

$$\frac{\partial E_z^0}{\partial y} + gE_y^0 = -jwmH_x^0$$
$$-\frac{\partial E_z^0}{\partial x} - gE_x^0 = -jwmH_y^0$$
$$\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = -jwmH_z^0$$

e

$$\frac{\partial H_z^0}{\partial y} + gH_y^0 = jweE_x^0$$
$$-\frac{\partial H_z^0}{\partial x} - gH_x^0 = jweE_y^0$$
$$\frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y} = jweE_z^0$$

$$H_x^0 = -\frac{1}{h^2}\left(g\,\frac{\partial H_z^0}{\partial x} - jwe\,\frac{\partial E_z^0}{\partial y} \right)$$
$$H_y^0 = -\frac{1}{h^2}\left(g\,\frac{\partial H_z^0}{\partial y} + jwe\,\frac{\partial E_z^0}{\partial x} \right)$$
$$E_x^0 = -\frac{1}{h^2}\left(g\,\frac{\partial E_z^0}{\partial x} + jwm\,\frac{\partial H_z^0}{\partial y} \right)$$
$$E_y^0 = -\frac{1}{h^2}\left(g\,\frac{\partial E_z^0}{\partial y} - jwm\,\frac{\partial H_z^0}{\partial x} \right)$$

DETERMINAÇÃO DE $\vec{E}$ E $\vec{H}$

1. Resolver

$$\nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$
$$\nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

$$h^2 = g^2 + w^2 me$$

2. Calcular

$$H_x^0 = -\frac{1}{h^2}\left(g\frac{\partial H_z^0}{\partial x} - jwe\frac{\partial E_z^0}{\partial y}\right)$$
$$H_y^0 = -\frac{1}{h^2}\left(g\frac{\partial H_z^0}{\partial y} + jwe\frac{\partial E_z^0}{\partial x}\right)$$
$$E_x^0 = -\frac{1}{h^2}\left(g\frac{\partial E_z^0}{\partial x} + jwm\frac{\partial H_z^0}{\partial y}\right)$$
$$E_y^0 = -\frac{1}{h^2}\left(g\frac{\partial E_z^0}{\partial y} - jwm\frac{\partial H_z^0}{\partial x}\right)$$

3. Obter

$$\vec{E}(x,y,z) = \vec{E}^0(x,y)e^{-gz}$$
$$\vec{H}(x,y,z) = \vec{H}^0(x,y)e^{-gz}$$

$$h^2 = g^2 + w^2 me \quad \Rightarrow \quad g = \sqrt{h^2 - w^2 me} = w\sqrt{me}\sqrt{\frac{h^2}{w^2 me} - 1}$$

FREQUÊNCIA DE CORTE

$$f_c = \frac{h}{2p\sqrt{me}}$$

$$g = w\sqrt{me}\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

$$g = w\sqrt{me}\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

- $f < f_c \;\Rightarrow\; g = a \;\Rightarrow$

$$\vec{E}(x,y,z) = \vec{E}^{\,0}(x,y)e^{-a\,z}$$
$$\vec{H}(x,y,z) = \vec{H}^{\,0}(x,y)e^{-a\,z}$$

modo evanescente

- $f > f_c \;\Rightarrow\; g = jb \;\Rightarrow$

$$\vec{E}(x,y,z) = \vec{E}^{\,0}(x,y)e^{-jb\,z}$$
$$\vec{H}(x,y,z) = \vec{H}^{\,0}(x,y)e^{-jb\,z}$$

modo em propagação

Modos em propagação

$$g = jb$$

$$b = b_m \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad , \quad b_m = w\sqrt{me}$$

$$l = \frac{l_m}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \quad , \quad l_m = \frac{2p}{b_m}$$

$$v_f = \frac{v_m}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \quad , \quad v_m = \frac{1}{\sqrt{me}}$$

$$v_g = v_m \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

Impedância de onda

- ondas planas propagando-se segundo +*z* num meio ilimitado

$$\vec{H} = \frac{1}{h}\left(\hat{z}\times\vec{E}\right)$$
$$\vec{E} = -h\left(\hat{z}\times\vec{H}\right)$$

$$h = \sqrt{m/e}$$

- ondas guiadas

ondas TEM ou TM: $\vec{H} = \frac{1}{Z}\left(\hat{z}\times\vec{E}\right)$

ondas TEM ou TE : $\vec{E} = -Z\left(\hat{z}\times\vec{H}\right)$

ou

$$H_x\hat{x} + H_y\hat{y} + H_z\hat{z} = \frac{1}{Z}\left(-E_y\hat{x} + E_x\hat{y}\right) \longrightarrow H_z = 0$$
$$E_x\hat{x} + E_y\hat{y} + E_z\hat{z} = -Z\left(-H_y\hat{x} + H_x\hat{y}\right) \longrightarrow E_z = 0$$

$$\Downarrow$$

$$Z = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

impedância de onda

Potência média propagada

$$P_{med} = \int_A \vec{S}_{med} \cdot d\vec{A}$$

$$d\vec{A} = dA\,\hat{z}$$

$$\vec{S}_{med} = \frac{1}{2} Re\left\{\vec{E} \times \vec{H}^*\right\}$$

$$P_{med} = \frac{1}{2}\int_A Re\left\{E_x H^*_y - E_y H^*_x\right\} dA$$

$$E_x = Z H_y$$

$$E_y = -Z H_x$$

$$P_{med} = \frac{1}{2}\int_A Re\left\{\frac{1}{Z}\right\}\left(\left|E_x\right|^2 + \left|E_y\right|^2\right) dA = \frac{1}{2}\int_A Re\{Z\}\left(\left|H_x\right|^2 + \left|H_y\right|^2\right) dA$$

Energia média armazenada por unidade de comprimento

$$W'_{med} = \int_A \left(w_{e,med} + w_{m,med} \right) dA$$

$$w_{e,med} = \frac{e}{4} \vec{E} \cdot \vec{E}^* = \frac{e}{4} \left(|E_x|^2 + |E_y|^2 + |E_z|^2 \right)$$

$$w_{m,med} = \frac{m}{4} \vec{H} \cdot \vec{H}^* = \frac{m}{4} \left(|H_x|^2 + |H_y|^2 + |H_z|^2 \right).$$

Velocidade de transporte de energia

$$v_{en} = \frac{P_{med}}{W'_{med}}$$

Ondas TEM

- $E_z = H_z = 0$

$$E_x^0 = -\frac{1}{h^2}\left(g\frac{\partial E_z^0}{\partial x} + jwm\frac{\partial H_z^0}{\partial y} \right) \qquad H_x^0 = -\frac{1}{h^2}\left(g\frac{\partial H_z^0}{\partial x} - jwe\frac{\partial E_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2}\left(g\frac{\partial E_z^0}{\partial y} - jwm\frac{\partial H_z^0}{\partial x} \right) \qquad H_y^0 = -\frac{1}{h^2}\left(g\frac{\partial H_z^0}{\partial y} + jwe\frac{\partial E_z^0}{\partial x} \right)$$

$$\Downarrow$$

$$h^2 = 0 \quad \Rightarrow \quad f_c = \frac{h}{2p\sqrt{me}} = 0$$

$$\Downarrow$$

$$g = jb$$
$$b = b_m$$
$$l = l_m$$
$$v_f = v_g = v_m$$

$$gE_y^0 = -jwmH_x^0 \qquad gH_y^0 = jweE_x^0$$

$$-gE_x^0 = -jwmH_y^0 \qquad -gH_x^0 = jweE_y^0$$

$$\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = 0 \qquad \frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y} = 0$$

$$\Downarrow$$

$$Z_{TEM} = \frac{jwm}{g} = \frac{g}{jwe} = \sqrt{\frac{m}{e}} = h$$

Ondas TM

- $E_z \neq 0$ e $H_z = 0$

$$\nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$H_x^0 = \frac{jwe}{h^2}\frac{\partial E_z^0}{\partial y}$$

$$H_y^0 = -\frac{jwe}{h^2}\frac{\partial E_z^0}{\partial x}$$

$$E_x^0 = -\frac{g}{h^2}\frac{\partial E_z^0}{\partial x}$$

$$E_y^0 = -\frac{g}{h^2}\frac{\partial E_z^0}{\partial y}$$

$$Z_{TM} = \frac{g}{jwe} = \frac{w\sqrt{me}\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}}{jwe} = -jh\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

- modos evanescentes: Z_{TM} imaginária $\Rightarrow$ não há propagação de energia

- modos em propagação: $Z_{TM} = h\sqrt{1-(f_c/f)^2}$ real e inferior a h

Ondas TE

- $E_z = 0$ e $H_z \neq 0$

$$\nabla^2_{xy} H^0_z + h^2 H^0_z = 0$$

$$H^0_x = -\frac{g}{h^2}\frac{\partial H^0_z}{\partial x}$$

$$H^0_y = -\frac{g}{h^2}\frac{\partial H^0_z}{\partial y}$$

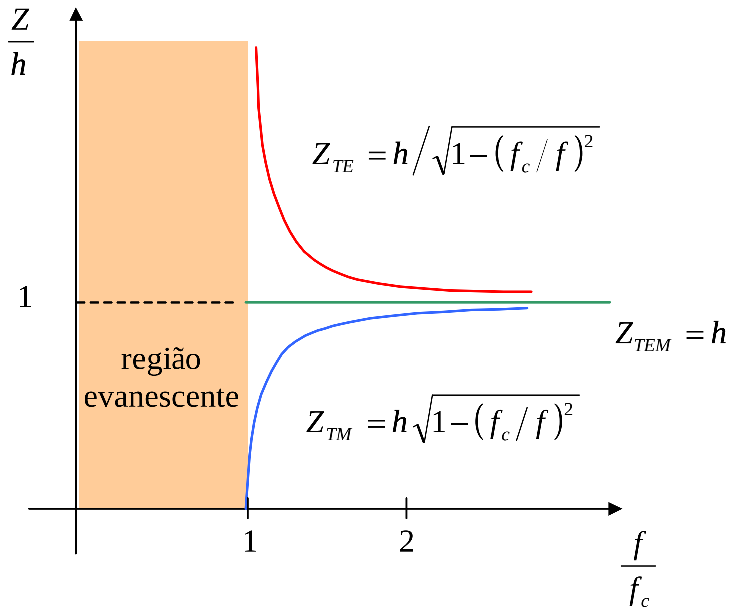
$$E^0_x = -\frac{jwm}{h^2}\frac{\partial H^0_z}{\partial y}$$

$$E^0_y = \frac{jwm}{h^2}\frac{\partial H^0_z}{\partial x}$$

$$Z_{TE} = \frac{jwm}{g} = \frac{jh}{\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}}$$

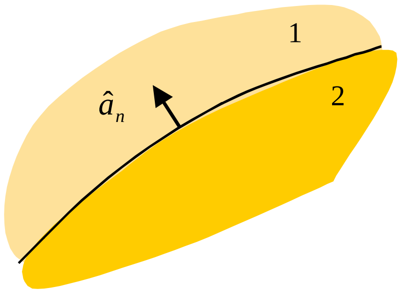
- modos evanescentes: Z_{TE} imaginária $\Longrightarrow$ não há propagação de energia

- modos em propagação: $Z_{TE} = h\Big/\sqrt{1-\left(f_c/f\right)^2}$ real e maior do que h



Z/h
1
região evanescente
$Z_{TE} = h/\sqrt{1-(f_c/f)^2}$
$Z_{TEM} = h$
$Z_{TM} = h\sqrt{1-(f_c/f)^2}$
1
2
f/f_c

Condições fronteira



$$\hat{a}_n \times \left(\vec{E}_1 - \vec{E}_2\right) = 0 \qquad\qquad \hat{a}_n \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0$$
$$\hat{a}_n \cdot \left(\vec{D}_1 - \vec{D}_2\right) = r_S \qquad\qquad \hat{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_S$$

$$\Updownarrow$$

$E_{\tan}$ contínuo

B_{norm} contínuo

D_{norm} contínuo se $r_S = 0$

$H_{\tan}$ contínuo se $\vec{J}_S = 0$

NOTAS:

- $\vec{D} = e\,\vec{E} \qquad\qquad \vec{B} = m\,\vec{H}$
- $\vec{E}_{cond} = \vec{D}_{cond} = \vec{B}_{cond} = \vec{H}_{cond} = 0$

GUIAS DE PLACAS PARALELAS

Guias metálicos

- Guias limitados por condutores perfeitos

Condições fronteira

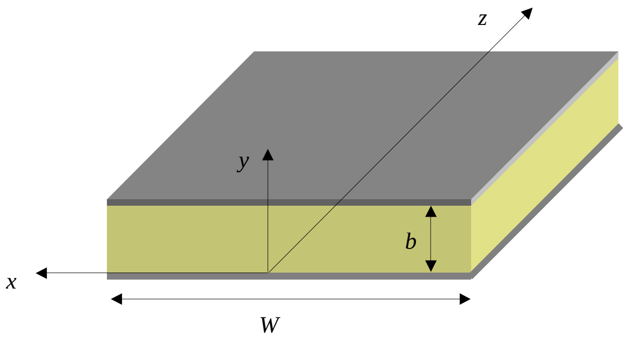
$E_{\tan}$ e B_{norm} contínuos

$$E_{condutor} = B_{condutor} = 0$$

$$\Downarrow$$

$E_{\tan} = H_{normal} = 0$ junto aos condutores

Guias de placas paralelas



- $W >> b \quad \Longrightarrow \quad \dfrac{\partial}{\partial x} = 0$

Condições fronteira

$$E_{\tan} = H_{normal} = 0 \quad \text{em} \quad y = 0 \text{ e } y = b$$

$$\Downarrow$$

$$E_x^0 = E_z^0 = H_y^0 = 0 \quad \text{em} \quad y = 0 \text{ e } y = b$$

Ondas TEM

$$E_z^0 = H_z^0 = 0$$

$$\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = 0 \qquad \frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y} = 0$$

$$\partial/\partial x = 0$$

$$\frac{dE_x^0}{dy} = \frac{dH_x^0}{dy} = 0$$

$$\Downarrow$$

E_x^0 e H_x^0 são constantes

$$E_x^0(0) = E_x^0(b) = 0$$

$$E_x^0 = Z_{TEM} H_y^0$$

$$E_y^0 = -Z_{TEM} H_x^0$$

$$Z_{TEM} = h$$

$$E_x^0 = 0$$

$$H_y^0 = 0$$

$E_y^0 =$ constante

$$\vec{E}^0 = E_0 \hat{y}$$

$$\vec{H}^0 = -\frac{E_0}{h} \hat{x}$$

Ondas TM

$$H_z^0 = 0$$

$$\frac{d^2 E_z^0}{dy^2} + h^2 E_z^0 = 0$$

$$h \neq 0$$

$$E_z^0(y) = A\sin(hy) + B\cos(hy)$$

$$E_z^0(0) = 0$$

$$E_z^0(b) = 0$$

$$B = 0$$

$$A\sin(hb) = 0 \quad \Leftrightarrow \quad hb = np \quad , \qquad n = 1,\ 2, \cdots$$

$$h = \frac{np}{b} \ , \qquad n = 1,\ 2, \cdots$$

$$E_z^0 = A_n \sin\left(\frac{np\ y}{b}\right)$$

$$h = \frac{np}{b}\ , \qquad n = 1,\ 2, \cdots$$

$$E_z^0 = A_n \sin\left(\frac{np\ y}{b}\right)$$

$$H_x^0 = \frac{jwe}{h^2}\frac{\partial E_z^0}{\partial y}$$

$$E_x^0 = -\frac{g}{h^2}\frac{\partial E_z^0}{\partial x}$$

$$H_y^0 = -\frac{jwe}{h^2}\frac{\partial E_z^0}{\partial x}$$

$$E_y^0 = -\frac{g}{h^2}\frac{\partial E_z^0}{\partial y}$$

MODO TM$_n$

$$E_z^0 = A_n \sin\left(\frac{np\ y}{b}\right)$$

$$H_x^0 = \frac{jweb}{np} A_n \cos\left(\frac{np\ y}{b}\right)$$

$$E_y^0 = -\frac{g\ b}{np} A_n \cos\left(\frac{np\ y}{b}\right)$$

$$H_y^0 = E_x^0 = 0$$

$$g = j\sqrt{w^2 me - \left(\frac{np}{b}\right)^2}$$

Ondas TE

$$E_z^0 = 0$$

$$\frac{d^2 H_z^0}{dy^2} + h^2 H_z^0 = 0$$

$$h \neq 0$$

$$H_z^0(y) = A\sin(hy) + B\cos(hy)$$

$$H_x^0 = -\frac{g}{h^2}\frac{\partial H_z^0}{\partial x}$$

$$E_x^0 = -\frac{jwm}{h^2}\frac{\partial H_z^0}{\partial y}$$

$$H_y^0 = -\frac{g}{h^2}\frac{\partial H_z^0}{\partial y}$$

$$E_y^0 = \frac{jwm}{h^2}\frac{\partial H_z^0}{\partial x}$$

$$H_y^0 = -\frac{g}{h}\left[A\cos(hy) - B\sin(hy)\right]$$

$$E_x^0 = -\frac{jwm}{h}\left[A\cos(hy) - B\sin(hy)\right]$$

$$E_y^0 = H_x^0 = 0$$

$$H_y^0 = -\frac{g}{h}\left[A\cos(hy) - B\sin(hy)\right]$$

$$E_x^0 = -\frac{jwm}{h}\left[A\cos(hy) - B\sin(hy)\right]$$

$$E_y^0 = H_x^0 = 0$$

$$E_x^0(0) = E_x^0(b) = 0$$

$$H_y^0(0) = H_y^0(b) = 0$$

$$A = 0$$

$$\sin(hb) = 0 \;\Leftrightarrow\; h = \frac{np}{b}\,, \quad n = 1,\, 2,\, 3,\, \cdots$$

MODO TE$_n$

$$H_z^0 = B_n \cos\left(\frac{np\,y}{b}\right)$$

$$H_y^0 = \frac{g\,b}{np} B_n \sin\left(\frac{np\,y}{b}\right)$$

$$E_x^0 = j\frac{wm\,b}{np} B_n \sin\left(\frac{np\,y}{b}\right)$$

$$H_x^0 = E_y^0 = 0$$

$$g = j\sqrt{w^2 me - \left(\frac{np}{b}\right)^2}$$

Frequência de corte

$$f_c = \frac{h}{2p\sqrt{me}}$$

$$h_{TEM} = 0$$

$$h_{TM,TE} = \frac{np}{b}$$

$$\left(f_c\right)_{TEM} = 0$$

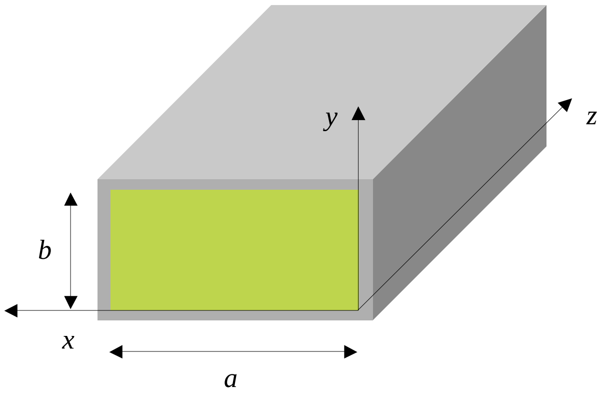
$$\left(f_c\right)_{TM,TE} = \frac{n}{2b\sqrt{me}}$$

⇩

Modo dominante num guia de placas paralelas é o modo TEM

GUIAS RECTANGULARES

Guias rectangulares



Condições fronteira

$E_{\tan}$ e B_{norm} contínuos

$E_{condutor} = B_{condutor} = 0$

$\Longrightarrow$ $E_{\tan} = H_{normal} = 0$ junto ao condutor

$\Downarrow$

$$E_x^0 = E_z^0 = H_y^0 = 0 \quad \text{em} \quad y = 0 \quad \text{ou} \quad y = b$$
$$E_y^0 = E_z^0 = H_x^0 = 0 \quad \text{em} \quad x = 0 \quad \text{ou} \quad x = a$$

Método da separação das variáveis

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} + h^2 y = 0$$

$$y(x,y) = X(x)Y(y)$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + h^2 = 0$$

$$-k_x^2 \qquad -k_y^2$$

$$\frac{\partial}{\partial y} = 0 \qquad \frac{\partial}{\partial x} = 0$$

$$\Downarrow$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k_x^2$$

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k_y^2$$

$$h^2 = k_x^2 + k_y^2$$

$$\frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0 \longrightarrow X(x) = A\sin(k_x x) + B\cos(k_x x)$$

$$\frac{d^2Y(y)}{dy^2} + k_y^2 Y(y) = 0 \longrightarrow Y(y) = C\sin(k_y y) + D\cos(k_y y)$$

$$\downarrow$$

$$\boxed{y\,(x, y) = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y) + D\cos(k_y y)]}$$

$$\boxed{h^2 = k_x^2 + k_y^2}$$

Ondas TM

$$H_z^0 = 0 \qquad \nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$E_z^0 = \underbrace{[A\sin(k_x x) + B\cos(k_x x)]}_{X(x)}\ \underbrace{[C\sin(k_y y) + D\cos(k_y y)]}_{Y(y)}$$

$$E_z^0(0, y) = E_z^0(a, y) = 0 \quad \rightarrow \quad B = 0 \quad \text{e} \quad A\sin(k_x a) = 0$$

$$E_z^0(x, 0) = E_z^0(x, b) = 0 \quad \rightarrow \quad D = 0 \quad \text{e} \quad C\sin(k_y b) = 0$$

$$k_x = \frac{mp}{a}\ ,\ m \ \text{inteiro} \quad \text{e} \quad X(x) = A\sin\left(\frac{mpx}{a}\right)$$

$$k_y = \frac{np}{b}\ ,\ n \ \text{inteiro} \quad \text{e} \quad Y(y) = C\sin\left(\frac{npy}{b}\right)$$

modo TM_{mn}

$$E_z^0 = E_{0,mn}\sin\left(\frac{mpx}{a}\right)\sin\left(\frac{npy}{b}\right)$$

$$H_x^0 = \frac{j\omega\varepsilon}{h^2}\frac{n\pi}{b}E_{0,mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$H_y^0 = -\frac{j\omega\varepsilon}{h^2}\frac{m\pi}{a}E_{0,mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_x^0 = -\frac{\gamma}{h^2}\frac{m\pi}{a}E_{0,mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_y^0 = -\frac{\gamma}{h^2}\frac{n\pi}{b}E_{0,mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$h^2 = \left(\frac{mp}{a}\right)^2 + \left(\frac{np}{b}\right)^2$$

Importante:

- $h \neq 0 \;\Rightarrow\; m \neq 0$ ou $n \neq 0$
- n=0 ou m =0 $\;\Rightarrow\;$ $\vec{E} = \vec{H} = 0$

$\Longrightarrow$ $n \geq 1$ e $m \geq 1$

Ondas TE

$E_z^0 = 0$

$$\nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

$$H_z^0 = \underbrace{[A\sin(k_x x) + B\cos(k_x x)]}_{X(x)} \underbrace{[C\sin(k_y y) + D\cos(k_y y)]}_{Y(y)}$$

$$E_x^0 = -j\frac{wm}{h^2} X(x)Y'(y)$$

$$E_y^0 = j\frac{wm}{h^2} X'(x)Y(y)$$

onde

$$X'(x) = k_x[A\cos(k_x x) - B\sin(k_x x)]$$
$$Y'(y) = k_y[C\cos(k_y y) - D\sin(k_y y)]$$

$E_y^0(0, y) = E_y^0(a, y) = 0$ → $A = 0$ e $k_x = \frac{mp}{a}$, m inteiro

$E_x^0(x,0) = E_x^0(x,b) = 0$ → $C = 0$ e $k_y = \frac{np}{b}$, n inteiro

modo TE$_{mn}$

$$H_z^0 = H_{0,mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$H_x^0 = \frac{\gamma}{h^2}\frac{m\pi}{a}H_{0,mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$H_y^0 = \frac{\gamma}{h^2}\frac{n\pi}{b}H_{0,mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_x^0 = \frac{j\omega\mu}{h^2}\frac{n\pi}{b}H_{0,mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_y^0 = -\frac{j\omega\mu}{h^2}\frac{m\pi}{a}H_{0,mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$h^2 = \left(\frac{mp}{a}\right)^2 + \left(\frac{np}{b}\right)^2$$

- $h \neq 0 \;\Rightarrow\; m \neq 0$ ou $n \neq 0$

Frequência de corte

$$h = \sqrt{\left(\frac{mp}{a}\right)^2 + \left(\frac{np}{b}\right)^2}$$

$$f_c = \frac{h}{2p\sqrt{me}}$$

$$f_c = \frac{1}{2\sqrt{me}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

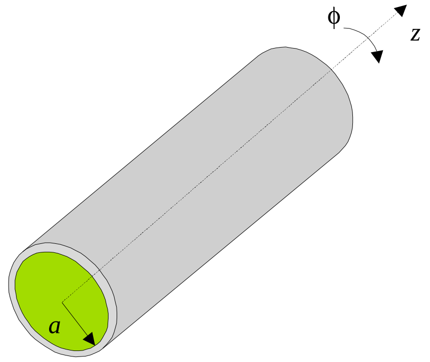
- modos TM: $n \geq 1$ e $m \geq 1$ $\Longrightarrow$ modo TM com frequência de corte mais baixa é modo TM_{11}

- modos TE: $m \neq 0$ ou $n \neq 0$ $\Longrightarrow$ se $a > b$, modo TE com a frequência de corte mais baixa é o modo TE_{10}

- $(f_c)_{TE_{10}} < (f_c)_{TM_{11}}$ $\Longrightarrow$ **modo TE_{10} é modo dominante nos guias rectangulares**

GUIAS CIRCULARES

Guias circulares



coordenadas cilíndricas $\Longrightarrow$

$$\vec{E} = \left(E_r^0 \hat{r} + E_f^0 \hat{f} + E_z^0 \hat{z}\right) e^{-g\,z}$$
$$\vec{H} = \left(H_r^0 \hat{r} + H_f^0 \hat{f} + H_z^0 \hat{z}\right) e^{-g\,z}$$

Condições fronteira

$E_{\tan} = H_{normal} = 0$ junto ao condutor

$\Downarrow$

$$E_f^0 = E_z^0 = H_r^0 = 0 \quad \text{em} \quad r = a$$

DETERMINAÇÃO DE $\vec{E}$ E $\vec{H}$

1. Resolver as equações

$$\nabla_{rf}^2 E_z^0 + h^2 E_z^0 = 0$$
$$\nabla_{rf}^2 H_z^0 + h^2 H_z^0 = 0$$

$$h^2 = g^2 + w^2 me$$
$$\nabla_{rf}^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial f^2}$$

2. Calcular

$$H_r^0 = -\frac{1}{h^2}\left(g\frac{\partial H_z^0}{\partial r} - \frac{jwe}{r}\frac{\partial E_z^0}{\partial f} \right)$$
$$H_f^0 = -\frac{1}{h^2}\left(\frac{g}{r}\frac{\partial H_z^0}{\partial f} + jwe\frac{\partial E_z^0}{\partial r} \right)$$
$$E_r^0 = -\frac{1}{h^2}\left(g\frac{\partial E_z^0}{\partial r} + \frac{jwm}{r}\frac{\partial H_z^0}{\partial f} \right)$$
$$E_f^0 = -\frac{1}{h^2}\left(\frac{g}{r}\frac{\partial E_z^0}{\partial f} - jwm\frac{\partial H_z^0}{\partial r} \right)$$

Equação de onda em coordenadas cilíndricas

$$\nabla^2_{rf} y + h^2 y = 0$$

$$\Updownarrow$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial y}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 y}{\partial f^2}+h^2 y = 0$$

$$y(r,f) = R(r)\Phi(f) \quad \longrightarrow$$

$$\downarrow$$

$$k_f^2 \longleftarrow \underbrace{\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{r}{R(r)}\frac{dR(r)}{dr}+h^2r^2}_{\frac{\partial}{\partial f}=0} = \underbrace{-\frac{1}{\Phi(f)}\frac{d^2\Phi(f)}{df^2}}_{\frac{\partial}{\partial r}=0} \longrightarrow k_f^2$$

$$\frac{d^2\Phi(f)}{df^2} + k_f^2\Phi(f) = 0$$

$$\Phi(f) = A\sin(k_f f) + B\cos(k_f f)$$

$$\Phi(f + 2p) = \Phi(f)$$

$$\Downarrow$$

$$\sin(k_f f + k_f 2p) = \sin(k_f f)$$
$$\cos(k_f f + k_f 2p) = \cos(k_f f)$$

$$\Downarrow$$

$$k_f = n, \; n \text{ inteiro}$$

$$\Phi(f) = A\sin(nf) + B\cos(nf)$$

$$\boxed{\Phi(f) = B\cos(nf)}$$

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{r}{R(r)}\frac{dR(r)}{dr}+h^2r^2=n^2$$

$\Downarrow$

$$r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}+\left(h^2r^2-n^2\right)R(r)=0$$

equação diferencial de Bessel

$\downarrow$

$$R(r)=C\,J_n(hr)+\;D\,N_n(hr)$$

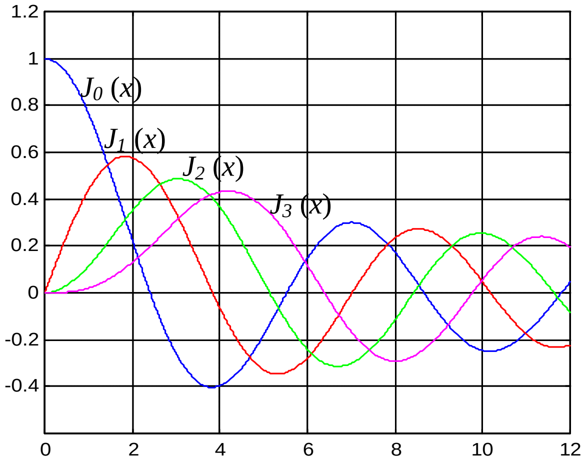
funções de Bessel de 1ª espécie

funções de Bessel de 2ª espécie

Funções de Bessel de 1ª espécie

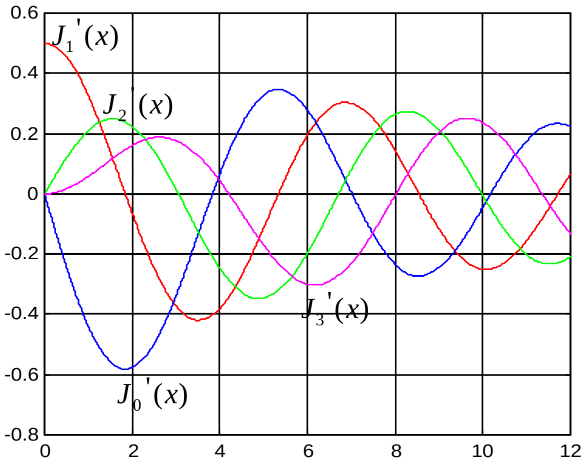
para *n* inteiro

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \, x^{n+2m}}{m!(m+n)!2^{n+2m}}$$



zero	$J_o(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$
1	2.4048	3.8317	5.1336	6.3802
2	5.5201	7.0156	8.4172	9.7610
3	8.6537	10.1735	11.6198	13.0152
4	11.7915	13.3237	14.7960	16.2235
5	14.9309	16.4706	17.9598	19.4094

$$J'_n(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

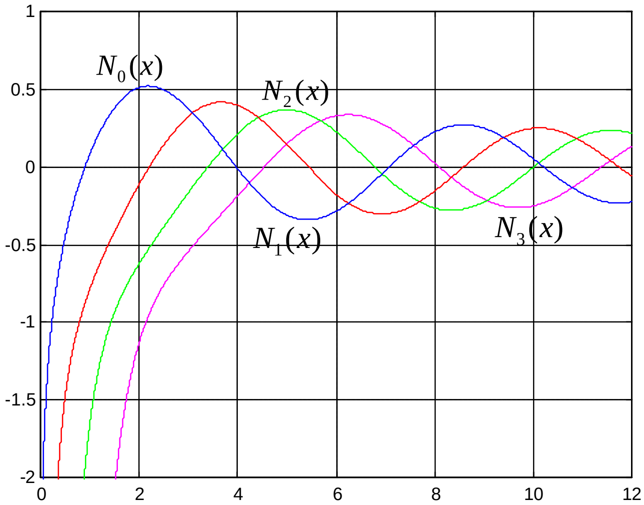


zero	$J'_o(x)$	$J'_1(x)$	$J'_2(x)$	$J'_3(x)$
1	3.8317	1.8412	3.0542	4.2012
2	7.0156	5.3314	6.7061	8.0152
3	10.1735	8.5363	9.9695	11.3459
4	13.3237	11.7060	13.1704	14.5858
5	16.4706	14.8636	16.3475	17.7887

Funções de Bessel de 2ª espécie

para *n* inteiro

$$N_n(x) = \lim_{p \to n} \frac{J_p(x)\cos(pp) - J_{-p}(x)}{\sin(pp)}$$



$$R(r) = C\,J_n(hr) + D\,N_n(hr)$$

toma valores infinitos em *r*=0

⇩

D=0

$$R(r) = CJ_n(hr)$$

⇩

$$\nabla^2_{rf} y + h^2 y = 0$$

$$y\,(r,f) = R(r)\Phi(f) = C_n J_n(hr)\cos(nf)$$

Ondas TM

$$H_z^0 = 0$$

$$\nabla_{rf}^2 E_z^0 + h^2 E_z^0 = 0$$

$$E_z^0 = C_n J_n(hr)\cos(nf)$$

$$H_r^0 = \frac{jwe}{h^2 r}\frac{\partial E_z^0}{\partial f}$$

$$H_f^0 = -\frac{jwe}{h^2}\frac{\partial E_z^0}{\partial r}$$

$$E_r^0 = -\frac{g}{h^2}\frac{\partial E_z^0}{\partial r}$$

$$E_f^0 = -\frac{g}{h^2 r}\frac{\partial E_z^0}{\partial f}$$

$$H_r^0 = -\frac{jwen}{h^2 r} C_n J_n(hr)\sin(nf)$$

$$H_f^0 = -\frac{jwe}{h} C_n J'_n(hr)\cos(nf)$$

$$E_r^0 = -\frac{g}{h} C_n J'_n(hr)\cos(nf)$$

$$E_f^0 = \frac{g\,n}{h^2 r} C_n J_n(hr)\sin(nf)$$

$$E_z^0(r = a, f) = 0 \quad \Longrightarrow \quad J_n(ha) = 0$$

$$J_n(ha) = 0$$

$$\downarrow$$

$$n = 0 \;\rightarrow\; h = \frac{2.4048}{a}\;;\; h = \frac{5.5201}{a}\;;\; h = \frac{8.6537}{a}\;;\;\cdots$$

$$n = 1 \;\rightarrow\; h = \frac{3.8317}{a}\;;\; h = \frac{7.0156}{a}\;;\; h = \frac{10.1735}{a}\;;\;\cdots$$

$$\vdots$$

modo TM$_{np}$ $\Longrightarrow$

$$h = h_{TM_{np}} = \frac{p - \text{ésimo zero de } J_n}{a}$$

Frequência de corte

$$(f_c)_{TM_{np}} = \frac{h_{TM_{np}}}{2p\sqrt{me}} = \frac{p - \text{ésimo zero de } J_n}{2p\; a\sqrt{me}}$$

menor zero de J_n é 2.4048 (n=0, p=1) $\Longrightarrow$

modo TM dominante é o modo TM$_{01}$

$$(f_c)_{TM_{01}} = \frac{2.4048}{2p\; a\sqrt{me}}$$

Ondas TE

$$E_z^0 = 0$$

$$\nabla_{rf}^2 H_z^0 + h^2 H_z^0 = 0$$

$$H_z^0 = C_n J_n(hr)\cos(nf)$$

$$H_r^0 = -\frac{g}{h^2}\frac{\partial H_z^0}{\partial r}$$

$$H_f^0 = -\frac{g}{h^2 r}\frac{\partial H_z^0}{\partial f}$$

$$E_r^0 = -\frac{jwm}{h^2 r}\frac{\partial H_z^0}{\partial f}$$

$$E_f^0 = \frac{jwm}{h^2}\frac{\partial H_z^0}{\partial r}$$

$$H_r^0 = -\frac{g}{h} C_n J'_n(hr)\cos(nf)$$

$$H_f^0 = \frac{g\,n}{h^2 r} C_n J_n(hr)\sin(nf)$$

$$E_r^0 = \frac{jwm\,n}{h^2 r} C_n J_n(hr)\sin(nf)$$

$$E_f^0 = \frac{jwm}{h} C_n J'_n(hr)\cos(nf)$$

$$E_f^0(r = a, f) = 0 \Longrightarrow J'_n(ha) = 0$$

$$J'_n(ha) = 0$$

$$n = 0 \rightarrow \quad h = \frac{3.8317}{a} \; ; \quad h = \frac{7.0156}{a} \; ; \quad \cdots$$

$$n = 1 \rightarrow \quad h = \frac{1.8412}{a} \; ; \quad h = \frac{5.3314}{a} \; ; \quad \cdots$$

$$\vdots$$

modo TE$_{np}$ $\Longrightarrow$

$$h = h_{TE_{np}} = \frac{p - \text{ésimo zero de } J'_n}{a}$$

Frequência de corte

$$(f_c)_{TE_{np}} = \frac{h_{TE_{np}}}{2p\sqrt{me}} = \frac{p - \text{ésimo zero de } J'_n}{2p\, a\sqrt{me}}$$

menor zero de J'_n é 1.8412 (n=1, p=1) $\Longrightarrow$

modo TE dominante é o modo TE$_{11}$

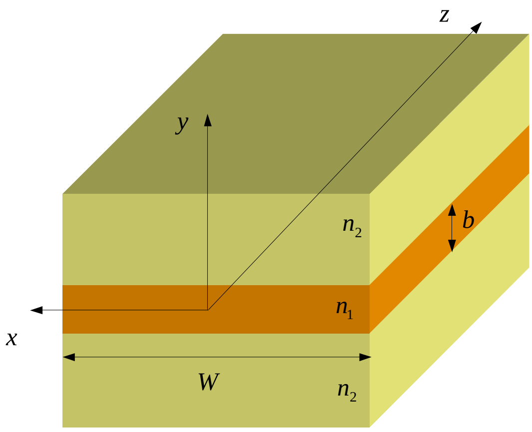
$$(f_c)_{TE_{11}} = \frac{1.8412}{2p\, a\sqrt{me}}$$

$$(f_c)_{TE_{11}} < (f_c)_{TM_{01}} \Longrightarrow$$

modo dominante num guia circular é o modo TE$_{11}$!

GUIAS DIELÉCTRICOS PLANARES

Guias dieléctricos planares



- $W >> b \;\Longrightarrow\; \dfrac{\partial}{\partial x} = 0$

Condições fronteira

$$r_s = 0$$

$$\vec{J}_s = 0 \quad \Longrightarrow \quad \begin{array}{l} E_{\tan} \text{ contínua} \\ B_{normal} \text{ contínua} \\ D_{normal} \text{ contínua} \\ H_{\tan} \text{ contínua} \end{array}$$

Equação de onda em guias dieléctricos planares

$$\frac{d^2 y}{dy^2} + h^2 y = 0 \qquad h^2 = \begin{cases} h_1^2 , & |y| \le \frac{b}{2} \\ h_2^2 , & |y| > \frac{b}{2} \end{cases}$$

- $h^2 > 0 \Rightarrow h$ real $\Rightarrow y = A\sin(hy) + B\cos(hy)$
- $h^2 < 0 \Rightarrow h = jn \Rightarrow y = Ce^{-n y} + De^{+n y}$

$$h_1 \text{ real}$$
$$h_2 = jn$$

$$\begin{cases} h_1^2 = g^2 + \left(\frac{w}{c} n_1\right)^2 \\ n^2 = -g^2 - \left(\frac{w}{c} n_2\right)^2 \end{cases} \Rightarrow n = \sqrt{\left(\frac{w}{c}\right)^2 \left(n_1^2 - n_2^2\right) - h_1^2}$$

- $g = jb \rightarrow b = \sqrt{\left(\frac{w}{c} n_1\right)^2 - h_1^2} = \sqrt{\left(\frac{w}{c} n_2\right)^2 + n^2}$

$$\Downarrow$$

$$\frac{w}{c} n_1 > b > \frac{w}{c} n_2$$

$$n_1 > n_2 \Leftarrow$$

$$y\,(y) = \begin{cases} De^{n\,y}\,, & y < -\dfrac{b}{2} \\ A\sin(h_1 y) + B\cos(h_1 y), & |y| \le \dfrac{b}{2} \\ Ce^{-n\,y}\,, & y > \dfrac{b}{2} \end{cases}$$

$y\,(y)$ contínua

em $y = \pm b/2$

$$y\,(y) = \begin{cases} \left[-A\sin\left(\dfrac{h_1 b}{2}\right) + B\cos\left(\dfrac{h_1 b}{2}\right)\right] e^{n\left(y+\frac{b}{2}\right)}, & y < -\dfrac{b}{2} \\ A\sin(h_1 y) + B\cos(h_1 y), & |y| \le \dfrac{b}{2} \\ \left[A\sin\left(\dfrac{h_1 b}{2}\right) + B\cos\left(\dfrac{h_1 b}{2}\right)\right] e^{-n\left(y-\frac{b}{2}\right)}, & y > \dfrac{b}{2} \end{cases}$$

modos pares

$$y_{\,par} = \begin{cases} B\cos\left(\dfrac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)}, & y < -\dfrac{b}{2} \\ B\cos(h_1 y), & |y| \le \dfrac{b}{2} \\ B\cos\left(\dfrac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)}, & y > \dfrac{b}{2} \end{cases}$$

modos ímpares

$$y_{\,ímpar} = \begin{cases} A\sin\left(\dfrac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)}, & y < -\dfrac{b}{2} \\ A\sin(h_1 y), & |y| \le \dfrac{b}{2} \\ A\sin\left(\dfrac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)}, & y > \dfrac{b}{2} \end{cases}$$

Ondas TM pares

$$H_z^0 = 0 \qquad E_z^0 = \psi_{par}$$

$$H_x^0 = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z^0}{\partial y}$$

$$E_y^0 = -\frac{\gamma}{h^2}\frac{\partial E_z^0}{\partial y}$$

$$E_x^0 = H_y^0 = 0$$

$|y| \le \frac{b}{2}$:

$$E_z^0 = B\cos(h_1 y)$$

$$H_x^0 = -\frac{j\omega\varepsilon_1}{h_1}B\sin(h_1 y)$$

$$E_y^0 = \frac{j\beta}{h_1}B\sin(h_1 y)$$

$y > \frac{b}{2}$:

$$E_z^0 = B\cos\left(\frac{h_1 b}{2}\right)e^{-\eta\left(y-\frac{b}{2}\right)}$$

$$H_x^0 = \frac{j\omega\varepsilon_2}{\eta}B\cos\left(\frac{h_1 b}{2}\right)e^{-\eta\left(y-\frac{b}{2}\right)}$$

$$E_y^0 = -\frac{j\beta}{\eta}B\cos\left(\frac{h_1 b}{2}\right)e^{-\eta\left(y-\frac{b}{2}\right)}$$

$y < -\frac{b}{2}$:

$$E_z^0 = B\cos\left(\frac{h_1 b}{2}\right)e^{\eta\left(y+\frac{b}{2}\right)}$$

$$H_x^0 = -\frac{j\omega\varepsilon_2}{\eta}B\cos\left(\frac{h_1 b}{2}\right)e^{\eta\left(y+\frac{b}{2}\right)}$$

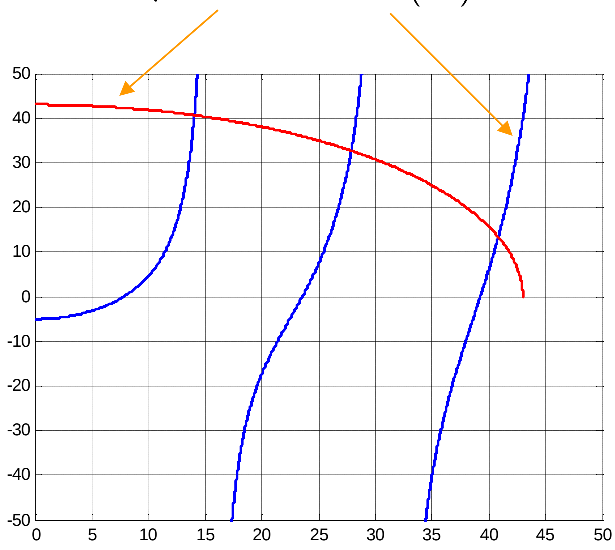
$$E_y^0 = \frac{j\beta}{\eta}B\cos\left(\frac{h_1 b}{2}\right)e^{\eta\left(y+\frac{b}{2}\right)}$$

$$\Rightarrow \quad \eta = -h_1\left(\frac{n_2}{n_1}\right)^2\cot\left(\frac{h_1 b}{2}\right)$$

$$\left(\frac{n_1}{n_2}\right)^2 \sqrt{\left(\frac{w}{c}\right)^2 \left(n_1^2 - n_2^2\right) - h_1^2} = -h_1 \cot\left(\frac{h_1 b}{2}\right)$$

EQUAÇÃO CARACTERÍSTICA

$$\sqrt{A^2 - x^2} = -x \cot(Bx)$$



valores característicos (h_1) em guias dieléctricos planares:

- **número finito** (correspondendo cada valor a um modo que se pode propagar no guia à frequência considerada);
- **dependem da frequência** de operação (valor A^2 é proporcional à frequência de operação w).

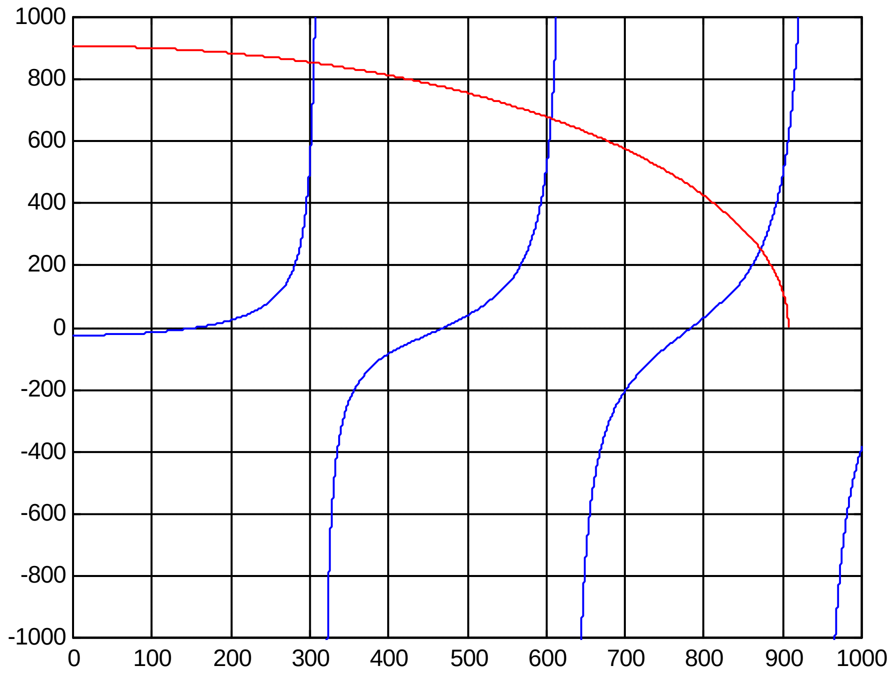
EXEMPLO

$$f = 25\ GHz$$

$$e = 4e_0$$

$$m = m_0$$

$$b = 2\ cm$$



$$h_{1,1} = 305.25$$

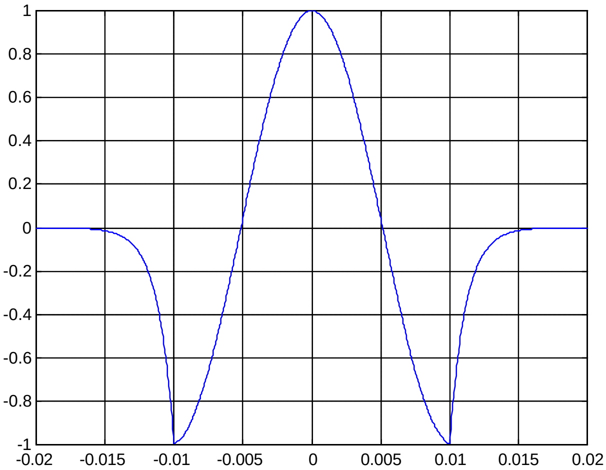
$$h_{1,2} = 606.22$$

$$h_{1,3} = 871.20$$

$TM_{par,1}$

$h_{1,1} = 305.25$

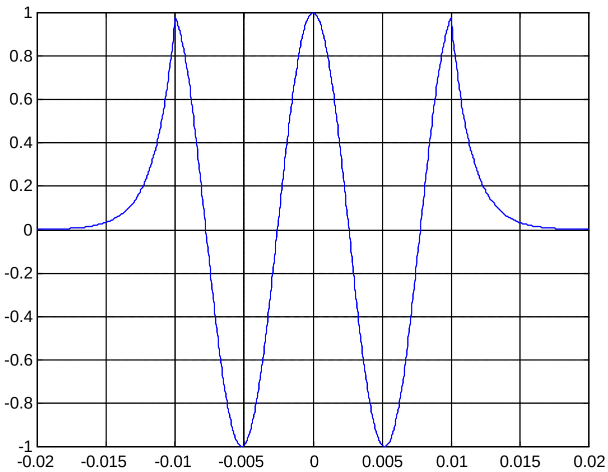
$n_1 = 853.98$



$TM_{par,2}$

$h_{1,2} = 606.22$

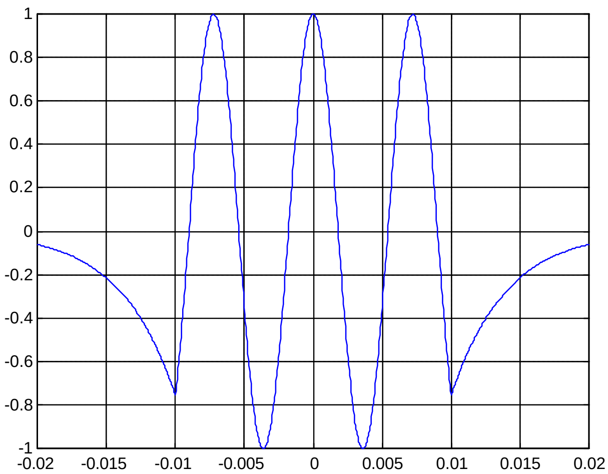
$n_2 = 674.51$



$TM_{par,3}$

$h_{1,3} = 871.20$

$n_3 = 251.98$

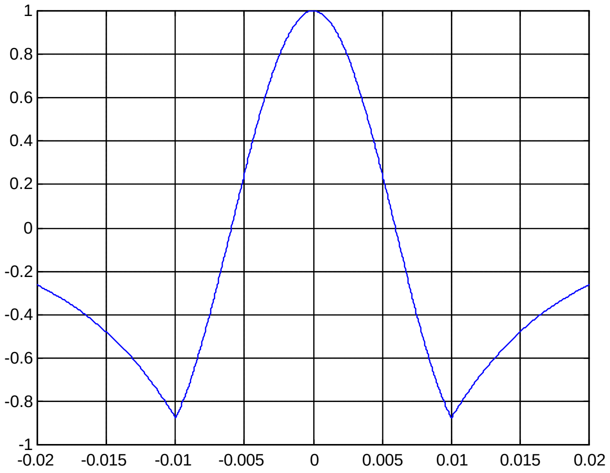


$$TM_{par,1}$$

$f = 8\ GHz$

$h_{d,1} = 264.03$

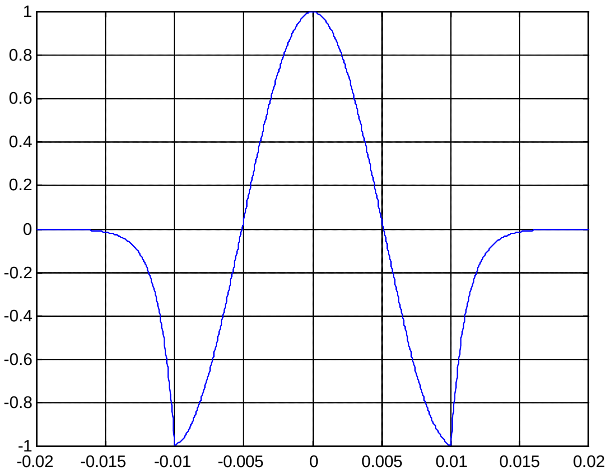
$n = 120.45$



$f = 25\ GHz$

$h_{d,1} = 305.25$

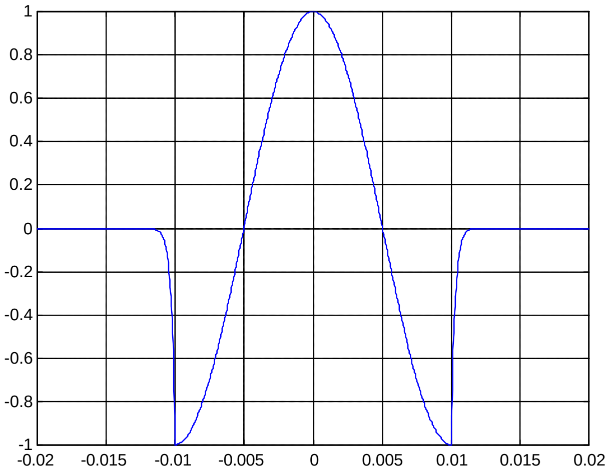
$n = 853.98$



$f = 100\ GHz$

$h_{d,1} = 312$

$n = 3614.16$



Frequência de corte

Condição de corte: $n = 0$

- para o corte:

$$n^2 = -g^2 - \left(\frac{w}{c} n_2\right)^2 = 0 \iff g^2 = -\left(\frac{w}{c} n_2\right)^2$$

$$h_1^2 = g^2 + \left(\frac{w}{c} n_1\right)^2 = \left(\frac{w}{c}\right)^2 \left(n_1^2 - n_2^2\right) \iff h_1 = \frac{w}{c}\sqrt{n_1^2 - n_2^2}$$

$$n = -h_1 \left(\frac{n_2}{n_1}\right)^2 \cot\left(\frac{h_1 b}{2}\right) = 0 \iff -\cot\left(\frac{h_1 b}{2}\right) = 0$$

$$\frac{h_1 b}{2} = \frac{w\, b\sqrt{n_1^2 - n_2^2}}{2c} = \left(n - \frac{1}{2}\right)p \quad , \quad n = 1,\ 2,\ \ldots$$

$$\left(f_c\right)_{TM\ par} = \frac{\left(n - \frac{1}{2}\right)c}{b\sqrt{n_1^2 - n_2^2}} \quad , \quad n = 1,\ 2,\ \ldots$$

$\Longrightarrow$ frequência de corte aumenta com a diminuição da largura do guia!

Modos TM ímpares

$$H_z^0 = 0 \qquad E_z^0 = y_{\text{ímpar}}$$

$|y| \le \frac{b}{2}$:

$$E_z^0 = A\sin(h_1 y)$$

$$H_x^0 = \frac{jwe_1}{h_1} A\cos(h_1 y)$$

$$E_y^0 = -\frac{jb}{h_1} A\cos(h_1 y)$$

$y > \frac{b}{2}$:

$$E_z^0 = A\sin\left(\frac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)}$$

$$H_x^0 = \frac{jwe_2}{n} A\sin\left(\frac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)}$$

$$E_y^0 = -\frac{jb}{n} A\sin\left(\frac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)}$$

$y < -\frac{b}{2}$:

$$E_z^0 = -A\sin\left(\frac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)}$$

$$H_x^0 = \frac{jwe_2}{n} A\sin\left(\frac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)}$$

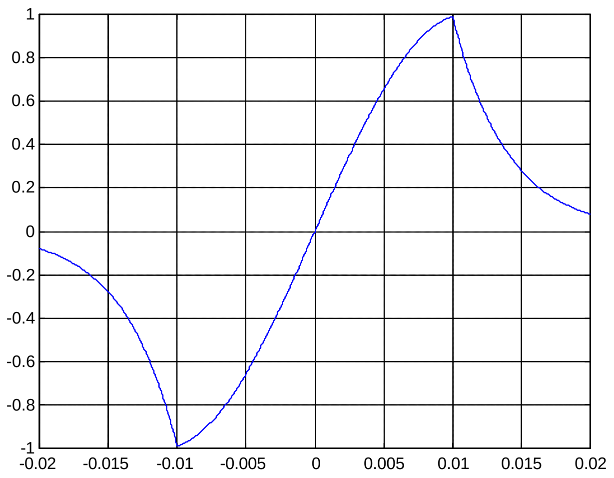
$$E_y^0 = -\frac{jb}{n} A\sin\left(\frac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)}$$

$$n = \frac{e_2}{e_1} h_1 \tan\left(\frac{h_1 b}{2}\right)$$

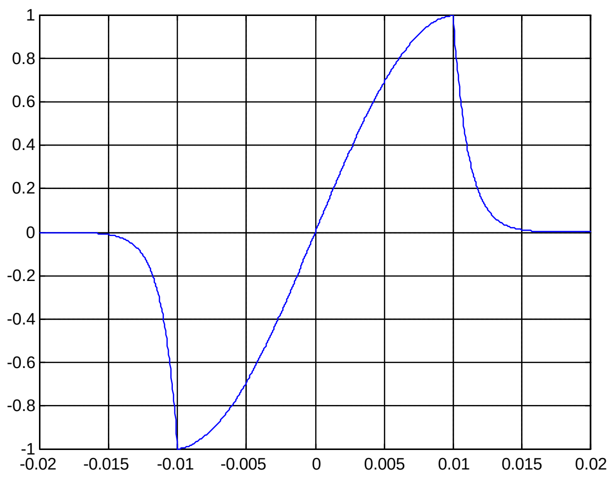
$$\sqrt{\left(\frac{w}{c}\right)^2 \left(n_1^2 - n_2^2\right) - h_1^2} = \frac{e_2}{e_1} h_1 \tan\left(\frac{h_1 b}{2}\right)$$

$$TM_{impar,1}$$

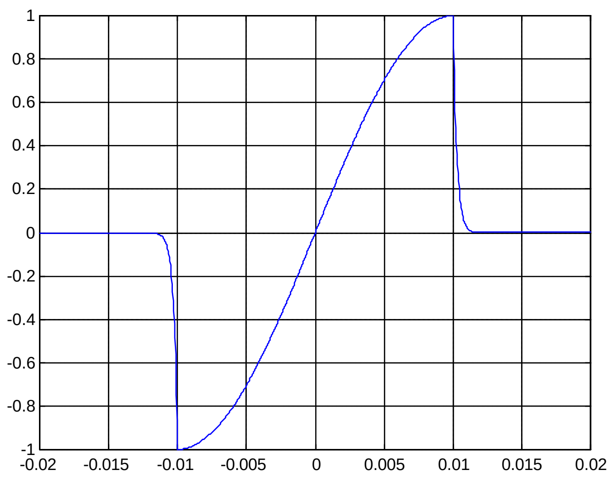
$$f = 8\,GHz$$
$$h_{1,1} = 143.01$$
$$n_1 = 252.52$$



$$f = 25\,GHz$$
$$h_{1,1} = 152.81$$
$$n_1 = 893.93$$



$$f = 100\,GHz$$
$$h_{1,1} = 156.0$$
$$n_1 = 3624.24$$



Frequência de corte

$$n = 0 \quad \longrightarrow \quad \frac{e_2}{e_1} h_1 \tan\left(\frac{h_1 b}{2}\right) = 0$$

$$\frac{h_1 b}{2} = (n-1)p \quad \Leftrightarrow \quad h_1 = \frac{(n-1)2p}{b} \quad , \quad n = 1,\ 2,\ \ldots$$

$$(h_1)_{\text{corte}} = \frac{w}{c}\sqrt{n_1^2 - n_2^2}$$

$$(f_c)_{TM\ ímpar} = \frac{(n-1)c}{b\sqrt{n_1^2 - n_2^2}} \quad , \quad n = 1,\ 2,\ \ldots$$

Ondas TE pares

$$E_z^0 = 0 \qquad H_z^0 = y_{\ par}$$

$$|y| \le \frac{b}{2}: \qquad \begin{aligned} H_z^0 &= B\cos(h_1 y) \\ H_y^0 &= \frac{jb}{h_1} B\sin(h_1 y) \\ E_x^0 &= \frac{jwm_0}{h_1} B\sin(h_1 y) \end{aligned}$$

$$y > \frac{b}{2}: \qquad \begin{aligned} H_z^0 &= B\cos\left(\frac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)} \\ H_y^0 &= -\frac{jb}{n} B\cos\left(\frac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)} \\ E_x^0 &= -\frac{jwm_0}{n} B\cos\left(\frac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)} \end{aligned}$$

$$y < -\frac{b}{2}: \qquad \begin{aligned} H_z^0 &= B\cos\left(\frac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)} \\ H_y^0 &= \frac{jb}{n} B\cos\left(\frac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)} \\ E_x^0 &= \frac{jwm_0}{n} B\cos\left(\frac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)} \end{aligned}$$

$$\sqrt{\left(\frac{w}{c}\right)^2 \left(n_1^2 - n_2^2\right) - h_1^2} = -\, h_1 \cot\left(\frac{h_1 b}{2}\right)$$

$$\left(f_c\right)_{TE\ par} = \frac{\left(n - \frac{1}{2}\right)c}{b\sqrt{n_1^2 - n_2^2}} \quad , \quad n = 1,\ 2,\ \ldots$$

$$\left(f_c\right)_{TE\ par} = \left(f_c\right)_{TM\ par}$$

Modos TE ímpares

$$E_z^0 = 0 \qquad H_z^0 = y \text{ ímpar}$$

$|y| \le \frac{b}{2}$:

$$H_z^0 = A\sin(h_1 y)$$
$$H_y^0 = -\frac{jb}{h_1} A\cos(h_1 y)$$
$$E_x^0 = -\frac{jwm_0}{h_1} A\cos(h_1 y)$$

$y > \frac{b}{2}$:

$$H_z^0 = A\sin\left(\frac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)}$$
$$H_y^0 = -\frac{jb}{n} A\sin\left(\frac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)}$$
$$E_x^0 = -\frac{jwm_0}{n} A\sin\left(\frac{h_1 b}{2}\right) e^{-n\left(y-\frac{b}{2}\right)}$$

$y < -\frac{b}{2}$:

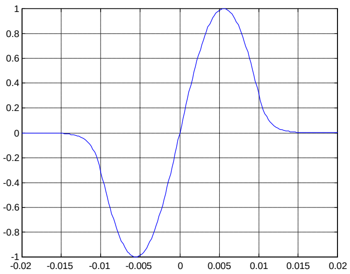
$$H_z^0 = -A\sin\left(\frac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)}$$
$$H_y^0 = -\frac{jb}{n} A\sin\left(\frac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)}$$
$$E_x^0 = -\frac{jwm_0}{n} A\sin\left(\frac{h_1 b}{2}\right) e^{n\left(y+\frac{b}{2}\right)}$$

$$\sqrt{\left(\frac{w}{c}\right)^2 \left(n_1^2 - n_2^2\right) - h_1^2} = -h_1 \tan\left(\frac{h_1 b}{2}\right)$$

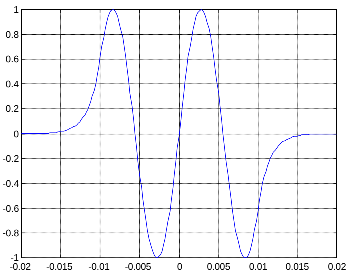
$$(f_c)_{TE \text{ ímpar}} = \frac{(n-1)c}{b\sqrt{n_1^2 - n_2^2}} \quad , \quad n = 1,\ 2,\ \ldots$$

$$(f_c)_{TE \text{ ímpar}} = (f_c)_{TM \text{ ímpar}}$$

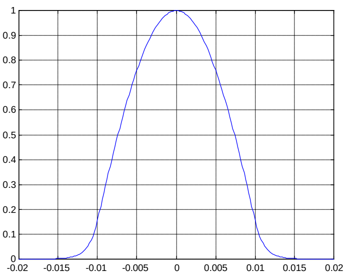
$TE_{par,\ 1}$



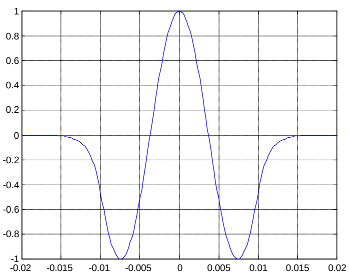
$TE_{par,\ 2}$



$TE_{ímpar,\ 1}$

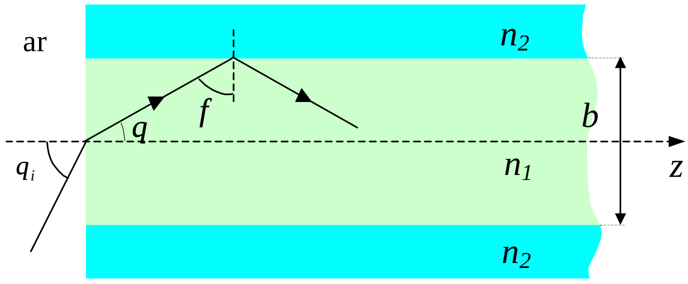


$TE_{ímpar,\ 2}$



MODOS		RELAÇÃO CARACTERTÍSTICA	FREQUENÊNCIA DE CORTE
PARES	TM	$n = -\left(\frac{n_2}{n_1}\right)^2 h_1 \cot\left(\frac{h_1 b}{2}\right)$	$f_c = \frac{\left(n - \frac{1}{2}\right)c}{b\sqrt{n_1^2 - n_2^2}}$
	TE	$n = -h_1 \cot\left(\frac{h_1 b}{2}\right)$	
ÍMPARES	TM	$n = \left(\frac{n_2}{n_1}\right)^2 h_1 \tan\left(\frac{h_1 b}{2}\right)$	$f_c = \frac{(n-1)c}{b\sqrt{n_1^2 - n_2^2}}$
	TE	$n = h_1 \tan\left(\frac{h_1 b}{2}\right)$	

Guias dieléctricos e reflexão interna total



$n_1 > n_2$

$$n_1 \cos(f) = \sin(q_i)$$

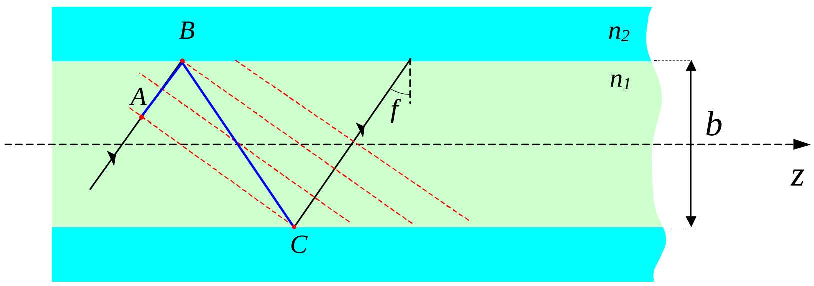
interface 1-2:

$$f > f_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \quad \Rightarrow \quad \textbf{Reflexão interna total}$$

onda guiada $\Rightarrow$ $f \geq f_c$ $\Rightarrow$ $\boxed{\sin(q_i) \leq \sqrt{n_1^2 - n_2^2}}$

- $\sqrt{n_1^2 - n_2^2}$ → **abertura numérica** (*NA*)
- $q_A = \sin^{-1}(NA)$ → **ângulo de aceitação**

Modos permitidos



- *A* e *C* ∈ mesma frente de onda $\Longrightarrow$ $d_C - d_A = 2np$

- $d_C = d_A + k_1 l_{AB} + d_r + k_1 l_{BC} + d_r$, $k_1 = \dfrac{w n_1}{c}$

$$\boxed{k_1(l_{AB} + l_{BC}) + 2d_r = 2np}$$

- $b = l_{BC}\cos(f)$ $\Longrightarrow$ $l_{BC} = \dfrac{b}{\cos(f)}$

- $l_{AB} = l_{BC}\sin(90^0 - 2f) = l_{BC}\cos(2f) = \dfrac{b\cos(2f)}{\cos(f)}$

$\Longrightarrow$ $$\boxed{2k_1 b\cos(f) + 2d_r = 2np\ , \quad n \ \text{inteiro}}$$

Determinação de d_r

- polarização $\perp$ $\Longleftrightarrow$ ondas TE
- polarização $||$ $\Longleftrightarrow$ ondas TM

$$\Gamma_{TE} = \frac{n_1 \cos(f) - \sqrt{n_2^2 - n_1^2 \sin^2(f)}}{n_1 \cos(f) + \sqrt{n_2^2 - n_1^2 \sin^2(f)}}$$

$$\Gamma_{TM} = \frac{-n_2^2 \cos(f) + n_1 \sqrt{n_2^2 - n_1^2 \sin^2(f)}}{n_2^2 \cos(f) + n_1 \sqrt{n_2^2 - n_1^2 \sin^2(f)}}$$

$$f > f_c$$

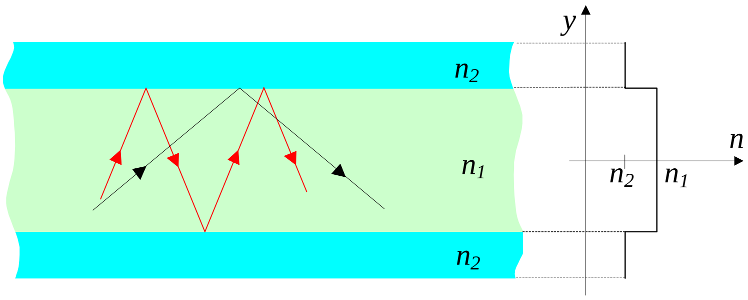
$$n_1^2 \sin^2(f) > n_2^2$$

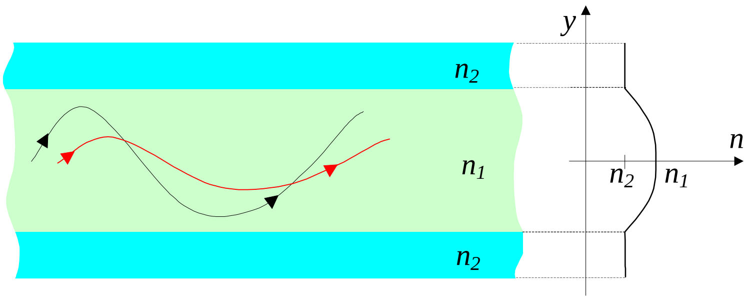
$$\Gamma = e^{j d_r}$$

$$d_{r,TE} = -2 \tan^{-1}\left(\frac{\sqrt{n_1^2 \sin^2(f) - n_2^2}}{n_1 \cos(f)} \right)$$

$$d_{r,TM} = p - 2 \tan^{-1}\left(\frac{n_1 \sqrt{n_1^2 \sin^2(f) - n_2^2}}{n_2^2 \cos(f)} \right)$$

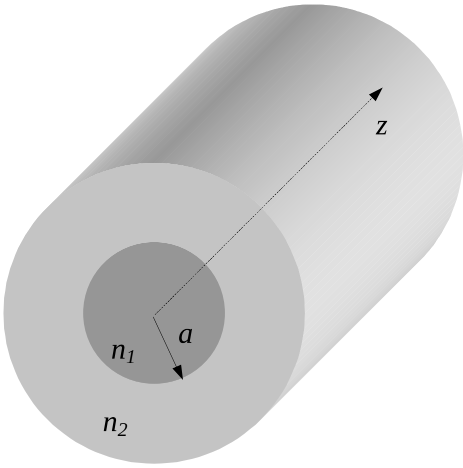
Índice de refracção gradual





GUIAS DIELÉCTRICOS CIRCULARES

GUIAS DIELÉCTRICOS CIRCULARES



núcleo	bainha
$\begin{cases} \nabla^2_{rf} E_z^0 + h_1^2 E_z^0 = 0 \\ \nabla^2_{rf} H_z^0 + h_1^2 H_z^0 = 0 \end{cases}$	$\begin{cases} \nabla^2_{rf} E_z^0 + h_2^2 E_z^0 = 0 \\ \nabla^2_{rf} H_z^0 + h_2^2 H_z^0 = 0 \end{cases}$
$h_1^2 = g^2 + \left(\frac{w\, n_1}{c}\right)^2$	$h_2^2 = g^2 + \left(\frac{w\, n_2}{c}\right)^2$

Equação de onda em guias dieléctricos circulares

$$\nabla^2_{rf} y + h^2 y = 0 \qquad h^2 = \begin{cases} h_1^2 , & r \le a \\ h_2^2 , & r > a \end{cases}$$

$$y(r,f) = R(r)\Phi(f)$$

$$\Phi(f) = Ae^{jnf}$$

$$r^2 \frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} + \left(h^2 r^2 - n^2\right)R(r) = 0$$

- $h^2 > 0 \Leftrightarrow$ *h* real $\qquad R(r) = B\, J_n(hr)$
- $h^2 < 0 \Leftrightarrow h = jn$

$$r^2 \frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} - \left(n^2 r^2 + n^2\right)R(r) = 0$$

equação diferencial de Bessel modificada

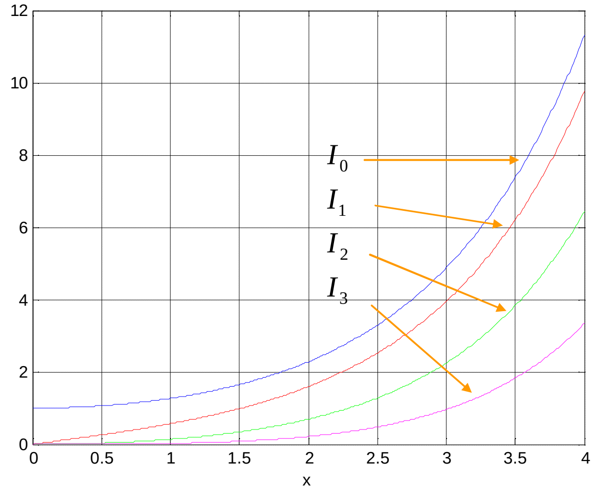
$$R(r) = C\, I_n(n\, r) + DK_n(n\, r)$$

funções de Bessel modificadas de 1ª e 2ª espécies

Funções de Bessel modificadas de 1ª espécie

Para *n* inteiro

$$I_n(x) = j^{-n} J_n(jx) = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!(n+k)!}$$



- $\lim_{x\to\infty} I_n(x) = \infty$

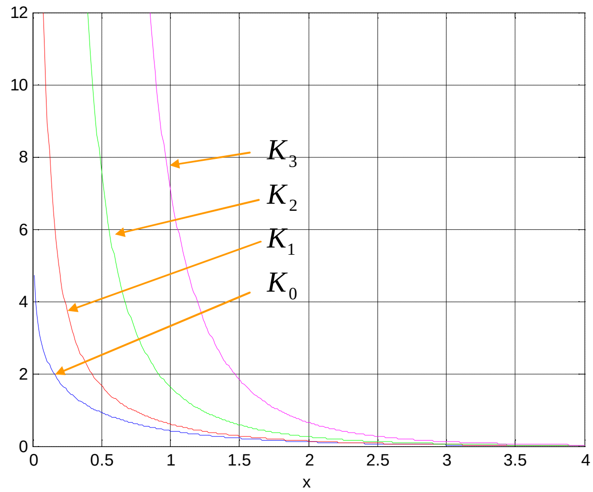
⇩

$I_n(x)$ não deve fazer parte da solução geral quando a região de interesse incluir o infinito!

Funções de Bessel modificadas de 2ª espécie

Para *n* inteiro

$$K_n(x) = \lim_{p \to n} \frac{p}{2\sin(pp)} \left[I_{-p}(x) - I_p(x) \right]$$



- $\lim_{x \to 0} K_n(x) = \infty$

$K_n(x)$ não deve fazer parte da solução geral quando a região de interesse incluir a origem!

onda guiada $\Longrightarrow$ h_1 real e $h_2 = jn$

$\Downarrow$

$$n = \sqrt{\left(\frac{w}{c}\right)^2 \left(n_1^2 - n_2^2\right) - h_1^2}$$

- $b = \sqrt{\left(\frac{w}{c} n_1\right)^2 - h_1^2} = \sqrt{\left(\frac{w}{c} n_2\right)^2 + n^2} \quad \Longrightarrow \quad \frac{w}{c} n_1 < b < \frac{w}{c} n_2$

$$\Longrightarrow \quad y(r, f) = \begin{cases} AJ_n(h_1 r) e^{jnf}, & r \leq a \\ BK_n(n\, r) e^{jnf}, & r > a \end{cases}$$

- núcleo: $E_z^0 = AJ_n(h_1 r) e^{jnf}$

  $H_z^0 = BJ_n(h_1 r) e^{jnf}$

- bainha: $E_z^0 = CK_n(n\, r) e^{jnf}$

  $H_z^0 = DK_n(n\, r) e^{jnf}$

- núcleo

$$E_z^0 = AJ_n(h_1 r)e^{jnf}$$
$$H_z^0 = BJ_n(h_1 r)e^{jnf}$$
$$H_r^0 = -\frac{1}{h_1^2}\left[jbh_1 BJ'_n(h_1 r) + \frac{we_1 n}{r} AJ_n(h_1 r) \right] e^{jnf}$$
$$H_f^0 = -\frac{1}{h_1^2}\left[-\frac{bn}{r} BJ_n(h_1 r) + jwe_1 h_1 AJ'_n(h_1 r) \right] e^{jnf}$$
$$E_r^0 = -\frac{1}{h_1^2}\left[jbh_1 AJ'_n(h_1 r) - \frac{wm_0 n}{r} BJ_n(h_1 r) \right] e^{jnf}$$
$$E_f^0 = -\frac{1}{h_1^2}\left[-\frac{bn}{r} AJ_n(h_1 r) - jwm_0 h_1 BJ'_n(h_1 r) \right] e^{jnf}$$

- bainha

$$E_z^0 = CK_n(n r)e^{jnf}$$
$$H_z^0 = DK_n(n r)e^{jnf}$$
$$H_r^0 = \frac{1}{n^2}\left[jbnDK'_n(nr) + \frac{we_2 n}{r} CK_n(nr) \right] e^{jnf}$$
$$H_f^0 = \frac{1}{n^2}\left[-\frac{bn}{r} DK_n(nr) + jwe_2 nCK'_n(nr) \right] e^{jnf}$$
$$E_r^0 = \frac{1}{n^2}\left[jbnCK'_n(nr) - \frac{wm_0 n}{r} DK_n(nr) \right] e^{jnf}$$
$$E_f^0 = \frac{1}{n^2}\left[-\frac{bn}{r} CK_n(nr) - jwm_0 nDK'_n(nr) \right] e^{jnf}$$

Condições fronteira

E_z^0 e E_f^0 contínuos em $r = a$

H_z^0 e H_f^0 contínuos em $r = a$

$$\Downarrow$$

$$AJ_n(h_1a) = CK_n(n\,a) \;\Leftrightarrow\; AJ_n(h_1a) - CK_n(n\,a) = 0$$

$$BJ_n(h_1a) - DK_n(n\,a) = 0$$

$$B\frac{b\,n}{h_1^2 a}J_n(h_1a) - A\frac{jwe_1}{h_1}J'_n(h_1a) + D\frac{b\,n}{n^2 a}K_n(n\,a) - C\frac{jwe_2}{n}K'_n(n\,a) = 0$$

$$A\frac{b\,n}{h_1^2 a}J_n(h_1a) + B\frac{jw\,m_0}{h_1}J'_n(h_1a) + C\frac{b\,n}{n^2 a}K_n(n\,a) + D\frac{jw\,m_0}{n}K'_n(n\,a) = 0$$

solução não trivial

$$\begin{vmatrix} J_n(h_1a) & 0 & K_n(n\,a) & 0 \\ 0 & J_n(h_1a) & 0 & K_n(n\,a) \\ -\frac{jwe_1}{h_1}J'_n(h_1a) & \frac{b\,n}{h_1^2 a}J_n(h_1a) & -\frac{jwe_2}{n}K'_n(n\,a) & \frac{b\,n}{n^2 a}K_n(n\,a) \\ \frac{b\,n}{h_1^2 a}J_n(h_1a) & \frac{jw\,m_0}{h_1}J'_n(h_1a) & \frac{b\,n}{n^2 a}K_n(n\,a) & \frac{jw\,m_0}{n}K'_n(n\,a) \end{vmatrix} = 0$$

$$\left(\frac{w}{c}\right)^2\left[\frac{J'_n(h_1a)}{h_1\,J_n(h_1a)} + \frac{K'_n(n\,a)}{n\,K_n(n\,a)}\right]\left[n_1^2\frac{J'_n(h_1a)}{h_1\,J_n(h_1a)} + n_2^2\frac{K'_n(n\,a)}{n\,K_n(n\,a)}\right] = \left(\frac{b\,n}{a}\right)^2\left(\frac{1}{h_1^2} + \frac{1}{n^2}\right)^2$$

equação característica para os modos TM, TE, EH e HE

***n*=0**

$$\begin{bmatrix} J_0(h_1 a) & 0 & K_0(n\,a) & 0 \\ 0 & J_0(h_1 a) & 0 & K_0(n\,a) \\ -\dfrac{jwe_1}{h_1} J'_0(h_1 a) & 0 & -\dfrac{jwe_2}{n} K'_0(n\,a) & 0 \\ 0 & \dfrac{jwm_0}{h_1} J'_0(h_1 a) & 0 & \dfrac{jwm_0}{n} K'_0(n\,a) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

$$\begin{bmatrix} J_0(h_1 a) & K_0(n\,a) \\ -\dfrac{e_1}{h_1} J'_0(h_1 a) & -\dfrac{e_2}{n} K'_0(n\,a) \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = 0$$

e

$$\begin{bmatrix} J_0(h_1 a) & K_0(n\,a) \\ \dfrac{1}{h_1} J'_0(h_1 a) & \dfrac{1}{n} K'_0(n\,a) \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = 0$$

solução não trivial
$B = D = 0$ possível
modos TM

$$\left[n_1^2 \frac{J_1(h_1 a)}{h_1\, J_0(h_1 a)} + n_2^2 \frac{K_1(n\,a)}{n\, K_0(n\,a)} \right] = 0$$

solução não trivial
$A = C = 0$ possível
modos TE

$$\left[\frac{J_1(h_1 a)}{h_1\, J_0(h_1 a)} + \frac{K_1(n\,a)}{n\, K_0(n\,a)} \right] = 0$$

Frequência de corte

n	**modo**	**condição de corte**
0	TE_{0p} TM_{0p}	$J_0(h_1 a) = 0$
1	HE_{1p} EH_{1p}	$J_1(h_1 a) = 0$
≥ 2	EH_{np}	$J_n(h_1 a) = 0$
	HE_{np}	$\left(\frac{n_1^2}{n_2^2} + 1\right) J_{n-1}(h_1 a) = \frac{h_1 a}{n-1} J_n(h_1 a)$

- modo HE_{11} tem frequência de corte nula
- modos seguintes: TE_{01} e TM_{01} (frequência de corte associada ao primeiro zero de J_0, em 2.405)

- parâmetro *V* ou frequência normalizada:

$$V^2 = \left(h_1^2 + n^2\right) a^2 = \left(\frac{w\,a}{c}\right)^2 \left(n_1^2 - n_2^2\right)$$

↓

$$V = \frac{2p\,a}{l_0} \sqrt{n_1^2 - n_2^2}$$

- $V > 2.405$ multimodo
- $V \leq 2.405$ monomodo.