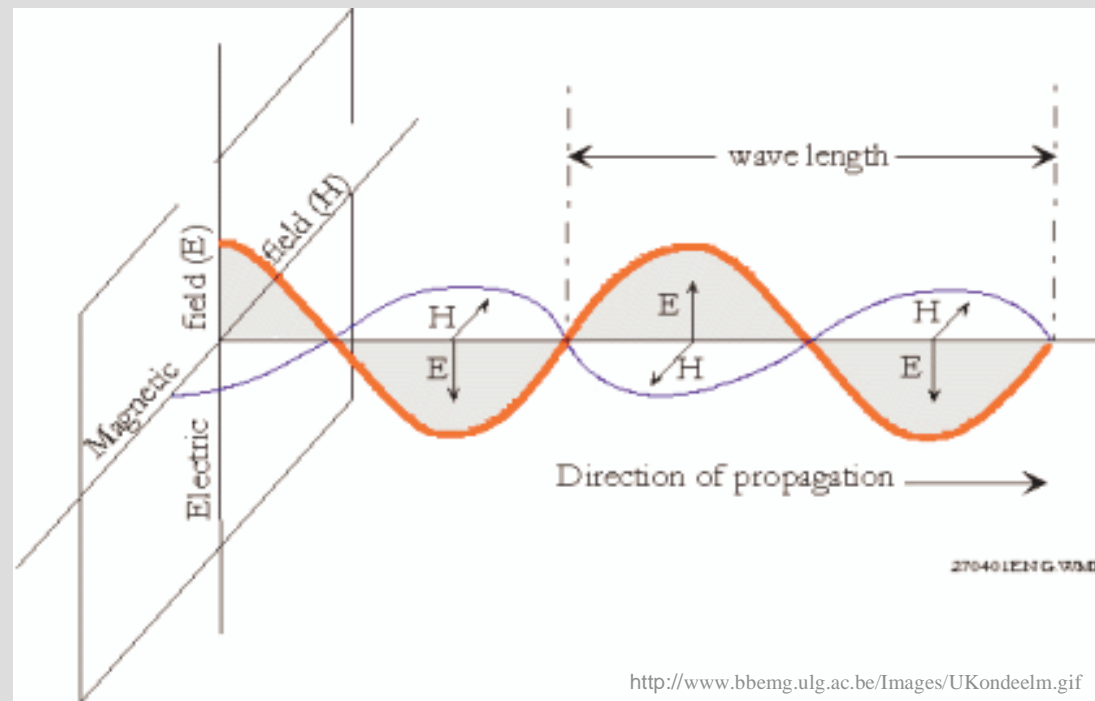


# Ondas electromagnéticas

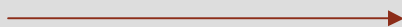


**MIEEC 2006/2007**

## disciplina final da área de Teoria da Electricidade

- **Circuitos** (*1º ano, tronco comum*)
  - Leis básicas dos circuitos eléctricos
- **Electromagnetismo** (*2º ano, tronco comum*)
  - Campos eléctrico e magnético
- **Ondas Electromagnéticas** (*2º ano, ramo Telecomunicações*)
  - Propagação de ondas electromagnéticas
    - meios infinitos
    - linhas de transmissão
    - guias de onda
  - Geração de ondas electromagnéticas
    - antenas

- **Teóricas**
  - 1 turma
  - 2 X 1h por semana
- **Teórico-práticas**
  - 4 turmas
  - 1 X 2h por semana
- **Laboratórios**
  - 4 turmas
  - 3 X 2h por semestre



**Lab1 (sala I220)**

- turmas 11 e 12 → dia 29/3
- turmas 13 e 14 → dia 12/4

**Lab2 (sala I220)**

- turmas 11 e 12 → dia 26/4
- turmas 13 e 14 → dia 3/5

**Lab3 (sala B236)**

- turmas 11 e 12 → dia 17/5
- turmas 13 e 14 → dia 24/5

- **Teóricas**
  - exposição e discussão da matéria
  - resolução de exercícios (?)
- **Teórico-práticas**
  - resolução de exercícios pelos docentes
  - resolução de exercícios pelos alunos
  - realização de microtestes
- **Laboratórios**
  - realização de trabalhos de acordo com guiões fornecidos
  - elaboração e entrega de relatórios



	teóricas	práticas	laboratórios	atendimento
<b>Aníbal Matos</b> gab I 316 anibal@fe.up.pt		turma 14		2ª feira, 8h – 9h
<b>Carlos Pintassilgo</b> gab I 115 cdp@fe.up.pt			todas as turmas	5ª feira, 10h – 11h
<b>Inês Carvalho</b> gab I 313 mines@fe.up.pt	todos os alunos	turmas 11 e 13		5ª feira, 9h – 12h
<b>Nuno Fidalgo</b> gab J 110 jfidalgo@fe.up.pt		turma 12		2ª feira, 15h – 16h

- **Exame final**
  - 70% da nota final
  - consulta de formulário fornecido
  - inclui pergunta sobre laboratório (apenas para alunos que frequentam disciplina)
- **Microtestes**
  - 20% da nota final
  - realizados nas aulas práticas sem aviso prévio
  - curta duração (aprox 15 min)
  - entre 7 a 9 durante o semestre
  - 2 piores não contam para nota
- **Componente laboratorial**
  - 10% da nota final

**nota de frequência**

- **Classificação de frequência - AD (0 – 20 valores)**

- Microtestes – M (0 – 4 valores)
  - média dos microtestes, excluindo os 2 piores
- Laboratórios – L (0 – 2 valores)
  - média das classificações em cada trabalho

$$AD = (M + L) \cdot \frac{20}{6}$$

- **Condições para obtenção de frequência**

- Não exceder limite de faltas
  - aulas TP (25% das previstas → 3 faltas)
  - Labs (25% dos previstos → 1 falta)
- Classificação mínima 30% **em cada** componente
  - Microtestes
  - Labs

- **Alunos com dispensa de frequência**

- Trabalhadores estudantes
- Para usar este direito **não se podem inscrever** nas turmas TP/Labs

Exame final – E (0 – 20 valores)

Nota de frequência – AD (0 – 20 valores)

- **Alunos a frequentar**

$$CF = 0.7 \cdot E + 0.3 \cdot AD$$

- **Alunos com dispensa de frequência**

$$CF = E$$

- **Melhoria de classificação**

$$CF = E$$

- **Linhas de transmissão**
  - Tensão, corrente e impedância ao longo da linha; adaptação; transitórios
- **Ondas electromagnéticas planas**
  - Propagação em meios infinitos e em meios com perdas; incidência em interfaces
- **Guias de onda e cavidades**
  - Guias metálicos e dieléctricos
- **Antenas e radiação**
  - Dipolos eléctrico e magnético; padrões de radiação; antenas finas e grupos de antenas
- **Métodos Numéricos**
  - Diferenças finitas; elementos finitos

	<b>Aulas Teóricas</b>	<b>Aulas Teórico-Práticas</b>
<b>Apresentação/Revisão</b>	<b>1</b>	<b>1</b>
<b>Linhas de transmissão</b>	<b>6</b>	<b>3</b>
<b>Ondas Electromagnéticas</b>	<b>5</b>	<b>3</b>
<b>Guias de Onda</b>	<b>7</b>	<b>2</b>
<b>Radiação</b>	<b>4</b>	<b>1</b>
<b>Métodos Numéricos</b>	<b>2</b>	<b>1</b>
<b>Total</b>	<b>25</b>	<b>11</b>

- **Livro recomendado**

- D. K. Cheng, 'Field and Wave Electromagnetics', Addison Wesley Publishing, 1989.

- **Material fornecido**

- Acetatos das aulas teóricas
- Folhas de problemas
- Apontamentos sobre guias de onda e cavidades
- Apontamentos sobre métodos numéricos
- Formulário



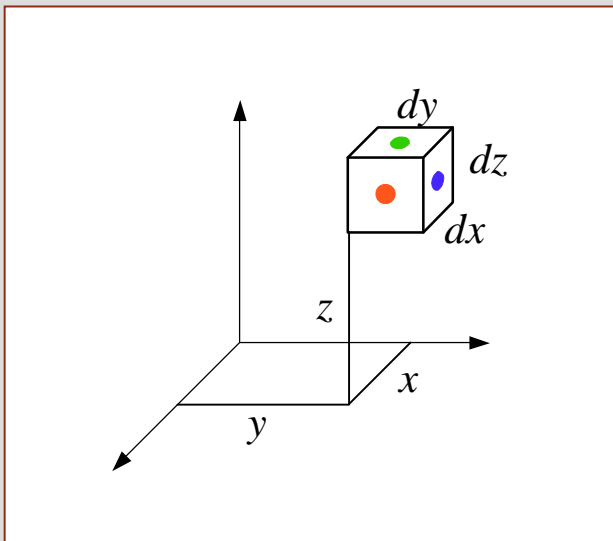
**disponível na página da disciplina:**

<http://www.fe.up.pt/~mines/OE/>

- **Sistemas de coordenadas**
  - Cartesiano
  - Cilíndrico
  - Esférico
- **Análise vectorial**
  - Gradiente
  - Rotacional
  - Laplaciano
- **Fasores**
- **Equações de Maxwell**



# Sistema de coordenadas cartesiano



$$d\vec{l} = dx \cdot \hat{x} + dy \cdot \hat{y} + dz \cdot \hat{z}$$

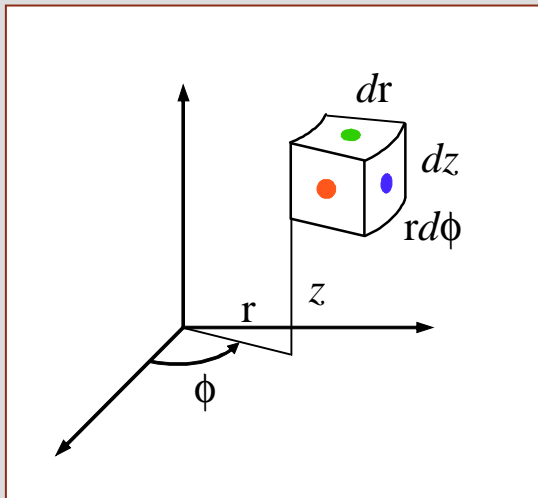
$$dA_x = dy dz$$

$$dA_y = dx dz$$

$$dA_z = dx dy$$

$$dV = dx dy dz$$

# Sistema de coordenadas cilíndrico



$$d\vec{l} = r \cdot \hat{r} + rd\phi \cdot \hat{\phi} + dz \cdot \hat{z}$$

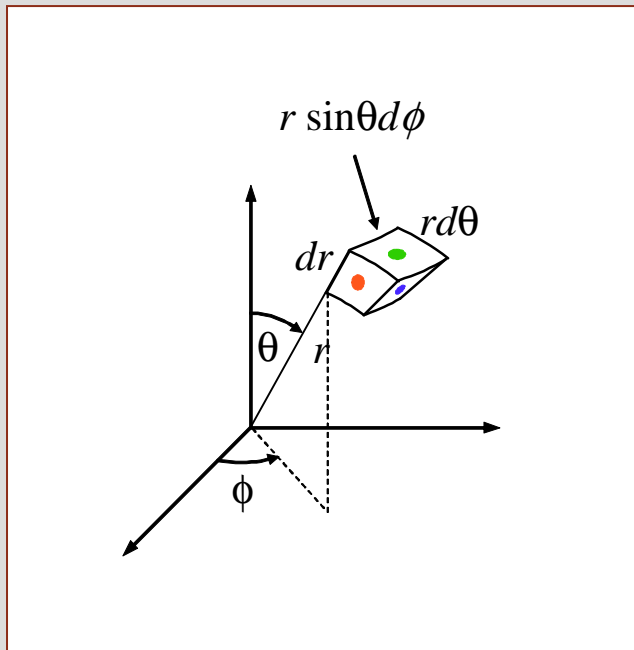
$$dA_z = r dr d\phi$$

$$dA_r = r d\phi dz$$

$$dA_\phi = dr dz$$

$$dV = r dr d\phi dz$$

# Sistema de coordenadas esférico



$$d\vec{l} = dr \cdot \hat{r} + r \sin\theta d\phi \cdot \hat{\phi} + r d\theta \cdot \hat{\theta}$$

$$dA_r = r^2 \sin\theta d\theta d\phi$$

$$dA_\theta = r \sin\theta dr d\phi$$

$$dA_\phi = r dr d\theta$$

$$dV = r^2 \sin\theta dr d\theta d\phi$$

- coordenadas cartesianas

$$\nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}$$

- coordenadas cilíndricas

$$\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z}$$

- coordenadas esféricas

$$\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}$$

- coordenadas cartesianas

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

- coordenadas cilíndricas

$$\nabla \times \vec{A} = \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

- coordenadas esféricas

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r \sin \theta A_\phi \end{vmatrix}$$

- coordenadas cartesianas

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- coordenadas cilíndricas

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

- coordenadas esféricas

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

- **Sinais harmónicos nos tempos**
  - de fácil geração
  - soluções de equações diferenciais lineares
  - permitem decomposição de sinais genéricos (série de Fourier,...)

- **Sinais sinusoidais são caracterizados por**  $i(t) = I_0 \cos(\omega t + \phi)$ 
  - amplitude
  - frequência
  - fase

- **Podemos escrever**  $i(t) = \text{Re}\{I_0 e^{j(\omega t + \phi)}\} = \text{Re}\{I e^{j\omega t}\}$

onde

$$I = I_0 e^{j\phi} \leftarrow \text{Fasor}$$

- **Linearidade**

$$av_1(t) + bv_2(t) \rightarrow aV_1 + bV_2$$

- **Derivação**

$$\frac{dx(t)}{dt} \rightarrow j\omega X$$

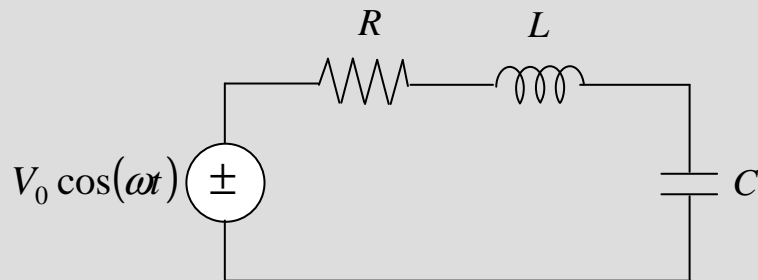
- **Integração**

$$\int x(t)dt \rightarrow \frac{X}{j\omega}$$



# Exemplo

## Circuito RLC série



resistência:  $v_R(t) = Ri(t) \rightarrow V_R = RI$

bobina:  $v_L(t) = L \frac{di(t)}{dt} \rightarrow V_L = j\omega L I$

condensador:  $i(t) = C \frac{dv_C(t)}{dt} \rightarrow V_C = \frac{I}{j\omega C}$

$$\Rightarrow V_0 = \left( R + j\omega L + \frac{1}{j\omega C} \right) I$$

$$\Rightarrow I = \frac{V_0}{R + j\omega L + \frac{1}{j\omega C}}$$

$$\Rightarrow i(t) = \operatorname{Re} \left\{ \frac{V_0 e^{j\omega t}}{R + j\omega L + \frac{1}{j\omega C}} \right\}$$

- Lei de Faraday:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- Lei de Ampère:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- Lei de Gauss:

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = \vec{J} + j\omega\epsilon\vec{E}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon}$$

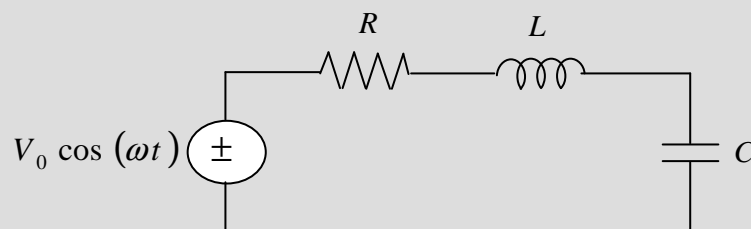
$$\nabla \cdot \vec{H} = 0$$

- notação fasorial
- meios LHI ( $\epsilon$ ,  $\mu$ )

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$

- Tempos de propagação são desprezáveis, sentindo-se de forma instantânea o efeito do sinal de entrada em todos os seus elementos



- Análise é válida apenas quando as dimensões dos diferentes elementos são muito menores do que o comprimento de onda do sinal de entrada

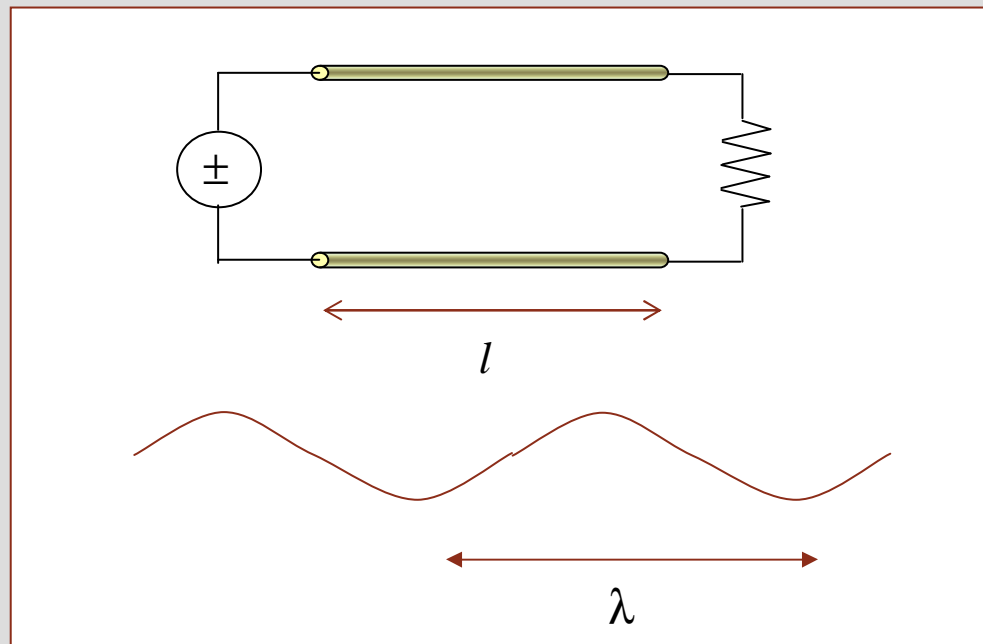
- Ex:

Para  $v=c$

$$f = 50 \text{ Hz} \rightarrow \lambda = 6000 \text{ km}$$

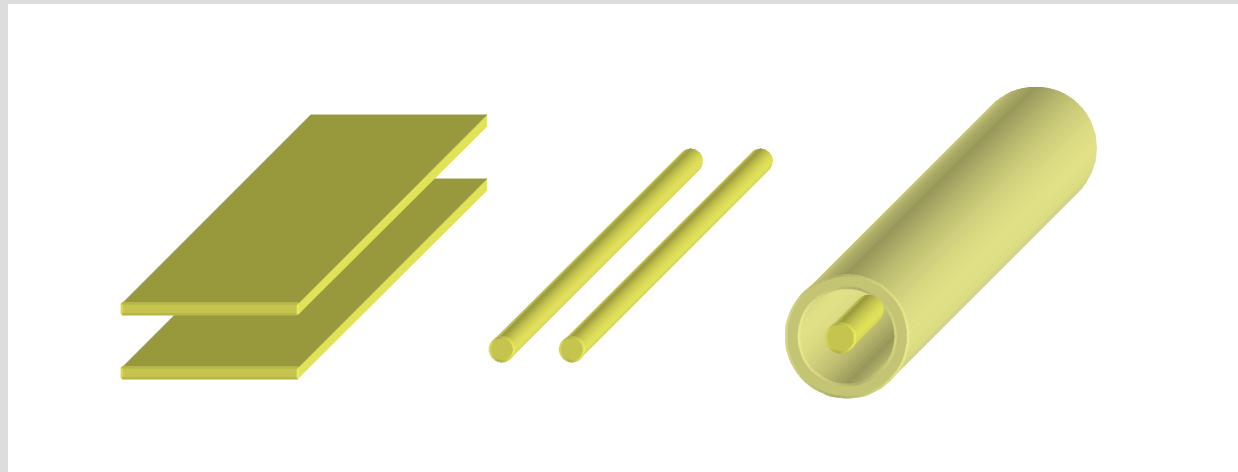
$$f = 2.4 \text{ GHz} \rightarrow \lambda = 12.5 \text{ cm}$$

- Quando as dimensões dos circuitos são comparáveis ao comprimento de onda dos sinais, é necessário considerar a variação destes ao longo do circuito



- A análise destes circuitos requer a utilização de elementos (resistências, bobinas, condensadores) distribuídos ao longo do circuito

## Linhas de transmissão



OE - MIEEC 2006/2007

## Equação de onda escalar

- A equação

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

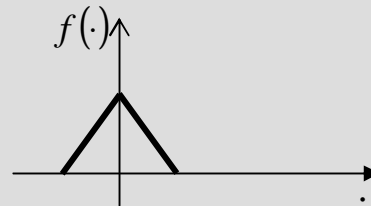
**EQUAÇÃO DE ONDA**

é satisfeita por qualquer função escalar  $u(x,t)$  do tipo

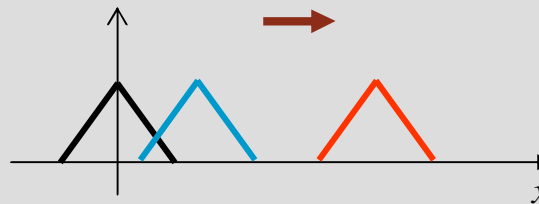
$$u(x,t) = f(x \pm ct)$$

## Exemplo

Seja



- se  $u(x,t)=f(x-ct)$

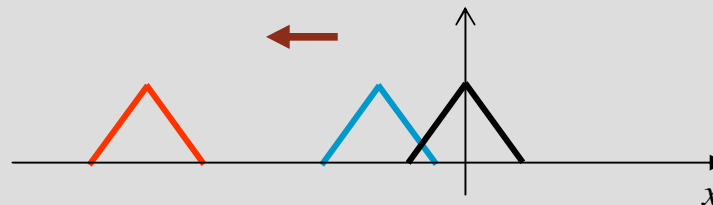


$$u(x, t = 0)$$

$$u(x, t_1) \quad \text{— blue —}$$

$$u(x, t_2) \quad \text{— red —}$$

- se  $u(x,t)=f(x+ct)$



$$u(x, t = 0)$$

$$u(x, t_1) \quad \text{— blue —}$$

$$u(x, t_2) \quad \text{— red —}$$

## Velocidade de propagação

- Para determinar a velocidade de propagação da onda é necessário considerar a velocidade com que um dado ponto do perfil  $f$  se propaga.

→ sendo  $x \pm ct = \text{const.}$ , tem-se  $dx \pm c dt = 0 \implies$

$$v = \left| \frac{dx}{dt} \right| = c$$



- Na equação

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

este termo está associado à velocidade de propagação



## Soluções harmónicas

$$u(x,t) = A \cos(\omega t - kx)$$

$$\omega = 2\pi f \quad k = \frac{2\pi}{\lambda}$$

solução da equação de onda  $\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 u(x,t)}{\partial t^2} = 0$

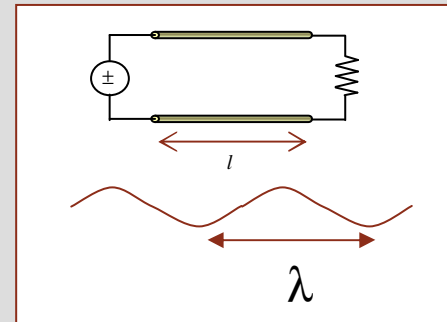
quando  $\omega = kv$

$$2\pi f = \frac{2\pi}{\lambda} v \Leftrightarrow v = \lambda f$$

## Modelo eléctrico da linha

### Linha de transmissão

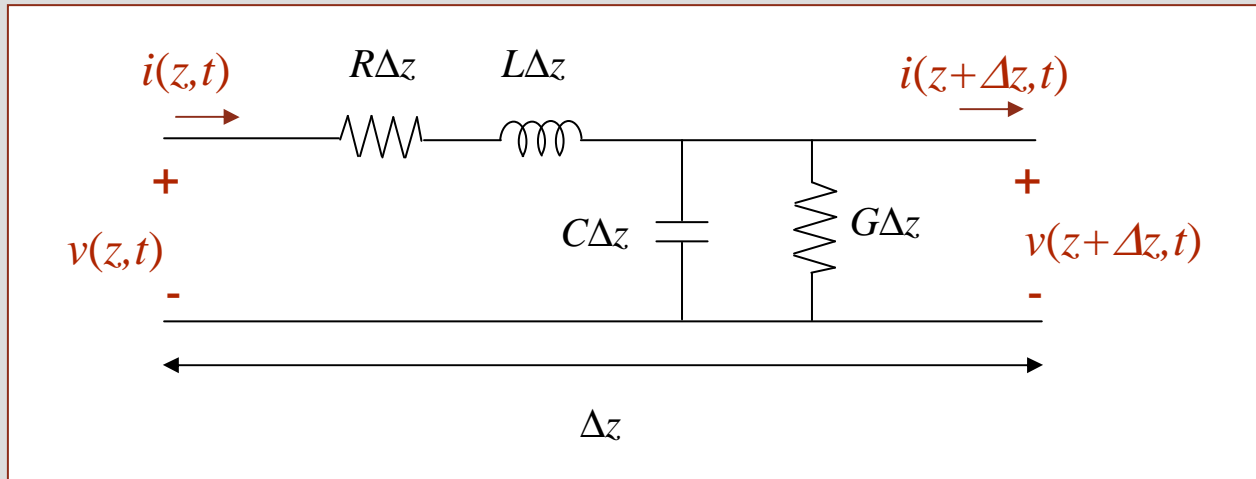
necessidade de considerar  
**elementos distribuídos**  
ao longo do circuito



R: resistência por unidade de comprimento  $\rightarrow \Omega/\text{m}$   
L: indutância por unidade de comprimento  $\rightarrow \text{H}/\text{m}$   
C: capacidade por unidade de comprimento  $\rightarrow \text{F}/\text{m}$   
G: condutância por unidade de comprimento  $\rightarrow \text{S}/\text{m}$

## Modelo eléctrico da linha

Para distância  $\Delta z$



$$v_R = R\Delta z i(z,t)$$

$$v_L = L\Delta z \frac{\partial i(z,t)}{\partial t}$$

$$i_G = G\Delta z v(z+\Delta z,t)$$

$$i_C = C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t}$$

## Equações gerais das linhas de transmissão

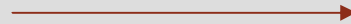
$$v(z,t) = R\Delta z i(z,t) + L\Delta z \frac{\partial i(z,t)}{\partial t} + v(z + \Delta z, t)$$

$$-i(z,t) + G\Delta z v(z + \Delta z, t) + C\Delta z \frac{\partial v(z + \Delta z, t)}{\partial t} + i(z + \Delta z, t) = 0$$



$$-\frac{\partial v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t}$$

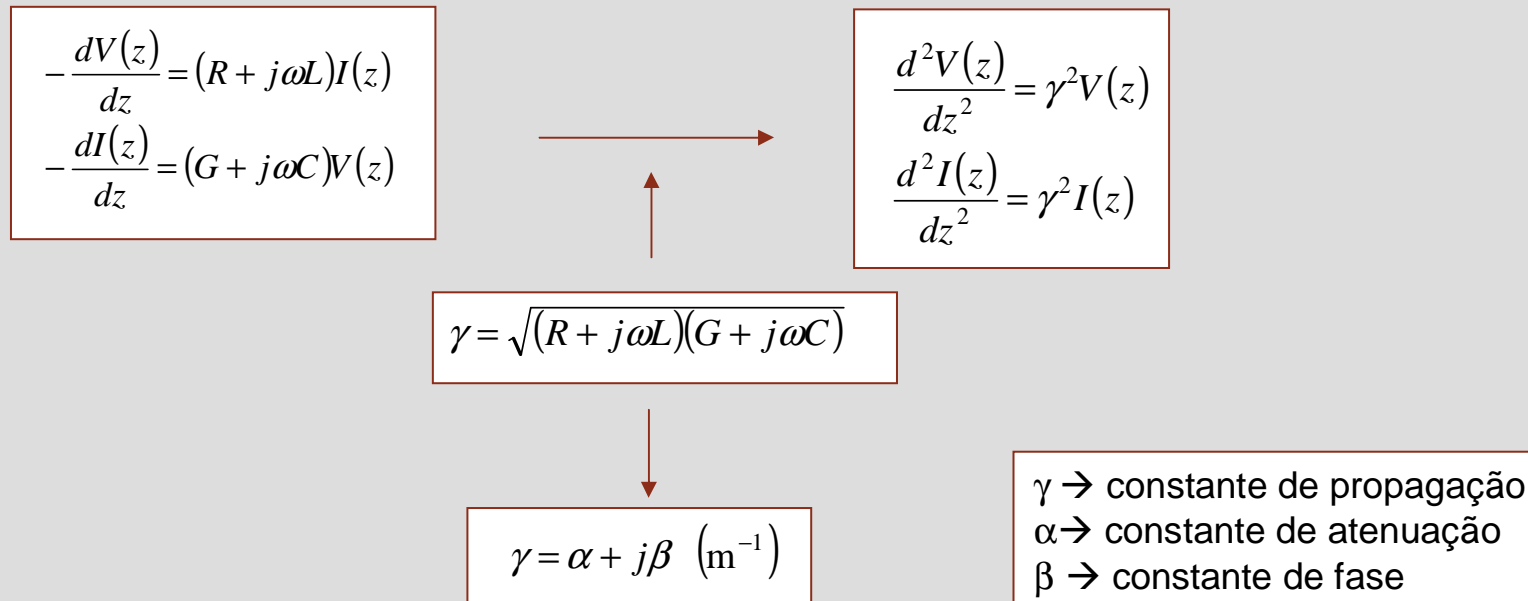
$$-\frac{\partial i(z,t)}{\partial z} = Gv(z,t) + C\frac{\partial v(z,t)}{\partial t}$$



$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z)$$

$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z)$$

## Corrente e tensão ao longo da linha



**Solução geral:**

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

## Corrente e tensão ao longo da linha

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z)$$

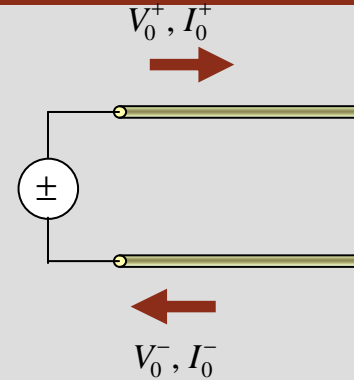
$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z)$$

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma}$$

tensão e corrente na linha  
completamente determinadas  
por 2 constantes

## Impedância característica

**Impedância característica:**  
quociente entre a tensão e a corrente para uma linha infinita



linha infinita  $\rightarrow$  não há reflexões

$$V_0^- = I_0^- = 0$$

$$\frac{V(z)}{I(z)} = \frac{V_0^+}{I_0^+}$$

$$Z_0 = \frac{V(z)}{I(z)} = \frac{V_0^+}{I_0^+} = \frac{R + j\omega L}{\gamma}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad (\Omega)$$

## Impedância característica

para uma linha de comprimento finito:

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0$$



$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$



$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$



## Características importantes – resumo

**Constante de propagação**

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \quad (\text{m}^{-1})$$

**Impedância característica**

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad (\Omega)$$

**Velocidade de propagação**

$$v = \frac{\omega}{\beta} \quad (\text{ms}^{-1})$$

**Comprimento de onda**

$$\lambda = \frac{2\pi}{\beta} \quad (\text{m})$$

Caso geral

- atenuação depende da frequência
- velocidade depende da frequência



**Distorção de sinais**

## Casos particulares – linhas sem perdas

Linha sem perdas  $\rightarrow R=G=0$

Constante de propagação

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}$$
$$\alpha = 0$$
$$\beta = \omega\sqrt{LC}$$

Impedância característica

$$Z_0 = \sqrt{\frac{L}{C}}$$

Velocidade de propagação

$$v = \frac{1}{\sqrt{LC}}$$

- não há atenuação
- velocidade constante
- impedância constante e real



**não há distorção**

## Casos particulares – linha sem distorção

Linha sem distorção  $\rightarrow R/L=G/C$

Constante de propagação

$$\gamma = \alpha + j\beta = (R + j\omega L)\sqrt{\frac{C}{L}}$$
$$\alpha = R\sqrt{\frac{C}{L}}$$
$$\beta = \omega\sqrt{LC}$$

Impedância característica

$$Z_0 = \sqrt{\frac{L}{C}}$$

Velocidade de propagação

$$v = \frac{1}{\sqrt{LC}}$$

- atenuação constante
- velocidade constante
- impedância constante e real



**não há distorção**

## Parâmetros das linhas de transmissão

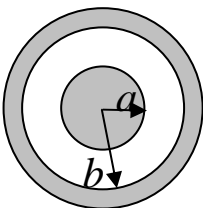
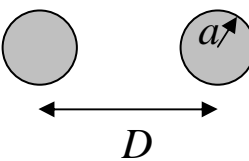
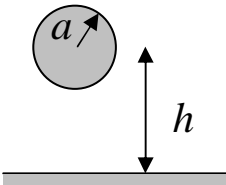
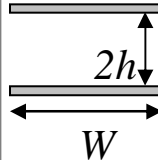
- O comportamento de uma dada linha depende da frequência de operação e dos seus parâmetros  $R$ ,  $L$ ,  $G$  e  $C$
- Estes parâmetros dependem da geometria da linha e dos materiais que a constituem

Seja

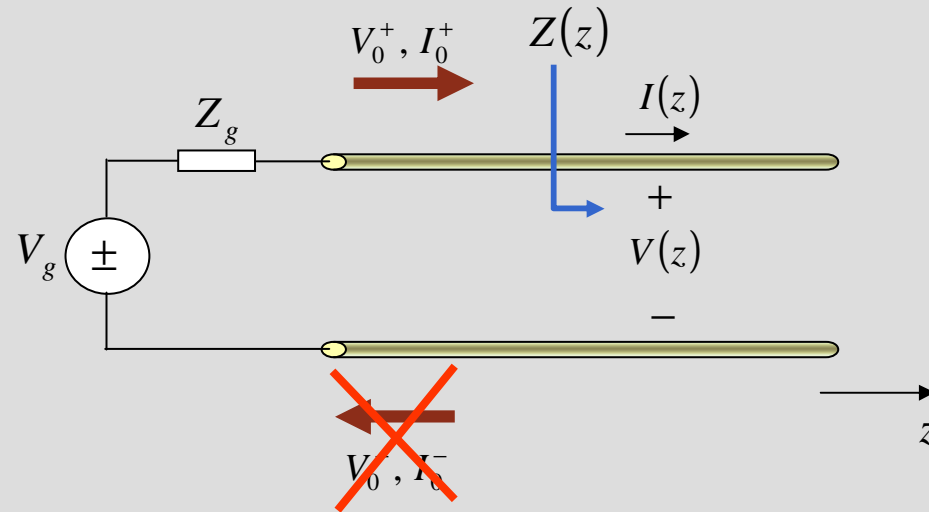
- $\sigma$  → condutividade do dieléctrico
- $\sigma_c$  → condutividade dos condutores
- $\varepsilon$  → permitividade eléctrica do dieléctrico
- $\mu$  → permeabilidade magnética do dieléctrico
- $\mu_c$  → permeabilidade magnética do condutor

$$\delta = \sqrt{\frac{1}{\pi f \mu_c \sigma_c}} \quad R_s = \frac{1}{\delta \sigma_c}$$

## Parâmetros das linhas de transmissão

				
	Coaxial	Bifilar	Condutor sobre terra	Tiras
$R(\Omega/m)$	$\frac{R_s}{2\pi} \left( \frac{1}{a} + \frac{1}{b} \right)$	$\frac{R_s}{\pi a} \frac{D/(2a)}{\sqrt{(D/(2a))^2 - 1}}$	$\frac{R_s}{2\pi a} \sqrt{\frac{h/a + 1}{h/a - 1}}$	$\frac{2R_s}{W}$
$L(H/m)$	$\frac{\mu}{2\pi} \ln \left( \frac{b}{a} \right)$	$\frac{\mu}{\pi} \operatorname{arcch} \left( \frac{D}{2a} \right)$	$\frac{\mu}{2\pi} \operatorname{arcch} \left( \frac{h}{a} \right)$	$\mu \frac{2h}{W}$
$G(S/m)$	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\operatorname{arcch}(D/2a)}$	$\frac{2\pi\sigma}{\operatorname{arcch}(h/a)}$	$\sigma \frac{W}{2h}$
$C(F/m)$	$\frac{2\pi\epsilon}{\ln(b/a)}$	$\frac{\pi\epsilon}{\operatorname{arcch}(D/2a)}$	$\frac{2\pi\epsilon}{\operatorname{arcch}(h/a)}$	$\epsilon \frac{W}{2h}$
$Z_0(\Omega)$	$\frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln \left( \frac{b}{a} \right)$	$\frac{1}{\pi} \sqrt{\frac{\mu}{\epsilon}} \operatorname{arcch} \left( \frac{D}{2a} \right)$	$\frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \operatorname{arcch} \left( \frac{h}{a} \right)$	$\frac{2h}{W} \sqrt{\frac{\mu}{\epsilon}}$

## Linhas de comprimento infinito

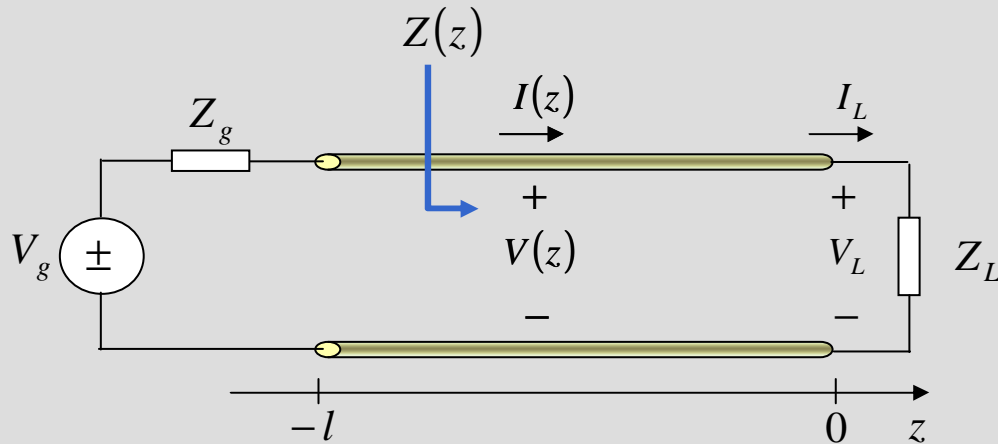


comprimento infinito → não há reflexões

$$\begin{aligned} V(z) &= V_0^+ e^{-\gamma z} \\ I(z) &= I_0^+ e^{-\gamma z} \end{aligned}$$

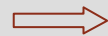
$$Z(z) = \frac{V(z)}{I(z)} = \frac{V_0^+}{I_0^+} = Z_0$$

## Linhas de comprimento finito



$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}$$

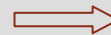
$$I(z) = \frac{V_o^+}{Z_0} e^{-\gamma z} - \frac{V_o^-}{Z_0} e^{\gamma z}$$



$$V_L = V(z=0) = V_o^+ + V_o^-$$

$$I_L = I(z=0) = \frac{V_o^+}{Z_0} - \frac{V_o^-}{Z_0}$$

$$V_L = Z_L I_L$$



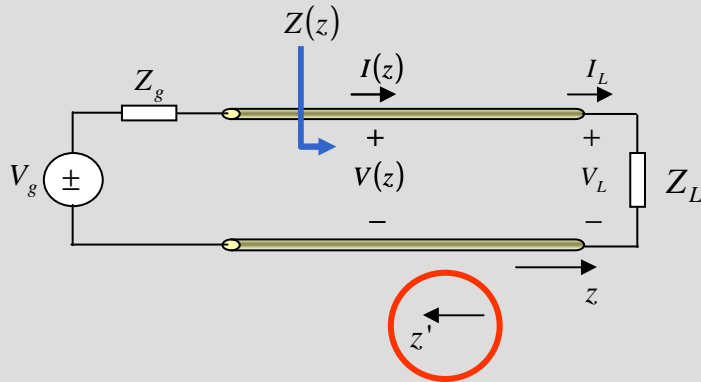
$$\begin{cases} Z_L I_L = V_o^+ + V_o^- \\ Z_0 I_L = V_o^+ - V_o^- \end{cases} \Leftrightarrow \begin{cases} V_o^+ = \frac{1}{2} I_L (Z_L + Z_0) \\ V_o^- = \frac{1}{2} I_L (Z_L - Z_0) \end{cases}$$



$$V(z) = \frac{1}{2} I_L [(Z_L + Z_0) e^{-\gamma z} + (Z_L - Z_0) e^{\gamma z}]$$

$$I(z) = \frac{1}{2Z_0} I_L [(Z_L + Z_0) e^{-\gamma z} - (Z_L - Z_0) e^{\gamma z}]$$

## Impedância ao longo da linha



$$V(z) = \frac{1}{2} I_L [(Z_L + Z_0)e^{-\gamma z} + (Z_L - Z_0)e^{\gamma z}]$$

$$I(z) = \frac{1}{2Z_0} I_L [(Z_L + Z_0)e^{-\gamma z} - (Z_L - Z_0)e^{\gamma z}]$$

$$\Rightarrow Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{(Z_L + Z_0)e^{-\gamma z} + (Z_L - Z_0)e^{\gamma z}}{(Z_L + Z_0)e^{-\gamma z} - (Z_L - Z_0)e^{\gamma z}}$$

$$\Rightarrow Z(z) = Z_0 \frac{(e^{-\gamma z} + e^{\gamma z})Z_L - (e^{-\gamma z} - e^{\gamma z})Z_0}{(e^{-\gamma z} + e^{\gamma z})Z_0 - (e^{-\gamma z} - e^{\gamma z})Z_L}$$

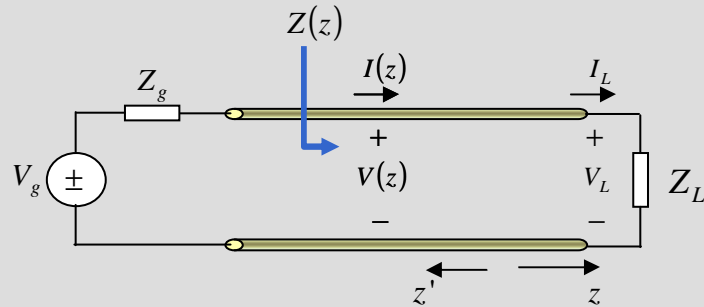
$$\Downarrow \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$Z(z') = Z_0 \frac{Z_L + Z_0 \tanh(\gamma z')}{Z_0 + Z_L \tanh(\gamma z')}$$

$$\leftarrow Z(z) = Z_0 \frac{Z_L - Z_0 \tanh(\gamma z)}{Z_0 - Z_L \tanh(\gamma z)}$$



## Impedância ao longo da linha



$$Z(z') = Z_0 \frac{Z_L + Z_0 \tanh(\gamma z')}{Z_0 + Z_L \tanh(\gamma z')}$$

linhas sem perdas  $\Rightarrow \gamma = j\beta$

$$\tanh(jx) = j \tan(x)$$

$$Z(z') = Z_0 \frac{Z_L + jZ_0 \tan(\beta z')}{Z_0 + jZ_L \tan(\beta z')}$$

linha de comprimento  $l$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

## Impedância de entrada - casos particulares

linha sem perdas, de linha de comprimento  $l$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

$$Z_L = Z_0 \quad \Rightarrow \quad Z_{in} = Z_0$$

$$Z_L = \infty \quad \Rightarrow \quad Z_{in} = -jZ_0 \cotg(\beta l)$$

$$Z_L = 0 \quad \Rightarrow \quad Z_{in} = jZ_0 \tan(\beta l)$$

$$l = n \frac{\lambda}{2} \quad \Rightarrow \quad Z_{in} = Z_L$$

$$l = (2n - 1) \frac{\lambda}{4} \quad \Rightarrow \quad Z_{in} = \frac{Z_0^2}{Z_L}$$

## Coeficiente de reflexão

Coeficiente de reflexão (tensão) → quociente entre tensão reflectida e tensão incidente

na carga:

$$\Gamma_L = \frac{V_{ref}(z=0)}{V_{inc}(z=0)} = \frac{V_o^-}{V_o^+}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$V_o^+ = \frac{1}{2} I_L (Z_L + Z_0)$$

$$V_o^- = \frac{1}{2} I_L (Z_L - Z_0)$$

Notas:

1. Em geral,  $\Gamma_L$  é complexo →  $\Gamma_L = |\Gamma_L| e^{j\theta_\Gamma}$

2. Para a corrente:

$$\Gamma_I = \frac{I_{ref}}{I_{inc}} = \frac{I_0^-}{I_0^+} = \frac{-\frac{V_0^-}{Z_0}}{\frac{V_0^+}{Z_0}} = -\frac{V_0^-}{V_0^+} = -\Gamma_L$$

Casos particulares:

$$Z_L = Z_0 \Rightarrow \Gamma_L = 0$$

**Não há reflexões**  
**Linha adaptada**

$$Z_L = \infty \Rightarrow \Gamma_L = 1$$

$$Z_L = 0 \Rightarrow \Gamma_L = -1$$

## Coeficiente de reflexão

Coeficiente de reflexão (tensão) → quociente entre tensão reflectida e tensão incidente

na carga:

$$\Gamma_L = \frac{V_{ref}(z=0)}{V_{inc}(z=0)} = \frac{V_o^-}{V_o^+}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L| e^{j\theta_\Gamma}$$

ao longo da linha:

$$\Gamma(z) = \frac{V_{ref}(z)}{V_{inc}(z)} = \frac{V_o^- e^{\gamma z}}{V_o^+ e^{-\gamma z}} = \Gamma_L e^{2\gamma z}$$

$$\Gamma(z') = \Gamma_L e^{-2\gamma z'}$$

$$z' = -z$$

linhas sem perdas:  $\gamma = j\beta \Rightarrow$

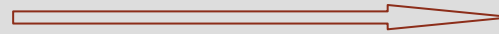
$$\Gamma(z') = |\Gamma_L| e^{j(\theta_\Gamma - 2\beta z')}$$

## Coeficiente de reflexão na carga

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1}$$

seja  $z_L = \frac{Z_L}{Z_0}$

impedância de carga  
normalizada



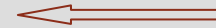
$$Z_L = R_L + jX_L$$

$$Z_0 = R_0 \text{ (linha sem perdas)}$$

$$z_L = r_L + jx_L, \quad r_L \geq 0$$



$$\Gamma_L = \frac{(r_L - 1) + jx_L}{(r_L + 1) + jx_L}$$



$$|\Gamma_L| = \frac{|(r_L - 1) + jx_L|}{|(r_L + 1) + jx_L|} \leq 1$$

$$|\Gamma_L|_{MAX} = 1 \Rightarrow r_L = 0$$

$$|\Gamma_L|_{min} = 0 \Rightarrow Z_L = Z_0$$

$$\longrightarrow 0 \leq |\Gamma_L| \leq 1$$

## Tensão ao longo da linha

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$V(z) = (V_0^+ - V_0^-) e^{-j\beta z} + V_0^- (e^{j\beta z} + e^{-j\beta z})$$

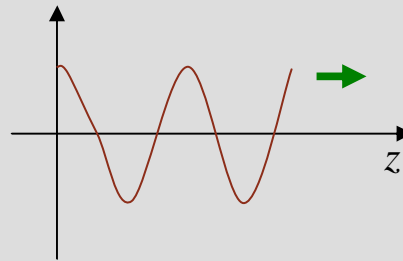
$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$V(z) = (V_0^+ - V_0^-) e^{-j\beta z} + 2V_0^- \cos(\beta z)$$

**onda móvel**      **onda estacionária**

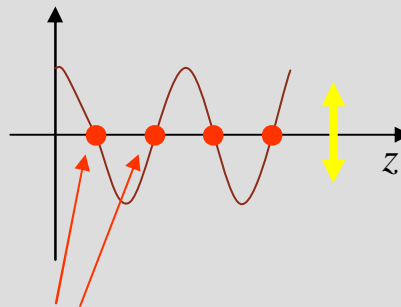
## Ondas móveis e estacionárias

•seja  $V(z) = Ae^{-j\beta z}$   $\longrightarrow$   $v(z,t) = \text{Re}\{Ae^{-j\beta z}e^{j\omega t}\} = \text{Re}\{Ae^{j(\omega t - \beta z)}\} = A\cos(\omega t - \beta z)$



onda móvel

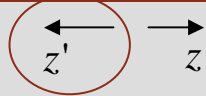
•seja  $V(z) = A\cos(\beta z)$   $\longrightarrow$   $v(z,t) = \text{Re}\{A\cos(\beta z)e^{j\omega t}\} = A\cos(\beta z)\cos(\omega t)$



onda estacionária

**nodos**  
(  $v=0$  para qualquer  $t$  )

## Tensão ao longo da linha



$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

onda móvel + onda estacionária

$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{j2\beta z})$$

$$\begin{aligned} V(z') &= V_0^+ e^{j\beta z'} (1 + \Gamma_L e^{-j2\beta z'}) \\ &= V_0^+ e^{j\beta z'} (1 + |\Gamma_L| e^{j(\theta_\Gamma - 2\beta z')}) \end{aligned}$$

$$\begin{aligned} |V(z')| &= |V_0^+| \sqrt{(1 + |\Gamma_L| \cos(\theta_\Gamma - 2\beta z'))^2 + (|\Gamma_L| \sin(\theta_\Gamma - 2\beta z'))^2} \\ &= |V_0^+| \sqrt{1 + |\Gamma_L|^2 + 2|\Gamma_L| \cos(\theta_\Gamma - 2\beta z')} \end{aligned}$$

periódico  
período =  $\lambda/2$



## Tensão ao longo da linha - exemplo

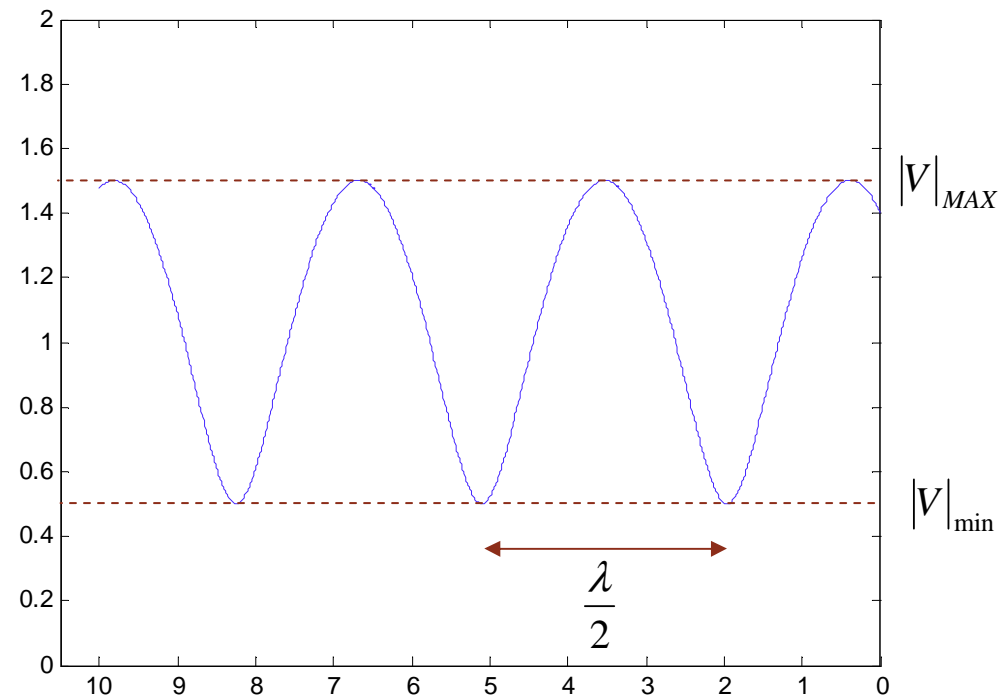
$$|V(z')| = |V_0^+| \sqrt{1 + |\Gamma_L|^2 + 2|\Gamma_L| \cos(\theta_\Gamma - 2\beta z')}$$

para

$$|V_0^+| = 1 \text{ V}$$

$$\Gamma_L = 0.5e^{j\frac{\pi}{4}}$$

$$\beta = 1 \text{ m}^{-1} \quad (\Rightarrow \lambda = 2\pi \text{ m})$$



## Máximos e mínimos de tensão

$$|V(z')| = |V_0^+| \sqrt{1 + |\Gamma_L|^2 + 2|\Gamma_L| \cos(\theta_\Gamma - 2\beta z')}$$

• **máximos de tensão:**  $\cos(\theta_\Gamma - 2\beta z') = +1$

• **localização:**  $\theta_\Gamma - 2\beta z'_M = -2n\pi$

$$z'_M = \frac{1}{2\beta} (2n\pi + \theta_\Gamma)$$

$z' \geq 0$   
 $n$  inteiro

• **valor:**  $|V|_{MAX} = |V_0^+| \sqrt{1 + |\Gamma_L|^2 + 2|\Gamma_L|}$

$$|V|_{MAX} = |V_0^+| (1 + |\Gamma_L|)$$

• **mínimos de tensão:**  $\cos(\theta_\Gamma - 2\beta z') = -1$

• **localização:**  $\theta_\Gamma - 2\beta z'_m = -(2n+1)\pi$

$$z'_m = \frac{1}{2\beta} [(2n+1)\pi + \theta_\Gamma]$$

$z' \geq 0$   
 $n$  inteiro

• **valor:**  $|V|_{min} = |V_0^+| \sqrt{1 + |\Gamma_L|^2 - 2|\Gamma_L|}$

$$|V|_{min} = |V_0^+| (1 - |\Gamma_L|)$$

## Tensão ao longo da linha - exemplo

$$|V(z')| = |V_0^+| \sqrt{1 + |\Gamma_L|^2 + 2|\Gamma_L| \cos(\theta_\Gamma - 2\beta z')}$$

para

$$|V_0^+| = 1 \text{ V}$$

$$\Gamma_L = 0.5e^{j\frac{\pi}{4}}$$

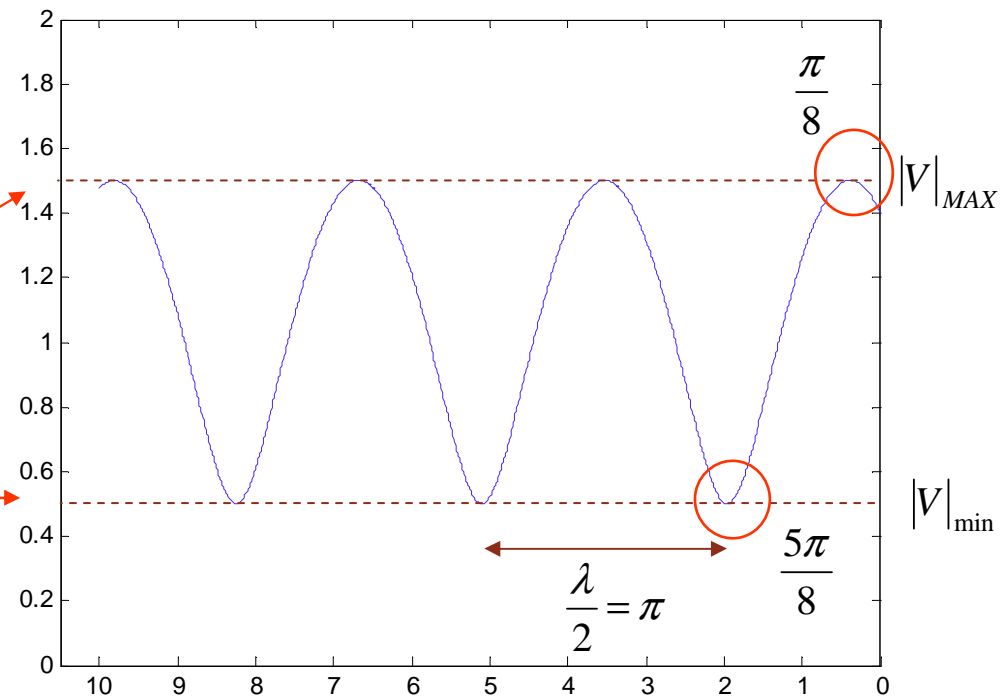
$$\beta = 1 \text{ m}^{-1} \quad (\Rightarrow \lambda = 2\pi \text{ m})$$

$$|V|_{MAX} = |V_0^+|(1 + |\Gamma_L|) = 1.5$$

$$|V|_{min} = |V_0^+|(1 - |\Gamma_L|) = 0.5$$

$$z'_M = n\pi + \frac{\pi}{8}$$

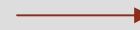
$$z'_m = n\pi + \frac{5\pi}{8}$$



## SWR

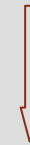
**SWR (Voltage Standing Wave Ratio) :**  
quociente entre tensão máxima e mínima

$$SWR = \frac{|V|_{MAX}}{|V|_{min}} = \frac{|V_0^+|(1 + |\Gamma_L|)}{|V_0^+|(1 - |\Gamma_L|)}$$



$$SWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

**Nota:**  $SWR \geq 1$



$$|\Gamma_L| = \frac{SWR - 1}{SWR + 1}$$

## SWR – casos particulares

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

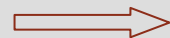
$$SWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

$$|\Gamma_L| = \frac{SWR - 1}{SWR + 1}$$

**Casos particulares:**

$Z_L = Z_0 \implies \Gamma_L = 0 \implies SWR = 1 \implies |V|_{MAX} = |V|_{min} \implies$  **não há onda estacionária**  
 $\implies$  **não há reflexões**

$$SWR = 1 \implies |\Gamma_L| = 0$$



**linha adaptada**  $\iff SWR = 1$

## SWR – casos particulares

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$SWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

$$|\Gamma_L| = \frac{SWR - 1}{SWR + 1}$$

Casos particulares:

$$Z_L = \infty \implies \Gamma_L = 1 \implies \boxed{SWR = \infty}$$

$$|V|_{MAX} = |V_0^+|(1 + |\Gamma_L|) = 2|V_0^+|$$

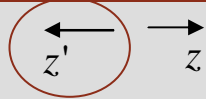
$$|V|_{min} = |V_0^+|(1 - |\Gamma_L|) = 0$$

$$Z_L = 0 \implies \Gamma_L = -1 \implies \boxed{SWR = \infty}$$

$$|V|_{MAX} = 2|V_0^+|$$

$$|V|_{min} = 0$$

## Corrente ao longo da linha



$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}$$

onda móvel + onda estacionária

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} (1 - \Gamma_L e^{j2\beta z})$$

$$\begin{aligned} I(z') &= \frac{V_0^+}{Z_0} e^{j\beta z'} (1 - \Gamma_L e^{-j2\beta z'}) \\ &= \frac{V_0^+}{Z_0} e^{j\beta z'} (1 - |\Gamma_L| e^{j(\theta_\Gamma - 2\beta z')}) \end{aligned}$$

$$\begin{aligned} |I(z')| &= \frac{|V_0^+|}{Z_0} \sqrt{(1 - |\Gamma_L| \cos(\theta_\Gamma - 2\beta z'))^2 + (|\Gamma_L| \sin(\theta_\Gamma - 2\beta z'))^2} \\ &= \frac{|V_0^+|}{Z_0} \sqrt{1 + |\Gamma_L|^2 - 2|\Gamma_L| \cos(\theta_\Gamma - 2\beta z')} \end{aligned}$$

periódico  
período =  $\lambda/2$

## Máximos e mínimos de corrente

$$|I(z')| = \frac{|V_0^+|}{Z_0} \sqrt{1 + |\Gamma_L|^2 - 2|\Gamma_L| \cos(\theta_\Gamma - 2\beta z')}$$

• **máximos de corrente:**  $\cos(\theta_\Gamma - 2\beta z') = -1$

• localização:  $\theta_\Gamma - 2\beta z' = -(2n+1)\pi$

$$z' = \frac{1}{2\beta} [(2n+1)\pi + \theta_\Gamma]$$

$z' \geq 0$   
 $n$  inteiro

• valor:  $|I|_{MAX} = \frac{|V_0^+|}{Z_0} \sqrt{1 + |\Gamma_L|^2 + 2|\Gamma_L|}$

$$|I|_{MAX} = \frac{|V_0^+|}{Z_0} (1 + |\Gamma_L|)$$

• **mínimos de corrente:**  $\cos(\theta_\Gamma - 2\beta z') = 1$

• localização:  $\theta_\Gamma - 2\beta z' = -2n\pi$

$$z' = \frac{1}{2\beta} (2n\pi + \theta_\Gamma)$$

$z' \geq 0$   
 $n$  inteiro

• valor:  $|I|_{min} = \frac{|V_0^+|}{Z_0} \sqrt{1 + |\Gamma_L|^2 - 2|\Gamma_L|}$

$$|I|_{min} = \frac{|V_0^+|}{Z_0} (1 - |\Gamma_L|)$$



## Tensão e corrente – localização de máximos e mínimos

$$|V(z')| = |V_0^+| \sqrt{1 + |\Gamma_L|^2 + 2|\Gamma_L| \cos(\theta_\Gamma - 2\beta z')}$$

$$|I(z')| = \frac{|V_0^+|}{Z_0} \sqrt{1 + |\Gamma_L|^2 - 2|\Gamma_L| \cos(\theta_\Gamma - 2\beta z')}$$

$$\cos(\theta_\Gamma - 2\beta z') = 1$$



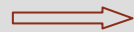
**máximos de tensão  
e  
mínimos de corrente**

$$z'_M = \frac{1}{2\beta} (2n\pi + \theta_\Gamma)$$

$$z' \geq 0$$

$n$  inteiro

$$\cos(\theta_\Gamma - 2\beta z') = -1$$



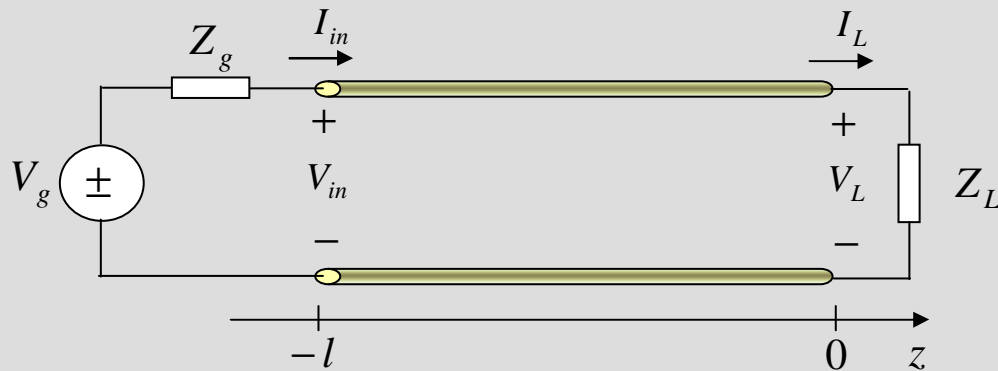
**mínimos de tensão  
e  
máximos de corrente**

$$z'_m = \frac{1}{2\beta} [(2n+1)\pi + \theta_\Gamma]$$

$$z' \geq 0$$

$n$  inteiro

## Linhas em circuitos



$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}$$

$$I(z) = \frac{V_o^+}{Z_0} e^{-\gamma z} - \frac{V_o^-}{Z_0} e^{\gamma z}$$

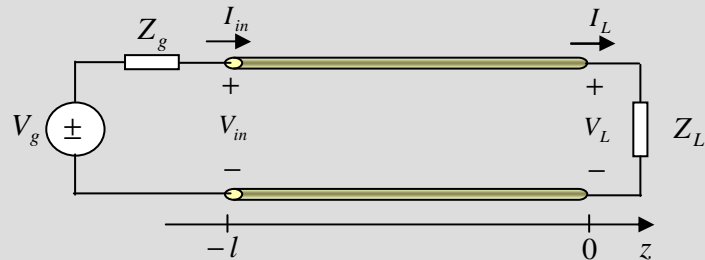
$$V_o^+ = \frac{1}{2} I_L (Z_L + Z_0)$$

$$\begin{aligned} V_g &= I_{in} Z_g + V_{in} \\ V_{in} &= V(z = -l) \\ I_{in} &= I(z = -l) \end{aligned}$$

$$\begin{aligned} V_{in} &= \frac{1}{2} I_L (Z_L + Z_0) e^{\gamma l} [1 + \Gamma_L e^{-2\gamma l}] \\ I_{in} &= \frac{1}{2Z_0} I_L (Z_L + Z_0) e^{\gamma l} [1 - \Gamma_L e^{-2\gamma l}] \end{aligned}$$

$$V_g = \frac{1}{2Z_0} I_L (Z_L + Z_0) e^{\gamma l} [Z_g (1 - \Gamma_L e^{-2\gamma l}) + Z_0 (1 + \Gamma_L e^{-2\gamma l})]$$

## Linhas em circuitos



$$V_g = \frac{1}{2Z_0} I_L (Z_L + Z_0) e^{\gamma l} \left[ Z_g (1 - \Gamma_L e^{-2\gamma l}) + Z_0 (1 + \Gamma_L e^{-2\gamma l}) \right]$$

$$\frac{1}{2} I_L (Z_L + Z_0) e^{\gamma l} = \frac{Z_0}{Z_g + Z_0} \left[ \frac{V_g}{1 - \Gamma_g \Gamma_L e^{-2\gamma l}} \right]$$

$$\frac{1}{2} I_L (Z_L + Z_0) e^{\gamma l} = \left[ \frac{Z_0 V_g}{Z_g + Z_0 + (Z_0 - Z_g) \Gamma_L e^{-2\gamma l}} \right]$$

$$V_0^+ = \frac{1}{2} I_L (Z_L + Z_0)$$

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}$$

**(coeficiente de reflexão no gerador)**

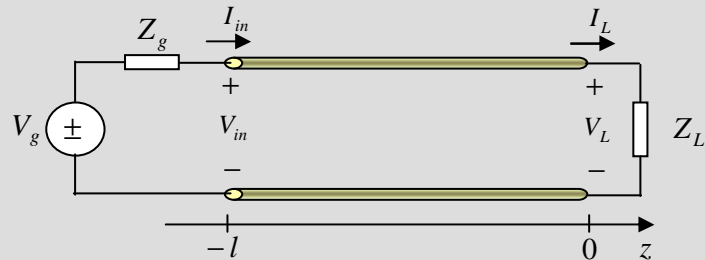
$$V(z) = \frac{Z_0 V_g e^{-\gamma l}}{Z_0 + Z_g} e^{-\gamma z} \left( \frac{1 + \Gamma_L e^{2\gamma z}}{1 - \Gamma_g \Gamma_L e^{-2\gamma l}} \right)$$

$$I(z) = \frac{V_g e^{-\gamma l}}{Z_0 + Z_g} e^{-\gamma z} \left( \frac{1 - \Gamma_L e^{2\gamma z}}{1 - \Gamma_g \Gamma_L e^{-2\gamma l}} \right)$$

**tensão e corrente definidas à custa de**

$$Z_0, \gamma, l, Z_L, V_g$$

## Linhas em circuitos



$$V(z) = \frac{Z_0 V_g e^{-\gamma l}}{Z_0 + Z_g} e^{-\gamma z} \left( \frac{1 + \Gamma_L e^{2\gamma z}}{1 - \Gamma_g \Gamma_L e^{-2\gamma l}} \right)$$

$$I(z) = \frac{V_g e^{-\gamma l}}{Z_0 + Z_g} e^{-\gamma z} \left( \frac{1 - \Gamma_L e^{2\gamma z}}{1 - \Gamma_g \Gamma_L e^{-2\gamma l}} \right)$$

$$V(z) = \frac{Z_0 V_g e^{-\gamma l}}{Z_0 + Z_g} e^{-\gamma z} (1 + \Gamma_L e^{2\gamma z}) (1 - \Gamma_g \Gamma_L e^{-2\gamma l})^{-1}$$

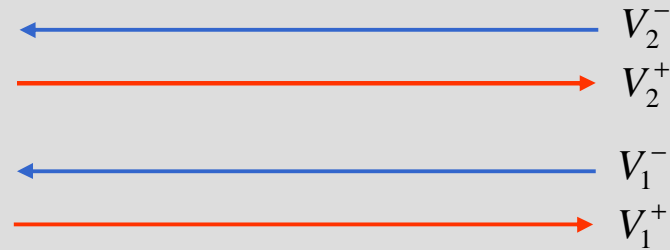
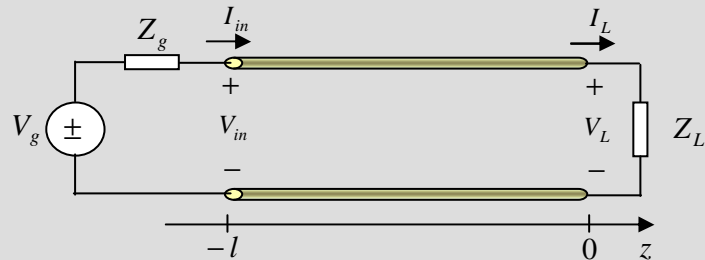


$$\longleftarrow \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$= \frac{Z_0 V_g e^{-\gamma l}}{Z_0 + Z_g} (e^{-\gamma z} + \Gamma_L e^{\gamma z}) \left[ 1 + \Gamma_g \Gamma_L e^{-2\gamma l} + (\Gamma_g \Gamma_L e^{-2\gamma l})^2 + \dots \right]$$

$$= \frac{Z_0 V_g e^{-\gamma l}}{Z_0 + Z_g} \left[ e^{-\gamma z} + \Gamma_L e^{\gamma z} + (\Gamma_g \Gamma_L e^{-2\gamma l}) e^{-\gamma z} + (\Gamma_g \Gamma_L e^{-2\gamma l}) \Gamma_L e^{\gamma z} + (\Gamma_g \Gamma_L e^{-2\gamma l})^2 e^{-\gamma z} + (\Gamma_g \Gamma_L e^{-2\gamma l})^2 \Gamma_L e^{\gamma z} + \dots \right]$$

## Linhas em circuitos



$$V(z) = \frac{Z_0 V_g e^{-\gamma l}}{Z_0 + Z_g} \left[ \begin{array}{cccccccc} e^{-\gamma z} & + & \Gamma_L e^{\gamma z} & + & (\Gamma_g \Gamma_L e^{-2\gamma l}) e^{-\gamma z} & + & (\Gamma_g \Gamma_L e^{-2\gamma l}) \Gamma_L e^{\gamma z} & + & (\Gamma_g \Gamma_L e^{-2\gamma l})^2 e^{-\gamma z} & + & (\Gamma_g \Gamma_L e^{-2\gamma l})^2 \Gamma_L e^{\gamma z} & + \dots \end{array} \right]$$

→   ←   →   ←   →   ←

$$V(z) = V_1^+ e^{-\gamma z} + V_1^- e^{\gamma z} + V_2^+ e^{-\gamma z} + V_2^- e^{\gamma z} + V_3^+ e^{-\gamma z} + V_3^- e^{\gamma z} + \dots$$

$$V_1^+ = \frac{Z_0 V_g e^{-\gamma l}}{Z_0 + Z_g}$$

$$V_2^+ = \Gamma_g e^{-2\gamma l} V_1^-$$

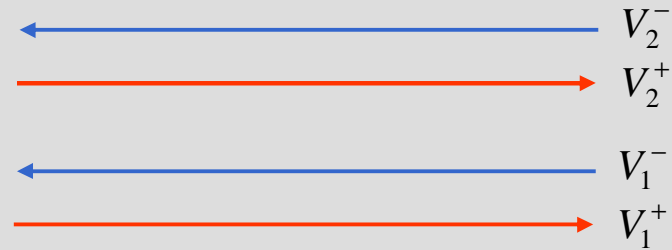
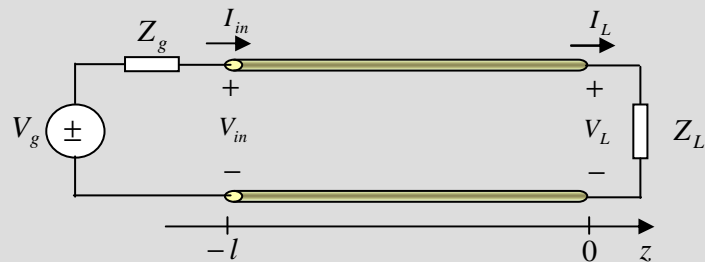
$$V_3^+ = \Gamma_g e^{-2\gamma l} V_2^-$$

$$V_1^- = \Gamma_L V_1^+$$

$$V_2^- = \Gamma_L V_2^+$$

$$V_3^- = \Gamma_L V_3^+$$

## Linhas em circuitos



$$V(z) = V_1^+ e^{-\gamma z} + V_1^- e^{\gamma z} + V_2^+ e^{-\gamma z} + V_2^- e^{\gamma z} + V_3^+ e^{-\gamma z} + V_3^- e^{\gamma z} + \dots$$

$$V(z) = (V_1^+ + V_2^+ + V_3^+ + \dots) e^{-\gamma z} + (V_1^- + V_2^- + V_3^- + \dots) e^{\gamma z} = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

## Potência na linha de transmissão – linha sem perdas

$$P_{med}(z') = \frac{1}{2} \operatorname{Re}\{V(z')I^*(z')\}$$

$$V(z') = V_0^+ e^{j\beta z'} (1 + |\Gamma_L| e^{j(\theta_\Gamma - 2\beta z')})$$

$$I(z') = \frac{V_0^+}{Z_0} e^{j\beta z'} (1 - |\Gamma_L| e^{j(\theta_\Gamma - 2\beta z')})$$

(linha sem perdas)

$$P_{med}(z') = \frac{1}{2} \operatorname{Re}\left\{ V_0^+ e^{j\beta z'} (1 + |\Gamma_L| e^{j(\theta_\Gamma - 2\beta z')}) \frac{(V_0^+)^*}{Z_0} e^{-j\beta z'} (1 - |\Gamma_L| e^{-j(\theta_\Gamma - 2\beta z')}) \right\}$$

$$= \frac{|V_0^+|^2}{2Z_0} \operatorname{Re}\left\{ 1 - |\Gamma_L|^2 + |\Gamma_L| (e^{j(\theta_\Gamma - 2\beta z')} - e^{-j(\theta_\Gamma - 2\beta z')}) \right\}$$

$$= \frac{|V_0^+|^2}{2Z_0} \operatorname{Re}\left\{ 1 - |\Gamma_L|^2 + 2j|\Gamma_L| \sin(\theta_\Gamma - 2\beta z') \right\}$$

$$P_{med}(z') = \frac{|V_0^+|^2}{2Z_0} (1 - |\Gamma_L|^2) = \text{constante}$$

incidente

reflectida

potência absorvida na carga



potência fornecida pelo gerador à linha

## Potência na linha de transmissão – caso geral

$$P_{med}(z') = \frac{1}{2} \operatorname{Re}\{V(z')I^*(z')\}$$

$$V(z') = V_0^+ e^{\alpha z'} e^{j\beta z'} (1 + |\Gamma_L| e^{-2\alpha z'} e^{j(\theta_\Gamma - 2\beta z')})$$

$$I(z') = \frac{V_0^+}{Z_0} e^{\alpha z'} e^{j\beta z'} (1 - |\Gamma_L| e^{-2\alpha z'} e^{j(\theta_\Gamma - 2\beta z')})$$

$$P_{med}(z') = \frac{1}{2} \operatorname{Re}\left\{ V_0^+ e^{\alpha z'} e^{j\beta z'} (1 + |\Gamma_L| e^{-2\alpha z'} e^{j(\theta_\Gamma - 2\beta z')}) \frac{(V_0^+)^*}{Z_0} e^{\alpha z'} e^{-j\beta z'} (1 - |\Gamma_L| e^{-2\alpha z'} e^{-j(\theta_\Gamma - 2\beta z')}) \right\}$$

$$= \frac{|V_0^+|^2}{2R_0} e^{2\alpha z'} \operatorname{Re}\left\{ 1 - |\Gamma_L|^2 e^{-4\alpha z'} + j2|\Gamma_L| e^{-2\alpha z'} \sin(\theta_\Gamma - 2\beta z') \right\}$$

se  $Z_0 = R_0$

$$P_{med}(z') = \frac{|V_0^+|^2}{2R_0} (e^{2\alpha z'} - |\Gamma_L|^2 e^{-2\alpha z'})$$

$$P_{med,L} = P_{med}(z'=0) = \frac{|V_0^+|^2}{2R_0} (1 - |\Gamma_L|^2)$$

$$P_{med,in} = P_{med}(z'=l) = \frac{|V_0^+|^2}{2R_0} (e^{2\alpha l} - |\Gamma_L|^2 e^{-2\alpha l})$$



## Impedância de carga $\Leftrightarrow$ coeficiente de reflexão

$$\Gamma_L = \frac{z_L - 1}{z_L + 1}$$

onde  $z_L = \frac{Z_L}{Z_0}$  (impedância de carga normalizada)

$$z_L = \frac{1 + \Gamma_L}{1 - \Gamma_L}$$

$$Z_L = R_L + jX_L$$

$$Z_0 = R_0 \text{ (linha sem perdas)}$$

$$z_L = r_L + jx_L$$

$$\Gamma_L = |\Gamma_L| e^{j\theta_\Gamma} = \Gamma_{re} + j\Gamma_{im}$$

$$r_L + jx_L = \frac{(1 + \Gamma_{re}) + j\Gamma_{im}}{(1 - \Gamma_{re}) - j\Gamma_{im}}$$

$$r_L = \frac{1 - \Gamma_{re}^2 - \Gamma_{im}^2}{(1 - \Gamma_{re})^2 + \Gamma_{im}^2}$$

$$x_L = \frac{2\Gamma_{im}}{(1 - \Gamma_{re})^2 + \Gamma_{im}^2}$$

## Impedância de carga $\Leftrightarrow$ coeficiente de reflexão

$$r_L = \frac{1 - \Gamma_{re}^2 - \Gamma_{im}^2}{(1 - \Gamma_{re})^2 + \Gamma_{im}^2}$$

$$x_L = \frac{2\Gamma_{im}}{(1 - \Gamma_{re})^2 + \Gamma_{im}^2}$$

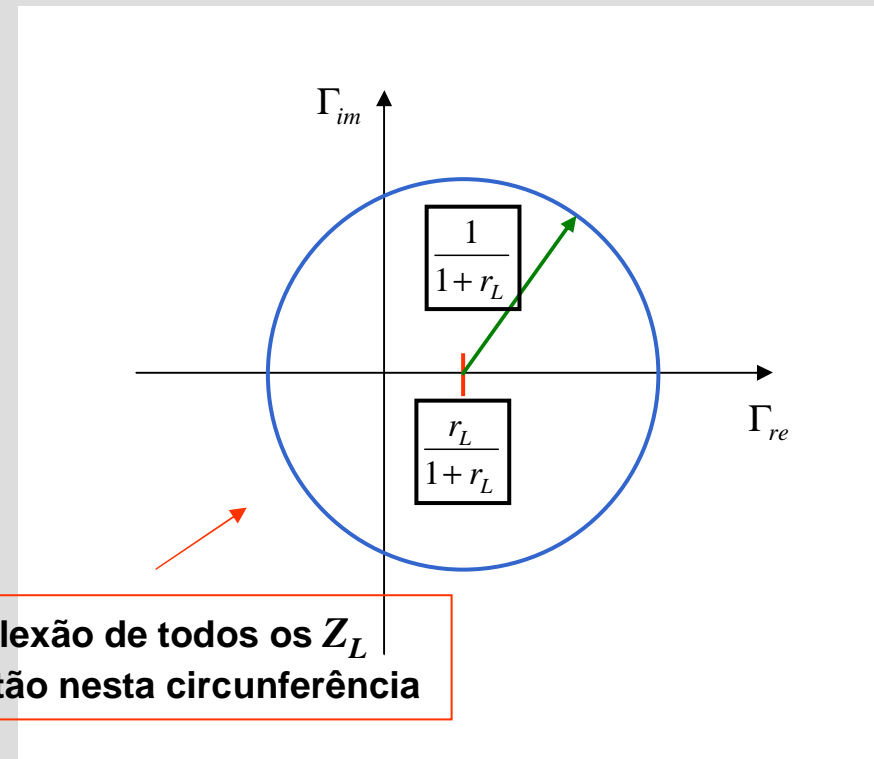
$$\left( \Gamma_{re} - \frac{r_L}{1 + r_L} \right)^2 + \Gamma_{im}^2 = \left( \frac{1}{1 + r_L} \right)^2$$

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$



**circunferência de raio  $1/(1 + r_L)$**

**centrada em  $\Gamma_{re} = r_L/(1 + r_L)$   
 $\Gamma_{im} = 0$**



**Os coeficientes de reflexão de todos os  $Z_L$  cuja parte real é  $r_L$  estão nesta circunferência**

## Impedância de carga $\leftrightarrow$ coeficiente de reflexão

$$\left( \Gamma_{re} - \frac{r_L}{1+r_L} \right)^2 + \Gamma_{im}^2 = \left( \frac{1}{1+r_L} \right)^2$$

Notas:

curva não depende de  $x_L$

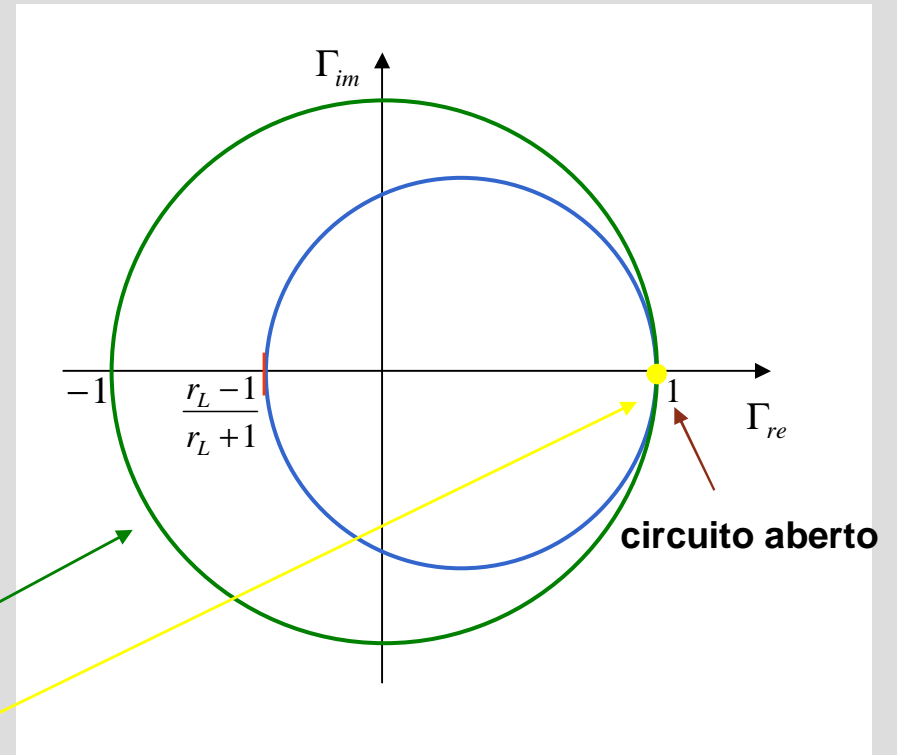
$$\Gamma_{im} = 0 \Rightarrow \Gamma_{re,e} = \frac{r_L - 1}{r_L + 1} \vee \Gamma_{re,d} = 1$$

$\Downarrow$

$$r_L = 0 \Rightarrow \Gamma_{re,e} = -1$$

$$r_L = \infty \Rightarrow \Gamma_{re,e} = 1$$

para qualquer  $Z_L$



## Impedância de carga $\Leftrightarrow$ coeficiente de reflexão

$$r_L = \frac{1 - \Gamma_{re}^2 - \Gamma_{im}^2}{(1 - \Gamma_{re})^2 + \Gamma_{im}^2}$$

$$x_L = \frac{2\Gamma_{im}}{(1 - \Gamma_{re})^2 + \Gamma_{im}^2}$$

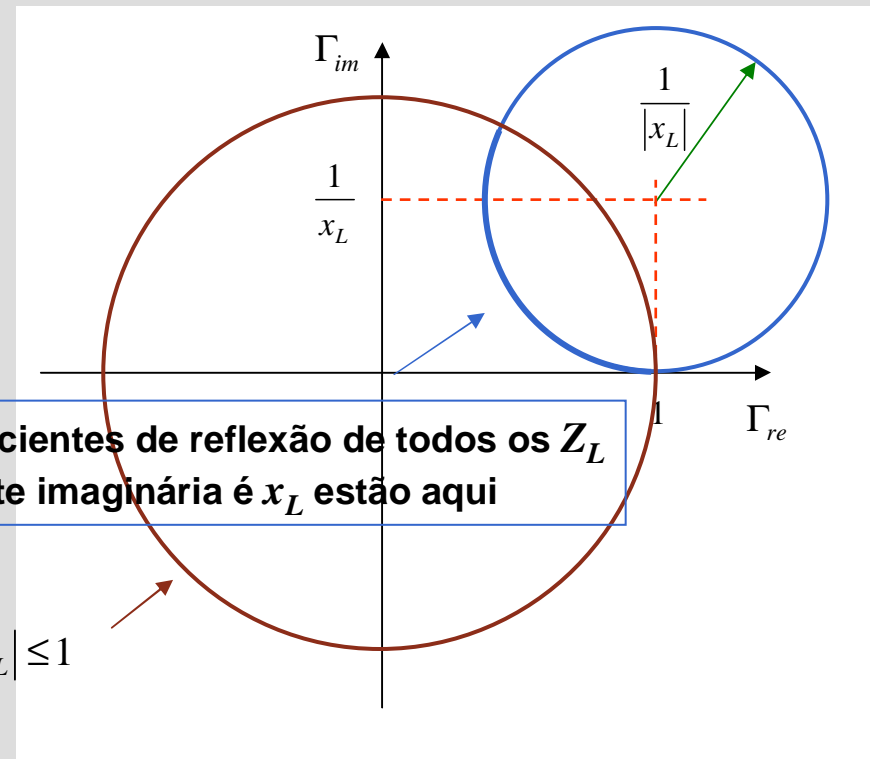
$$(\Gamma_{re} - 1)^2 + \left(\Gamma_{im} - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2$$

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$



**circunferência de raio**  $1/|x_L|$

**centrado em**  $\Gamma_{re} = 1$   
 $\Gamma_{im} = 1/x_L$



## Impedância de carga $\Leftrightarrow$ coeficiente de reflexão

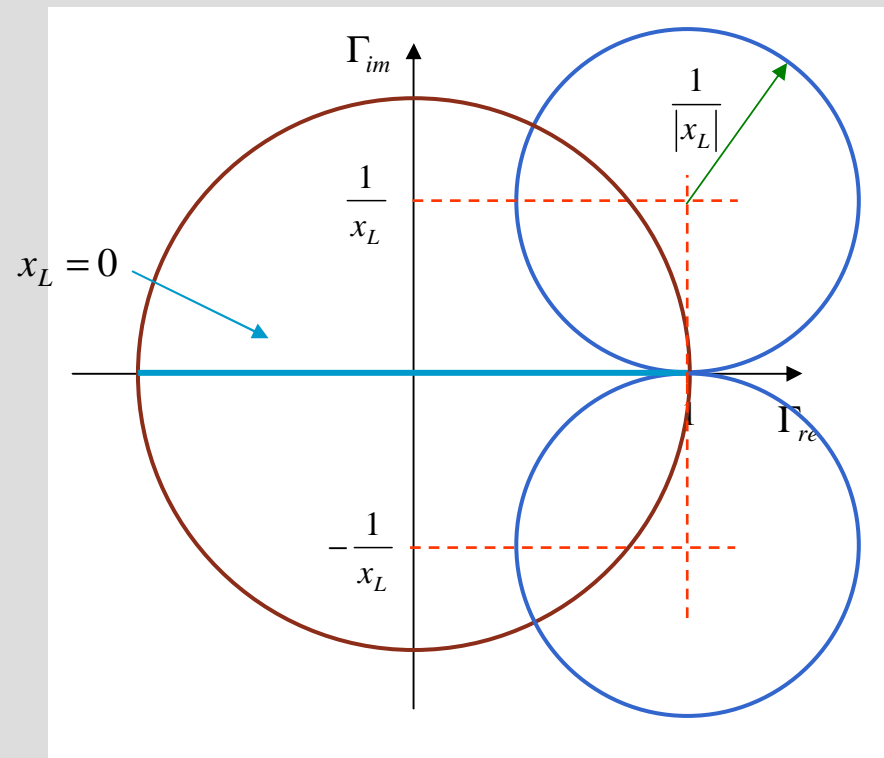
$$(\Gamma_{re} - 1)^2 + \left( \Gamma_{im} - \frac{1}{x_L} \right)^2 = \left( \frac{1}{x_L} \right)^2$$

### Notas:

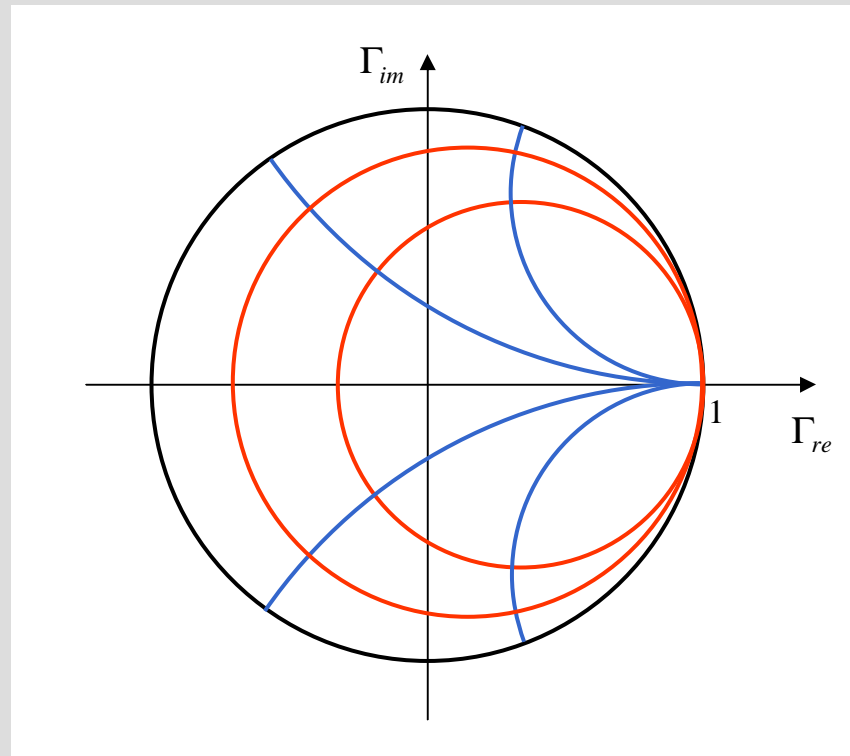
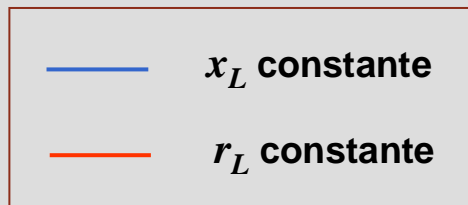
curva não depende de  $r_L$

curvas simétricas para  $x_L < 0$

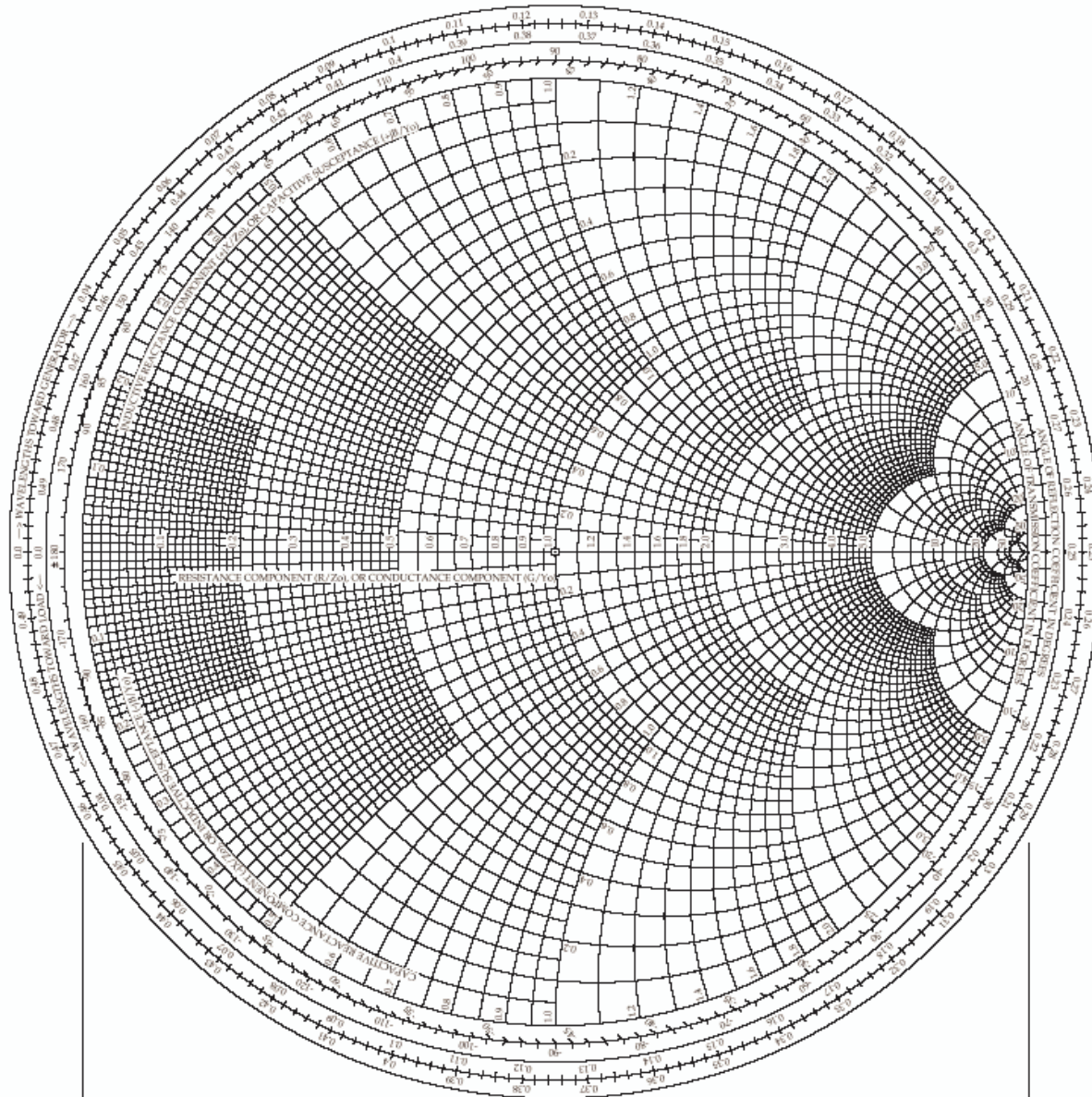
$x_L = 0 \Rightarrow$  raio infinito



## Diagrama de Smith



# Diagrama de Smith



## Diagrama de Smith

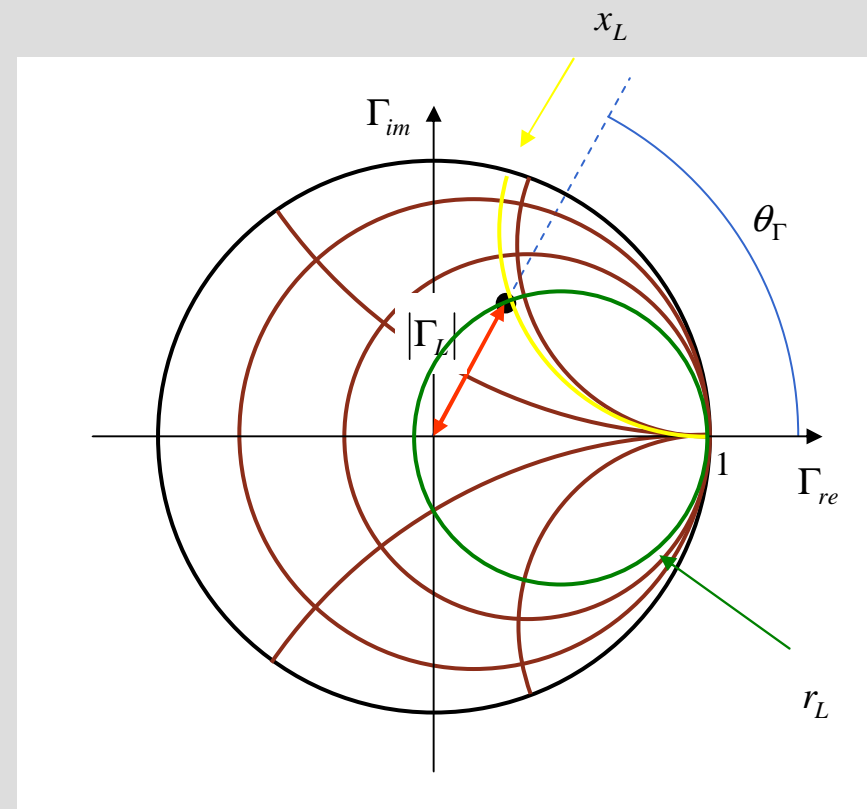
•a partir de:

$Z_L$  → ponto no diagrama ( intersecção de curvas referentes a  $r_L$  e  $x_L$  )

↓  
 $|\Gamma_L|$  e  $\theta_\Gamma$

•a partir de:

$\Gamma_L$  →  $r_L$  e  $x_L$





## Impedância ao longo da linha

ao longo da linha:

$$\Gamma(z) = \frac{V_{ref}(z)}{V_{inc}(z)} = \frac{V_o^- e^{z}}{V_o^+ e^{-z}} = \Gamma_L e^{2z}$$

$$\Gamma(z') = \Gamma_L e^{-2z'}$$

$$z' = -z$$

linhas sem perdas:  $\gamma = j\beta \Rightarrow$

$$\Gamma(z') = |\Gamma_L| e^{j(\theta_\Gamma - 2\beta z')}$$

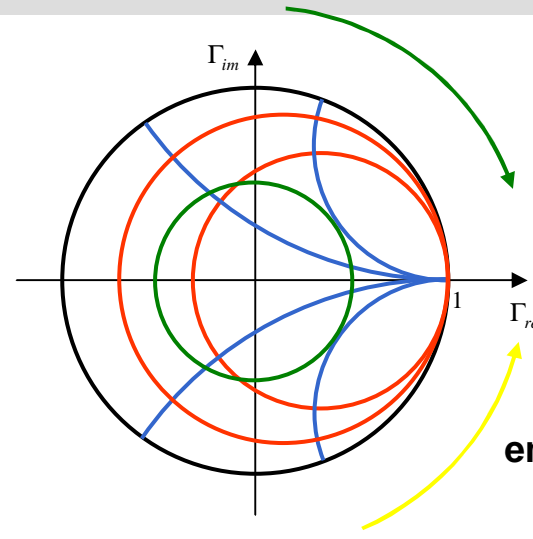
módulo constante  
fase diminui com  $z'$

**Nota:**

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$



diagrama de Smith pode ser usado para determinar  $Z(z)$  a partir de  $\Gamma(z)$



em direcção ao gerador

em direcção à carga

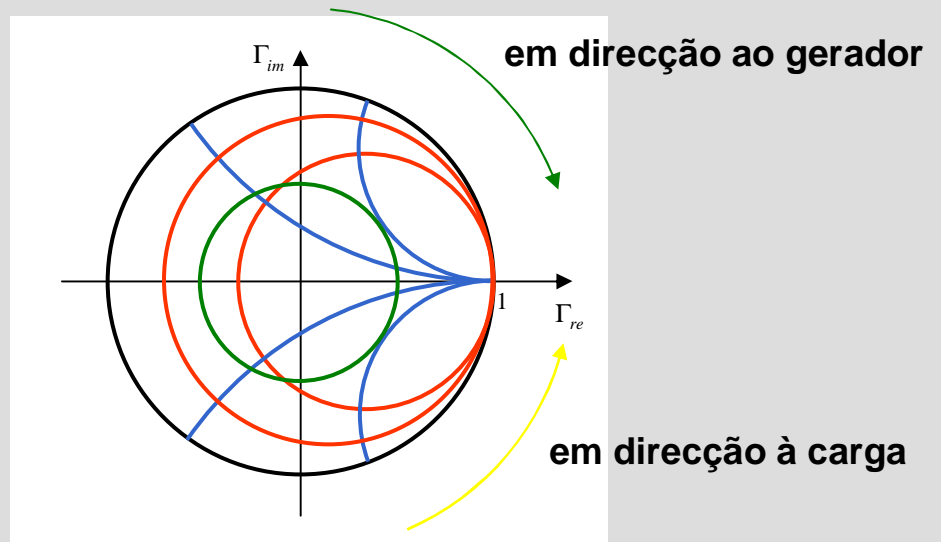
## Impedância ao longo da linha

No diagrama de Smith as distâncias medem-se em fracções de  $\lambda$

$$\Gamma(z') = |\Gamma_L| e^{j(\theta_r - 2\beta z')} \quad \Rightarrow \quad \text{quando } 2\beta z' = 2\pi \quad \longrightarrow \quad \text{posição inicial}$$

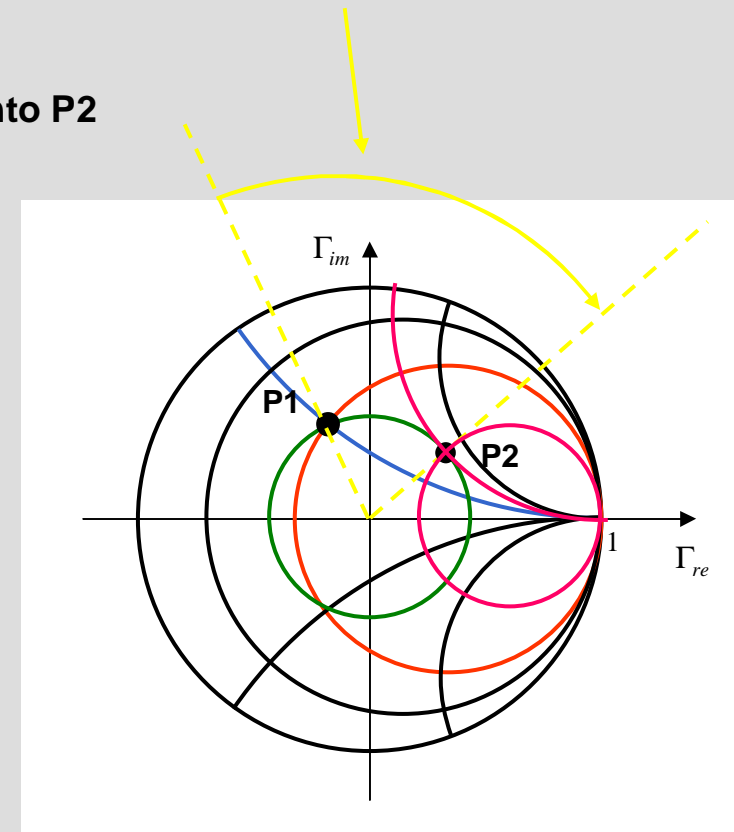
$$z' = \frac{2\pi}{2\beta} = \frac{\lambda}{2}$$

$\Rightarrow$  a uma volta completa ( $360^\circ$ )  
corresponde distância =  $\lambda/2$



## Impedância de entrada

1. marcar no diagrama ponto correspondente à impedância de carga normalizada,  $z_L \rightarrow$  ponto P1
2. desenhar circunferência centrada na origem que passa por P1
3. traçar segmento de recta que passa pela origem e por P1
4. traçar segmento de recta que passa na origem e corresponde a rotação de  $l$  em direcção ao gerador
5. intersecção da circunferência com este segmento  $\rightarrow$  ponto P2
6.  $Z_{in} = z_{in} \cdot Z_0$ , onde  $z_{in}$  é directamente obtido de P2



## Admitância

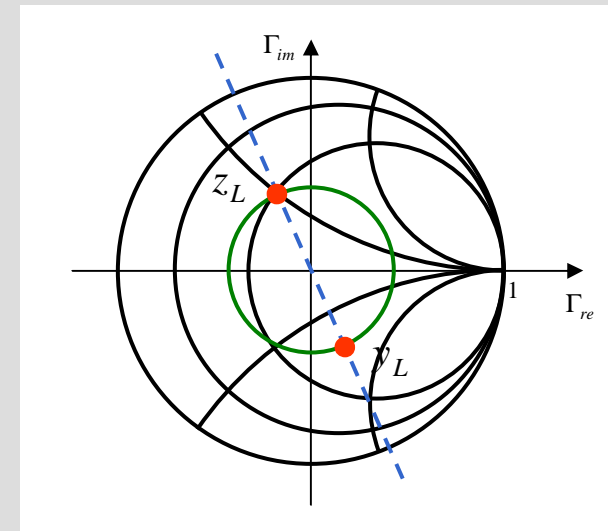
$$Z(z') = Z_0 \frac{Z_L + jZ_0 \tan(\beta z')}{Z_0 + jZ_L \tan(\beta z')} \longrightarrow Z\left(z' = \frac{\lambda}{4}\right) = Z_0 \frac{Z_L + jZ_0 \tan(\pi/2)}{Z_0 + jZ_L \tan(\pi/2)} = \frac{Z_0^2}{Z_L}$$

$$z(\lambda/4) = y_L$$

$$\frac{Z(\lambda/4)}{Z_0} = \frac{Z_0}{Z_L}$$

$$\begin{aligned} \lambda/2 &\Leftrightarrow 360^\circ \\ &\downarrow \\ \lambda/4 &\Leftrightarrow 180^\circ \end{aligned}$$

1. marcar  $z_L$
2. rodar  $180^\circ$



## Localização de máximos e mínimos

$$|V(z')| = |V_0^+| \sqrt{1 + |\Gamma_L|^2 + 2|\Gamma_L| \cos(\theta_\Gamma - 2\beta z')}$$

$$|I(z')| = \frac{|V_0^+|}{Z_0} \sqrt{1 + |\Gamma_L|^2 - 2|\Gamma_L| \cos(\theta_\Gamma - 2\beta z')}$$

$\cos(\theta_\Gamma - 2\beta z') = 1 \rightarrow$  **máximos de tensão e mínimos de corrente**

$\cos(\theta_\Gamma - 2\beta z') = -1 \rightarrow$  **mínimos de tensão e máximos de corrente**

$$\Gamma(z') = |\Gamma_L| e^{j(\theta_\Gamma - 2\beta z')}$$



**máximos de tensão quando**  $\angle \Gamma(z') = 2n\pi$

**mínimos de tensão quando**  $\angle \Gamma(z') = (2n + 1)\pi$

## Localização de máximos e mínimos

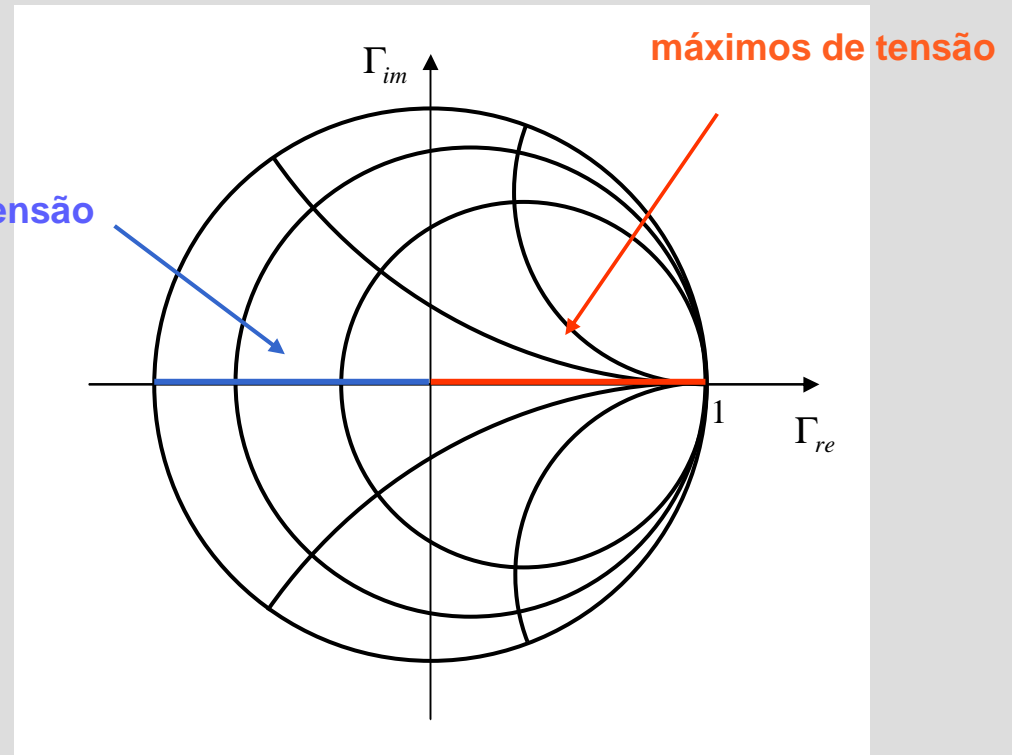
máximos de tensão quando  $\angle\Gamma(z') = 2n\pi$

mínimos de tensão quando  $\angle\Gamma(z') = (2n+1)\pi$

### Notas:

1. máximos e mínimos ocorrem quando a impedância da linha é real
2. como seria de esperar, pontos de máximos (mínimos) estão separados por  $n\lambda/2$

mínimos de tensão



## Adaptação de linhas de transmissão

linha sem perdas:

$$P_{med} = \frac{|V_0^+|^2}{2Z_0} (1 - |\Gamma_L|^2)$$

onde  $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$

$\Gamma_L = 0$

potência entregue à carga é máxima

é vantajoso que a linha esteja adaptada

$$Z_L = Z_0$$

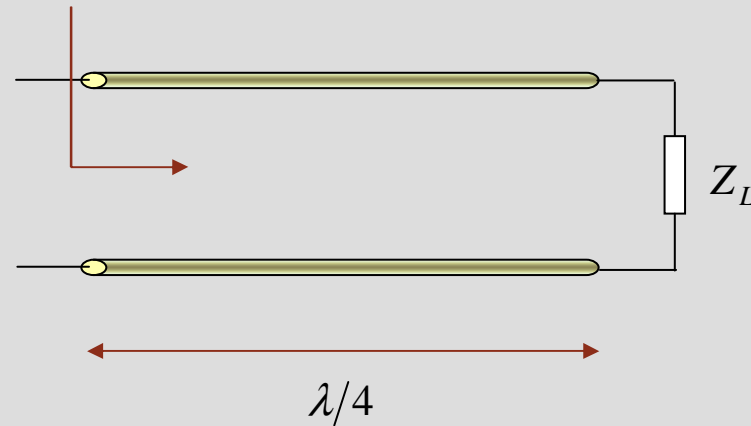
adaptação de linhas:

1. entrega de potência máxima à carga
2. evitar distorção dos sinais a transmitir

## Adaptação com linhas $\lambda/4$

$$Z(z') = Z_0 \frac{Z_L + jZ_0 \tan(\beta z')}{Z_0 + jZ_L \tan(\beta z')}$$

$$Z\left(z' = \frac{\lambda}{4}\right) = Z_0 \frac{Z_L + jZ_0 \tan(\pi/2)}{Z_0 + jZ_L \tan(\pi/2)} = \frac{Z_0^2}{Z_L}$$

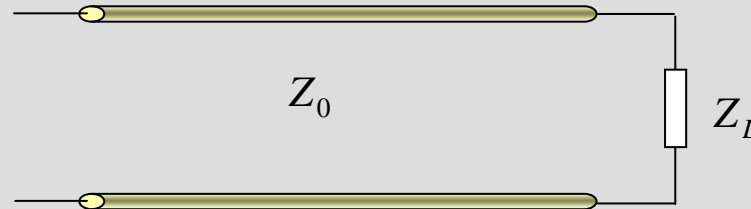


linha de quarto de comprimento  
pode ser usada para adaptar  
impedâncias

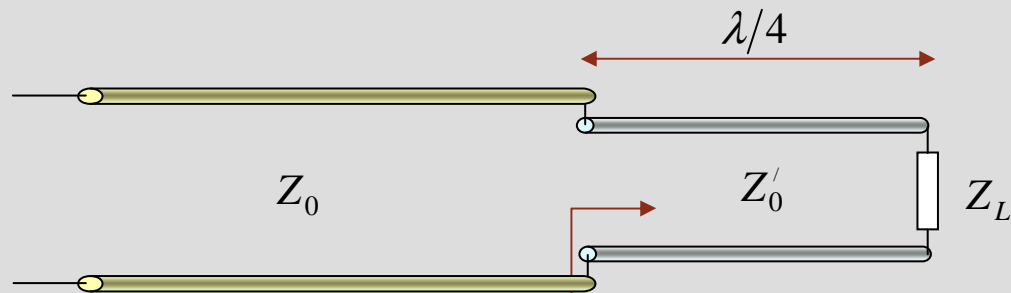


## Adaptação com linhas $\lambda/4$

linha desadaptada  $Z_L \neq Z_0$



linha adaptada

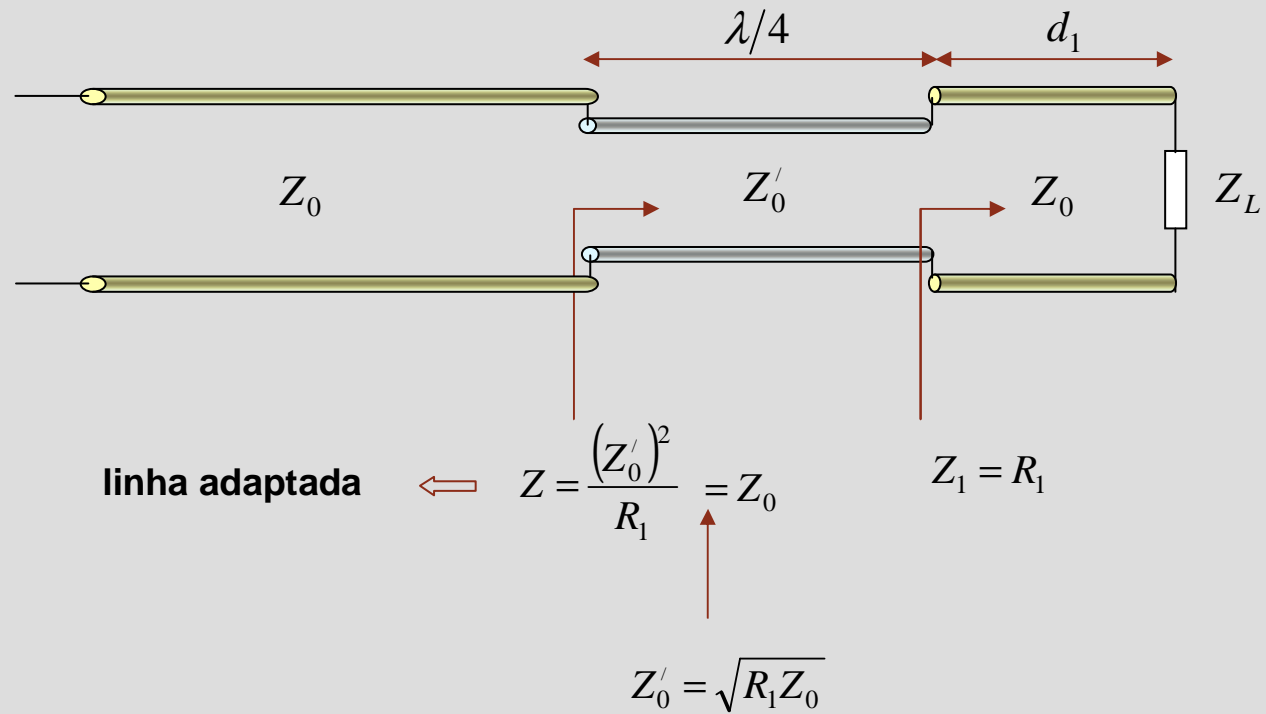


$$Z = \frac{(Z'_0)^2}{Z_L} = Z_0$$

$$Z'_0 = \sqrt{Z_L Z_0}$$

Nota:  $Z_L$  tem de ser real

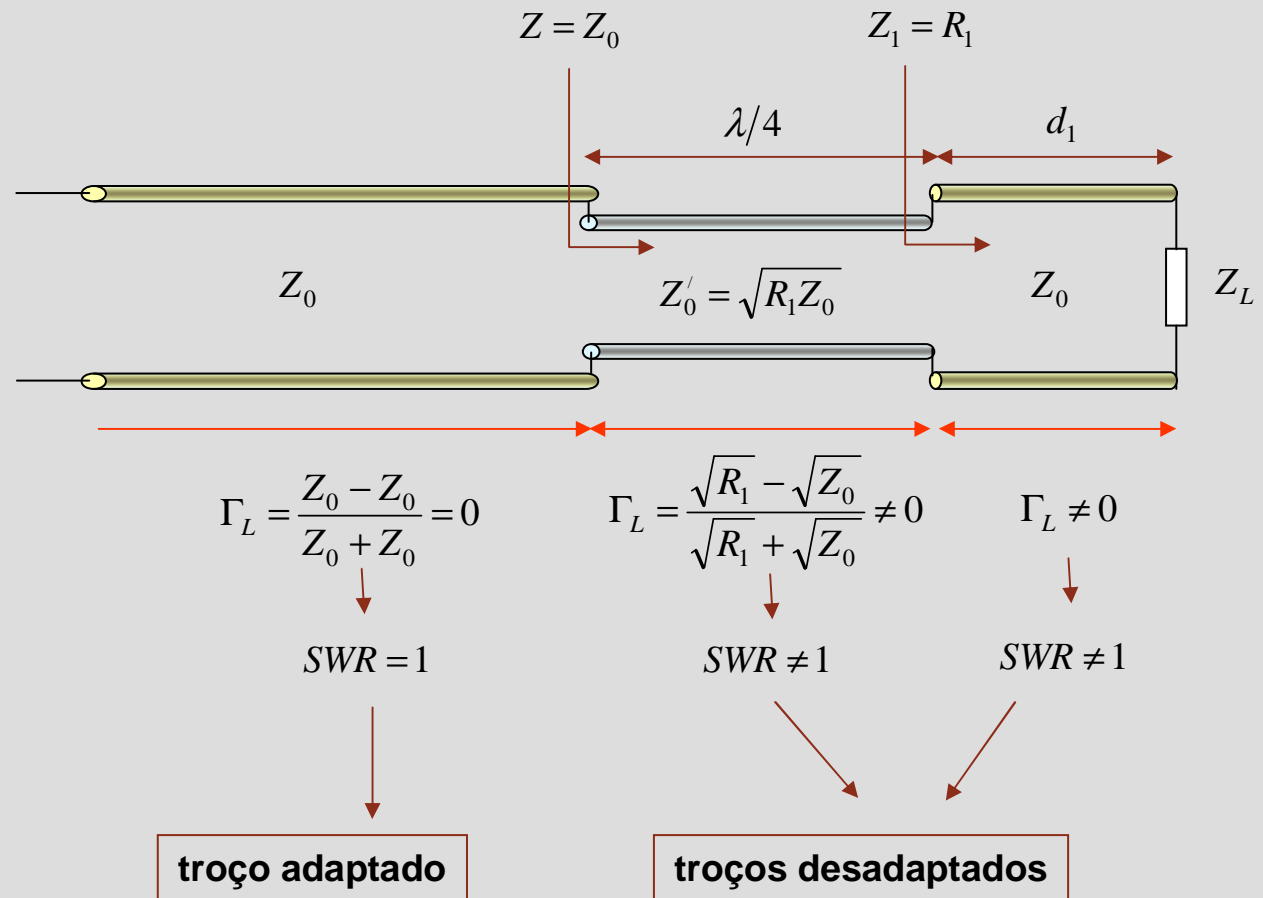
## Adaptação com linhas $\lambda/4$ – $Z_L$ complexa



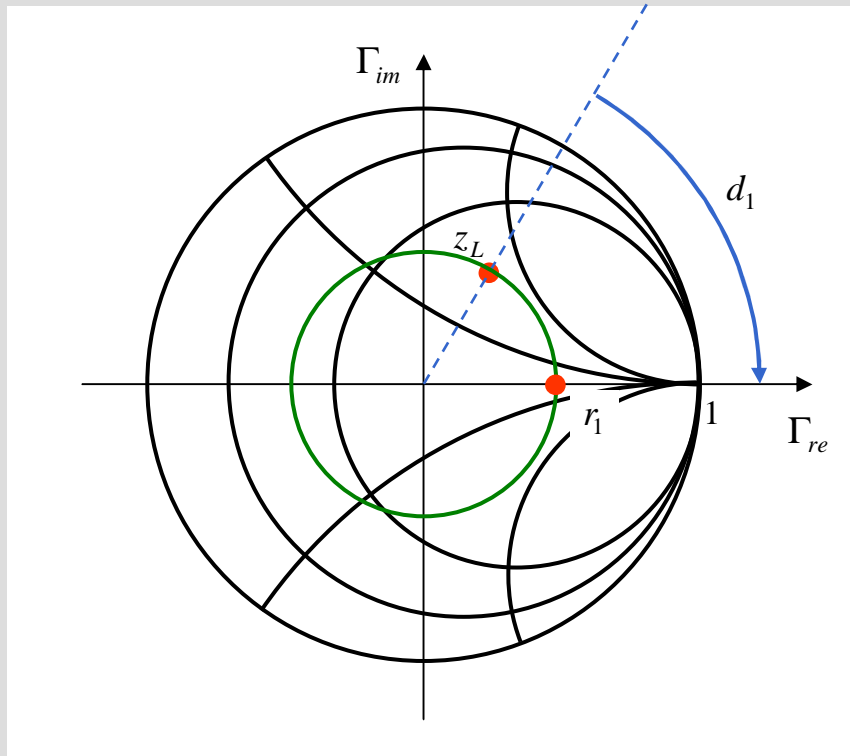
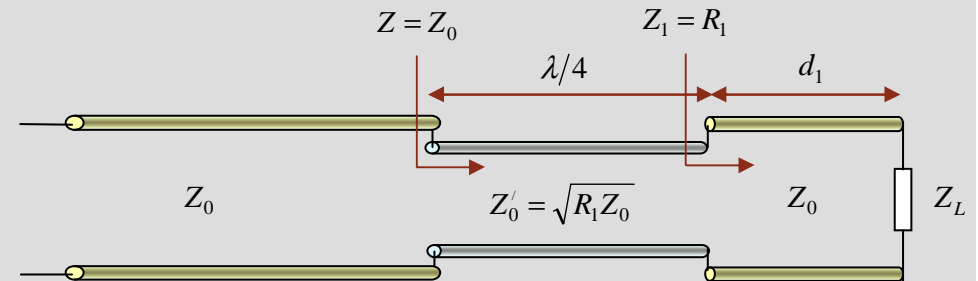
## Adaptação com linhas $\lambda/4$ – SWR

$$SWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$



## Adaptação com linhas $\lambda/4$ – diagrama de Smith



$$z_L = \frac{Z_L}{Z_0}$$

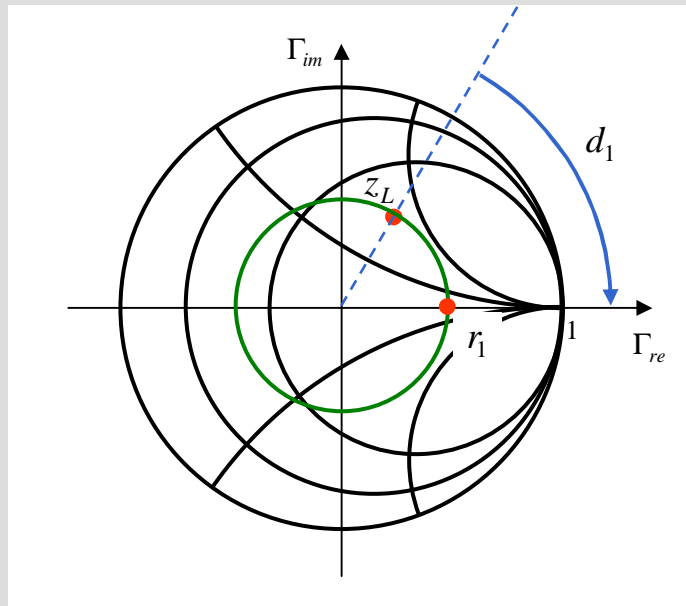
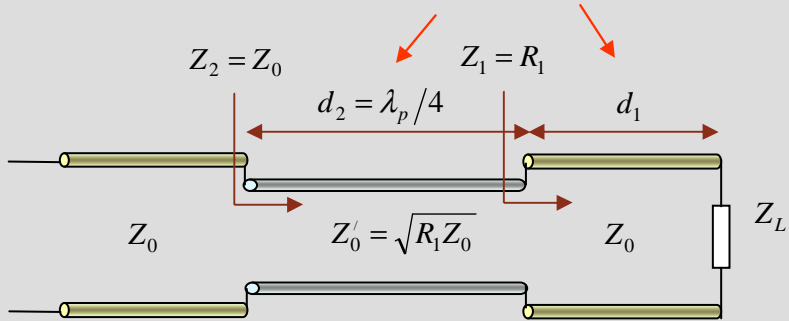
$$R_1 = r_1 Z_0$$

$$\downarrow$$

$$Z_0' = \sqrt{R_1 Z_0}$$

## Adaptação com linhas $\lambda/4$ – frequência diferente

projecto:  $f = f_p \rightarrow \lambda_p = \frac{v}{f_p}$



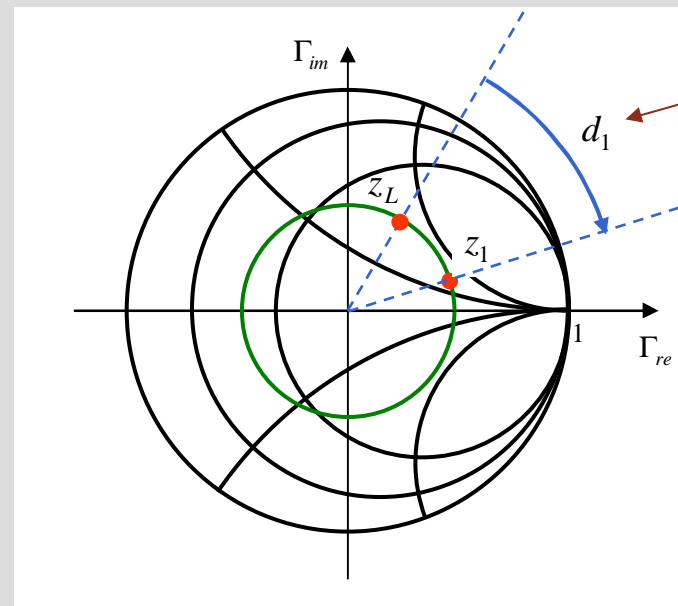
frequência diferente:  $f = f_d \neq f_p \rightarrow$

$$\lambda_d = \frac{v}{f_d}$$

$$Z'_0 \neq \sqrt{R_1 Z_0}$$

$$d_2 \neq \lambda_d/4$$

$Z_2 \neq Z_0$   
**linha desadaptada**



$$z_1 \neq r_1$$

## Adaptação com elementos reactivos

**Nota:**  
elementos em paralelo → trabalhar com admitâncias

se 
$$Y_1 = \frac{1}{Z_0} + jB_1$$
  
$$B_a = -B_1$$

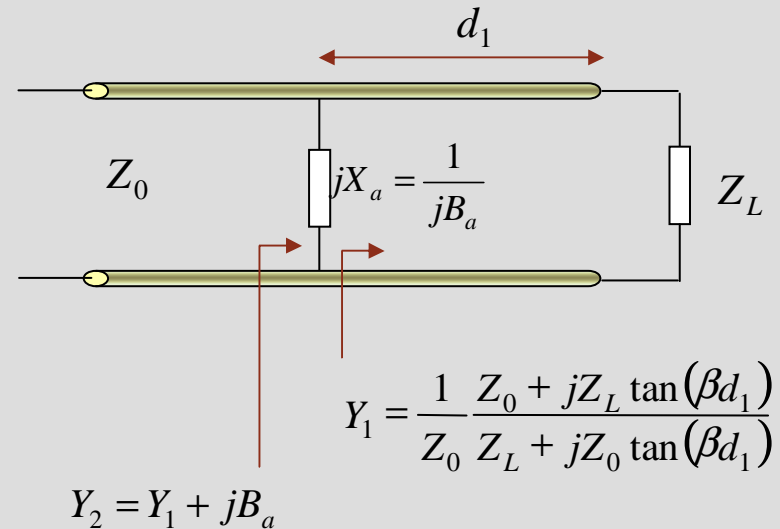
⇒ 
$$Y_2 = \frac{1}{Z_0}$$

↕

linha adaptada ↔  $Z_2 = Z_0$

⇒ Dimensionar adaptador → determinar  $X_a$  e  $d_1$

para que  $\text{Im}\{Y_2\} = 0$       para que  $\text{Re}\{Y_1\} = \frac{1}{Z_0}$



$X_a > 0 \rightarrow$  indutância  $\rightarrow L = \frac{X_a}{\omega}$

$X_a < 0 \rightarrow$  capacidade  $\rightarrow C = \frac{1}{\omega |X_a|}$

## Adaptação com elementos reactivos – diagrama de Smith

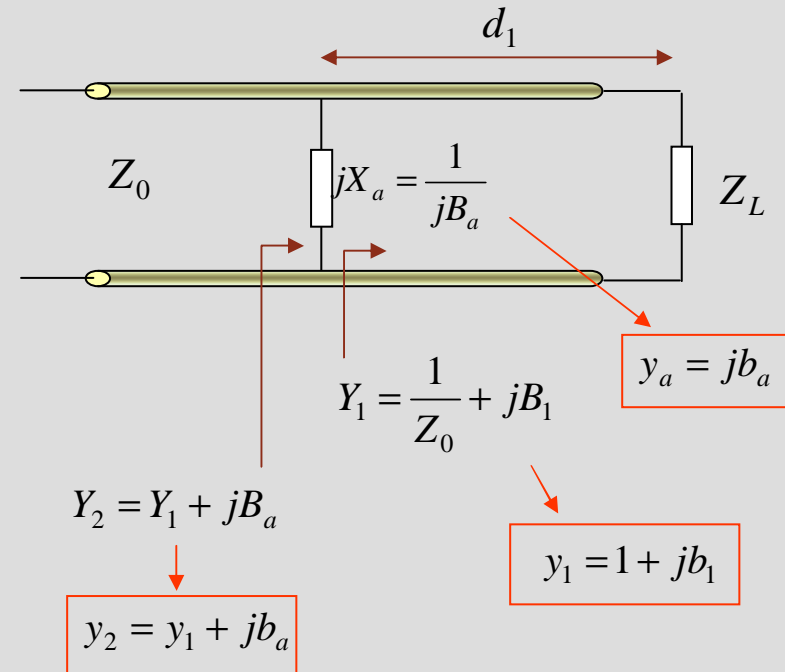
admitâncias normalizadas:

$$z = \frac{Z}{Z_0} \rightarrow y = \frac{1}{z} = \frac{Z_0}{Z} = YZ_0$$

condição de adaptação:  $Z_2 = Z_0 \rightarrow y_2 = 1$

$$\begin{aligned} y_2 &= 1 \\ y_2 &= y_1 + jb_a \\ y_1 &= 1 + jb_1 \end{aligned}$$

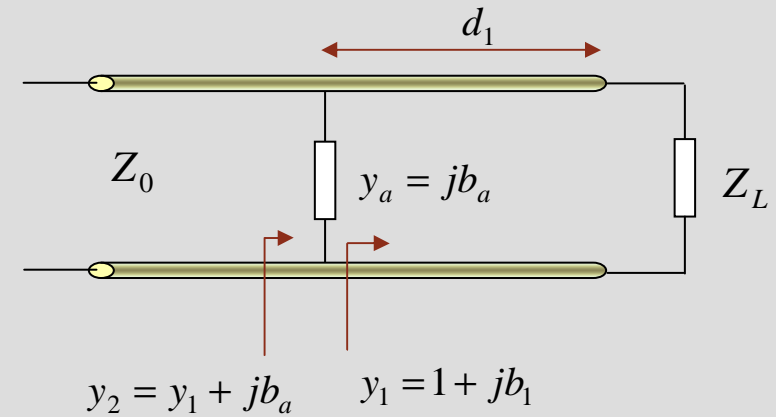
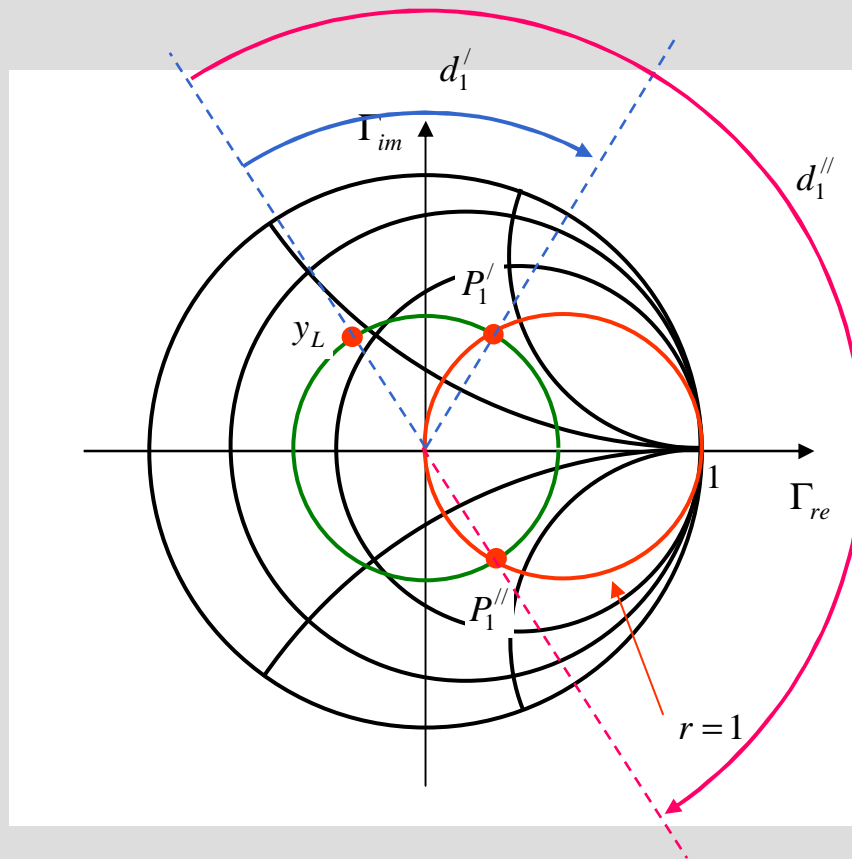
$$\begin{aligned} b_a &= -b_1 \\ d_1 \text{ tal que } \operatorname{Re}(y_1) &= 1 \end{aligned}$$



## Adaptação com elementos reactivos – diagrama de Smith

$$b_a = -b_1$$

$$d_1 \text{ tal que } \operatorname{Re}(y_1) = 1$$



curva  $r = 1$  é intersectada em dois pontos



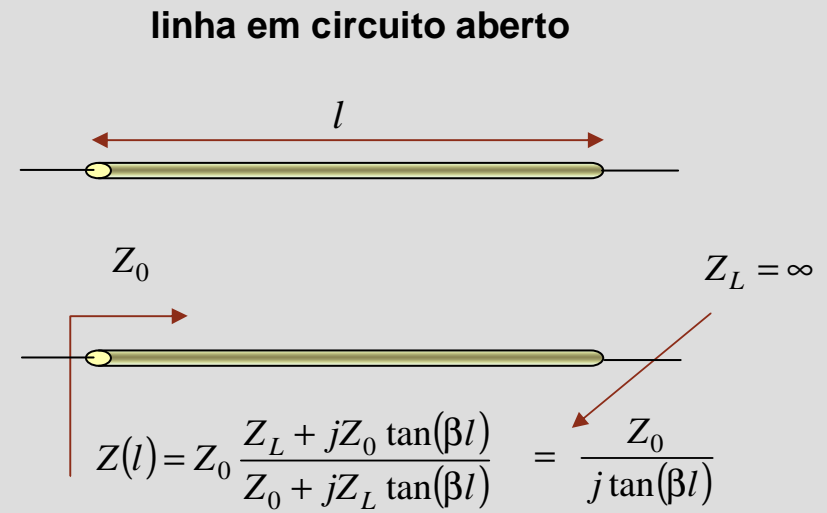
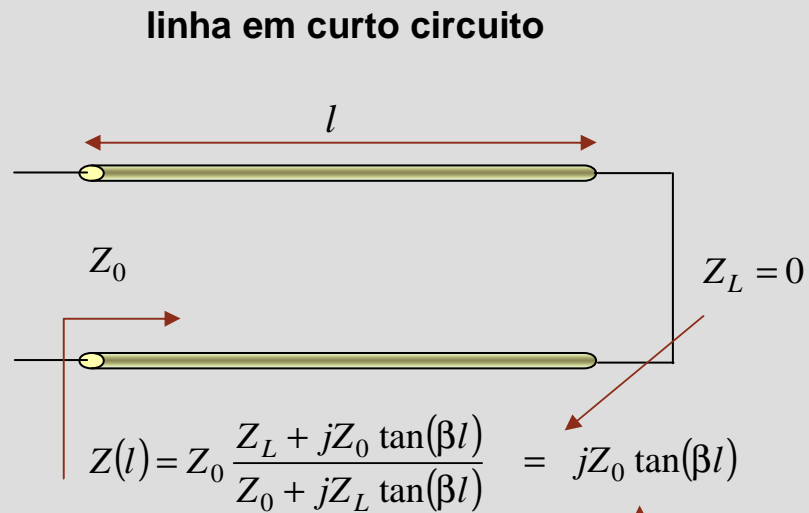
duas soluções possíveis

$$P_1' \text{ e } P_1'' \rightarrow y_1 = 1 + jb_1$$

$$b_a = -jb_1 \rightarrow X_a = -\frac{Z_0}{b_a}$$



## Linhas como elementos reactivos



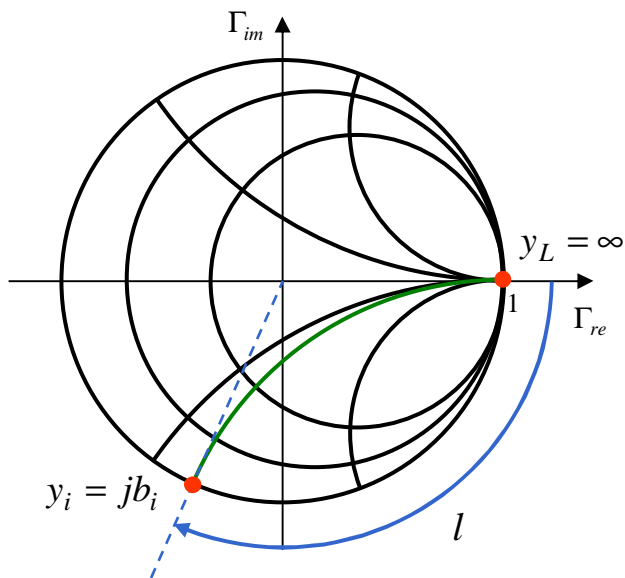
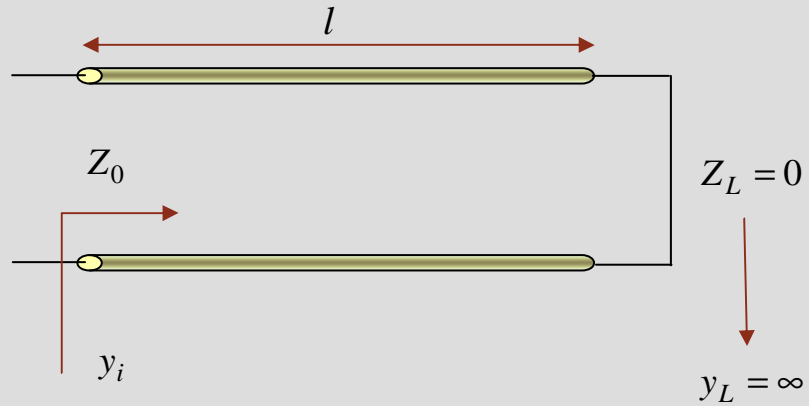
impedâncias de entrada  
puramente imaginárias

**STUBs**

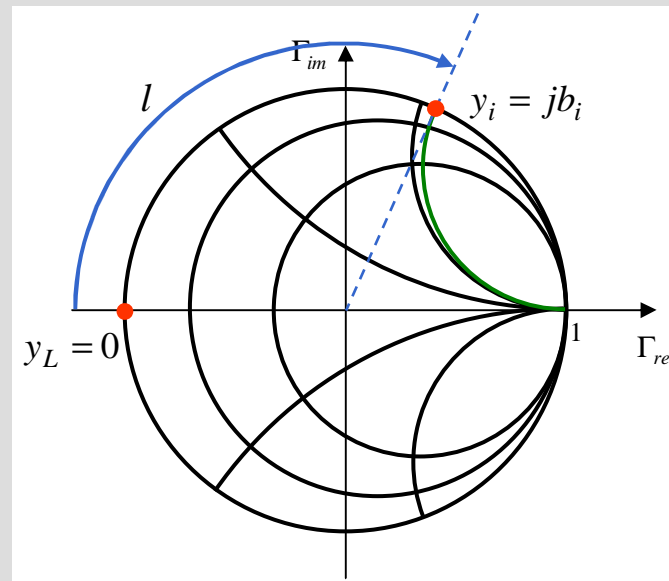
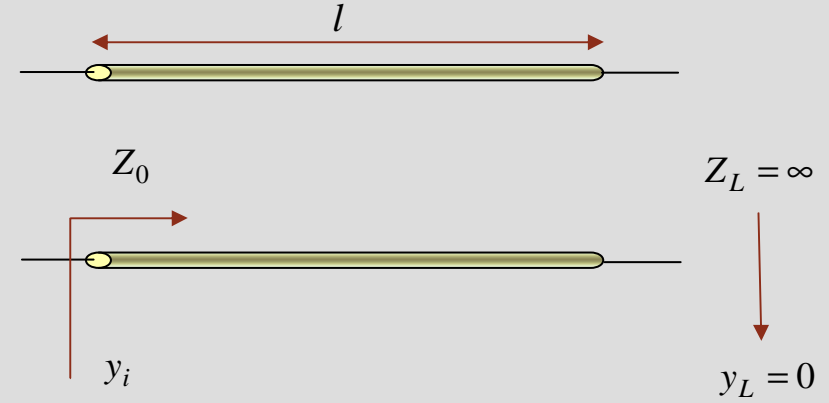
linhas em curto circuito ou circuito aberto podem ser usadas como elementos reactivos em adaptação

## Linhas como elementos reactivos – admitância de entrada

linha em curto circuito



linha em circuito aberto



## Adaptação com *stub* simples

**stub em paralelo**  $\Rightarrow y_2 = y_1 + y_s$

$y_s$  **imaginário**  $\Rightarrow \text{Re}(y_2) = \text{Re}(y_1)$

**linha adaptada**  $\Rightarrow y_2 = 1$

$$\text{Re}(y_1) = 1$$

$$\text{Im}(y_s) = -\text{Im}(y_1)$$

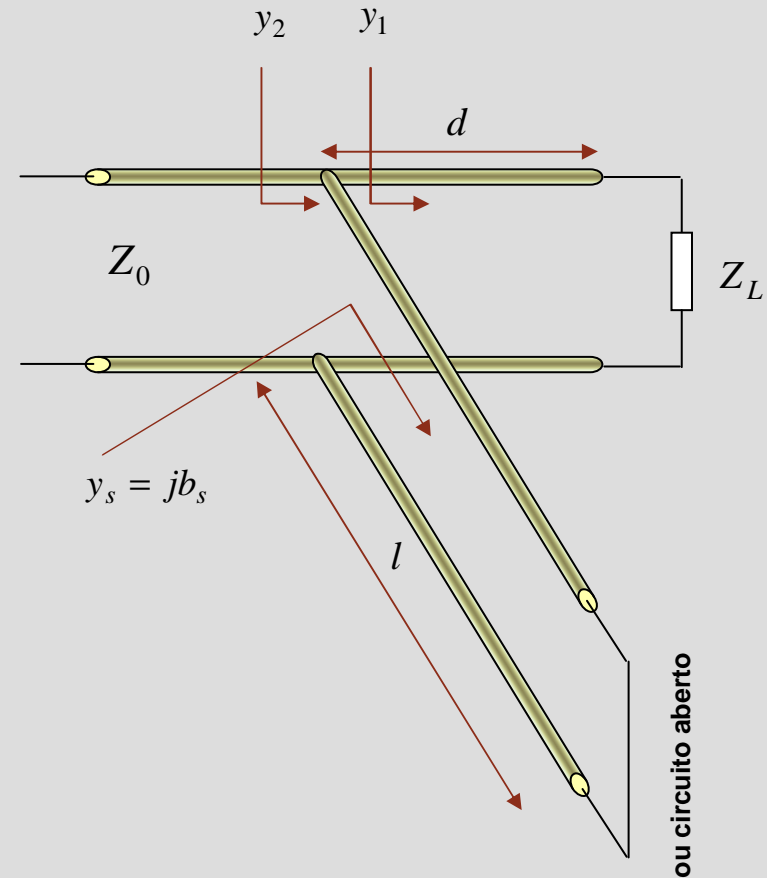
$$y_1 = 1 + jb_1$$

$$b_s = -b_1$$

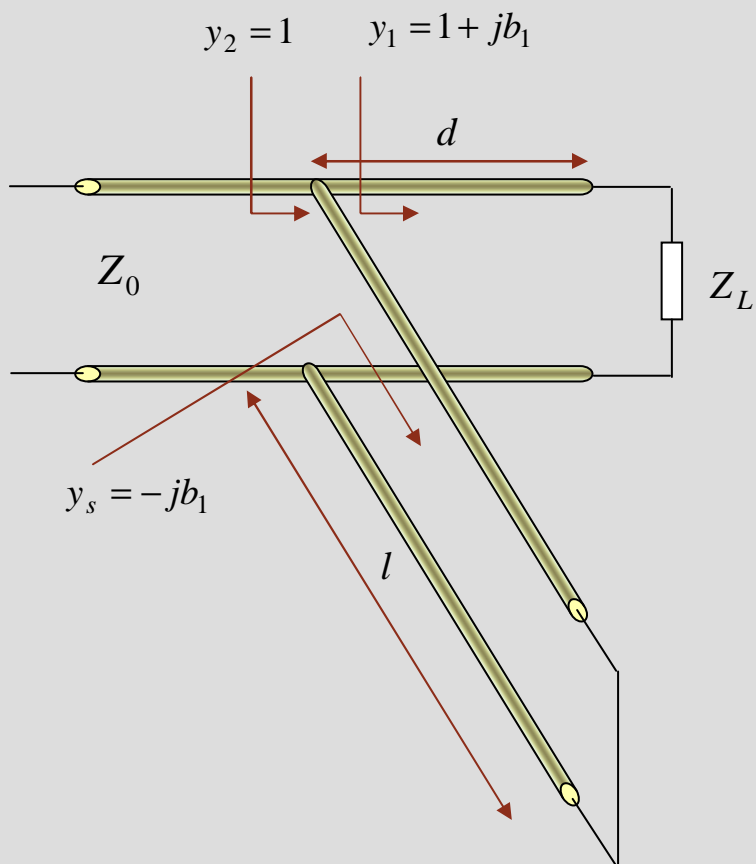
**projecto de adaptação:**

1. determinar  $d$  tal que  $\text{Re}(y_1) = 1$

2. determinar  $l$  tal que  $\text{Im}(y_s) = -\text{Im}(y_1)$

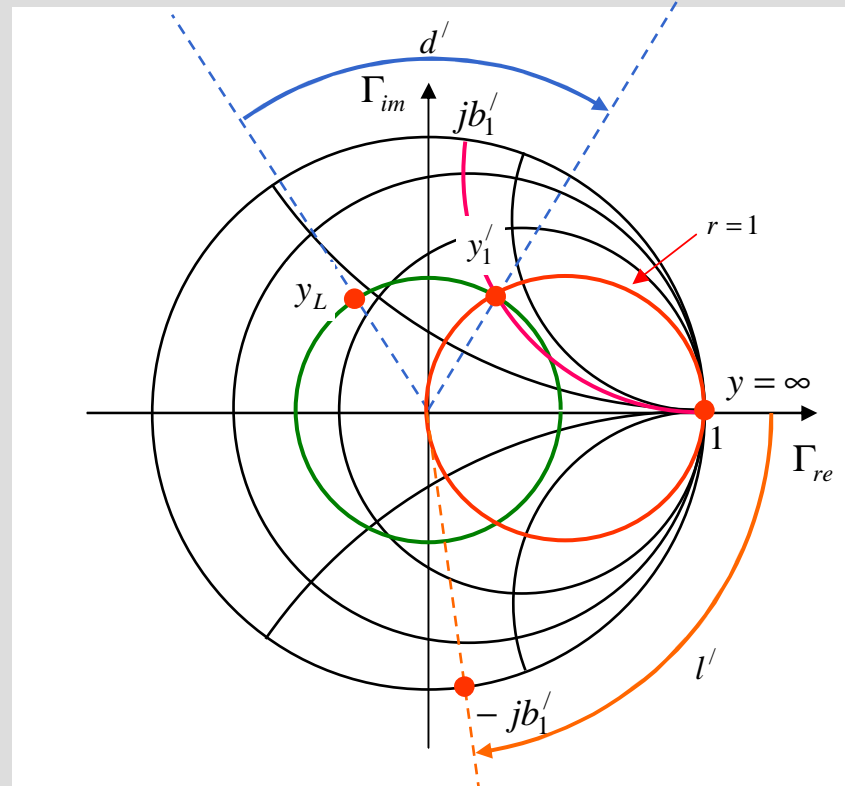


## Adaptação com *stub* simples – diagrama de Smith



### projecto de adaptação

1. determinar  $d$  tal que  $y_1 = 1 + jb_1$
2. determinar  $l$  tal que  $y_s = -jb_1$

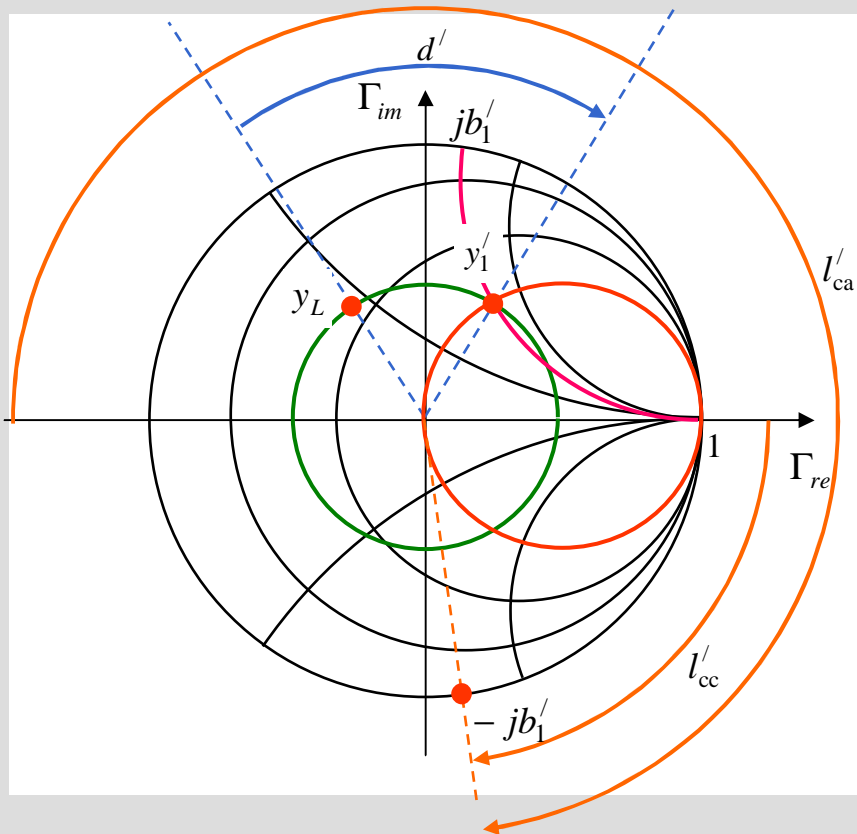


### Notas

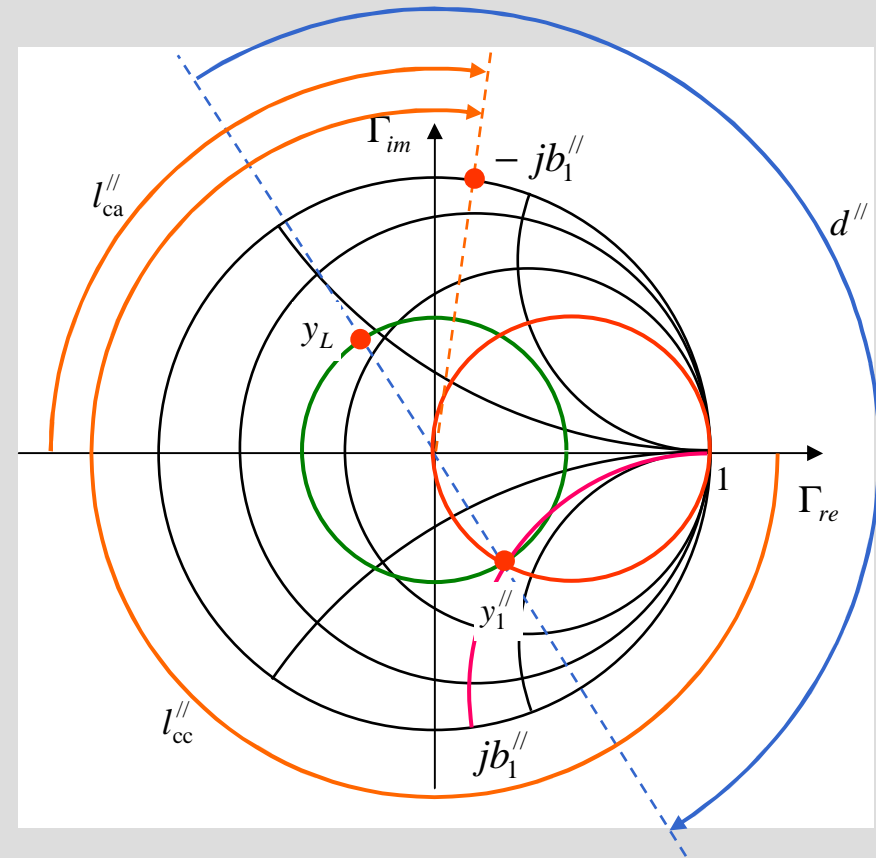
- curva  $r = 1$  intersectada em dois pontos  $\rightarrow$  duas soluções (apenas uma apresentada na figura)
- para stub em circuito aberto  $l$  mede-se a partir de  $y = 0$

# Adaptação com *stub* simples – diagrama de Smith

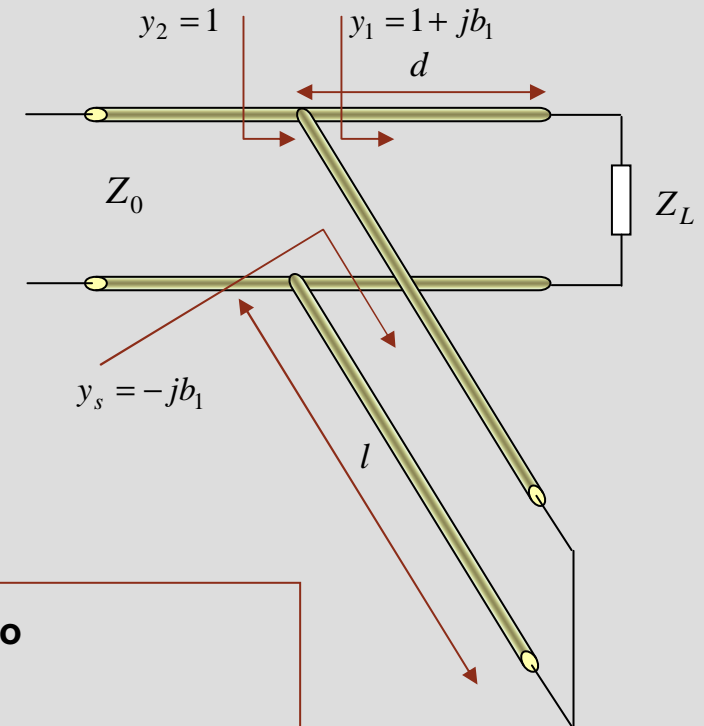
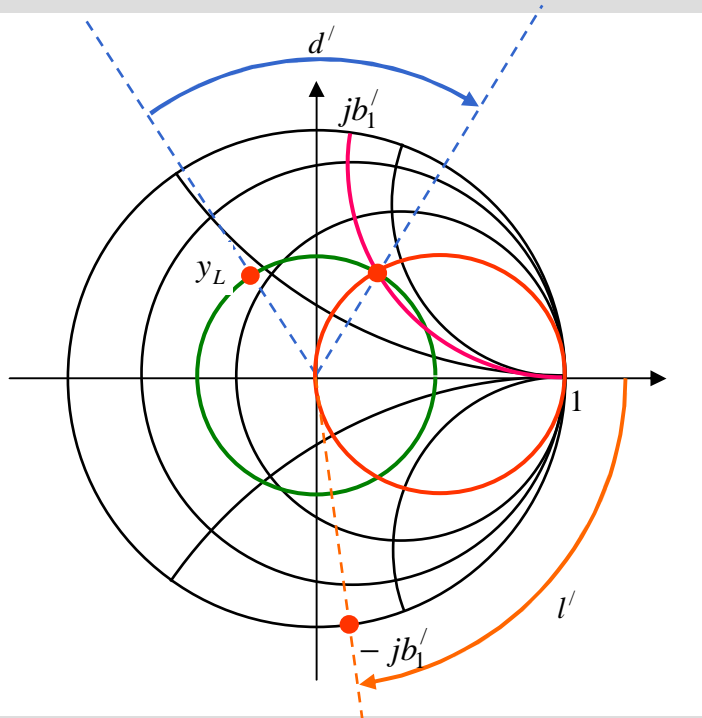
solução 1



solução 2



## Adaptação com *stub* simples – resumo



### projecto de adaptação

1. marcar  $y_L$
2. determinar  $d$  tal que  $y_1 = 1 + jb_1$
3. determinar  $l$  tal que  $y_s = -jb_1$



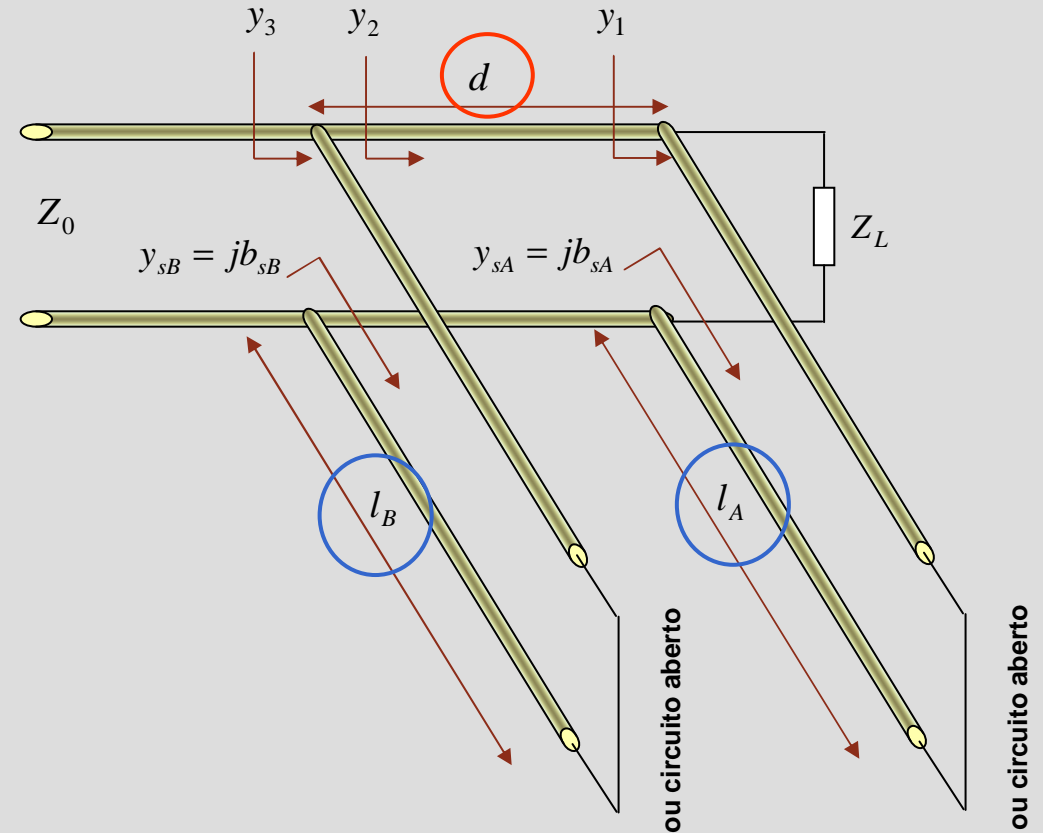
**dificuldade: nem sempre é possível colocar o stub na posição desejada**

## Adaptação com *stub* duplo

quando a posição para colocar o *stub* está pré-definida, é ainda possível adaptar a linha de transmissão usando o método do *stub* duplo

se  $d$  é fixo

projecto do duplo *stub* → dimensionar  $l_A$  e  $l_B$



## Adaptação com *stub* duplo

stubs em paralelo  $\Rightarrow y_1 = y_L + y_{sA}$

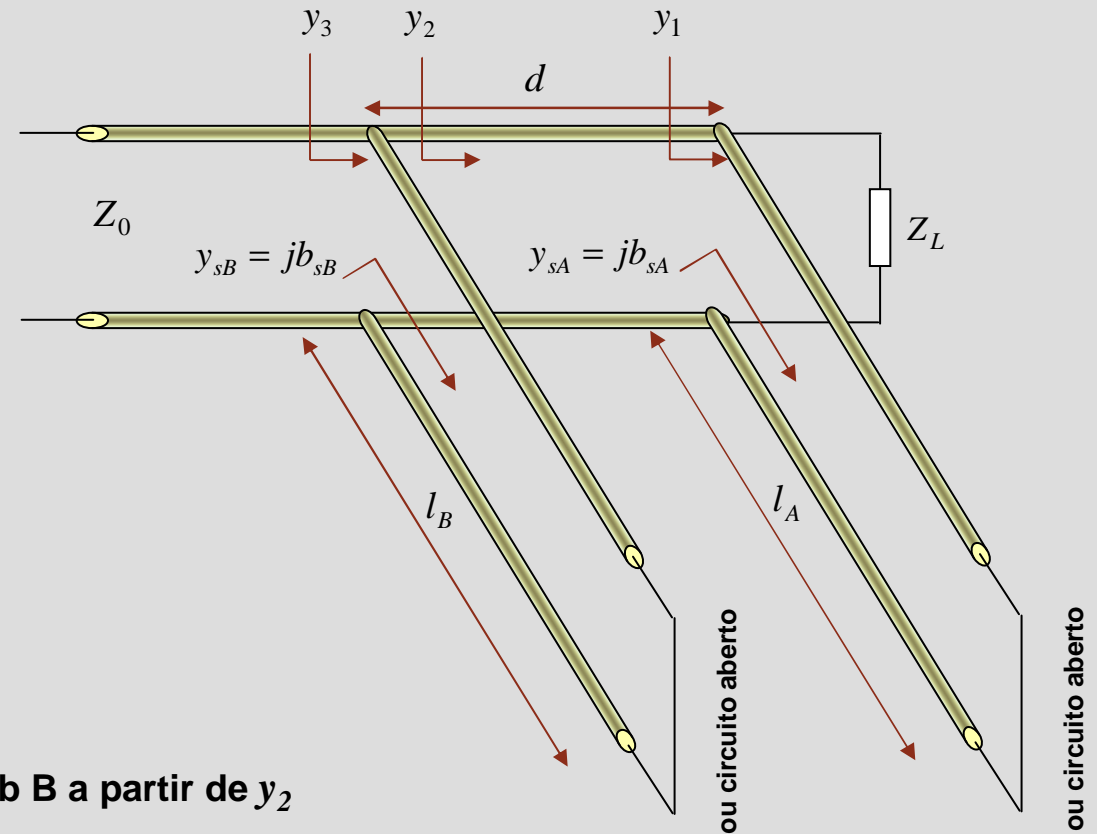
$$y_3 = y_2 + y_{sB}$$

$y_s$  imaginário  $\Rightarrow \text{Re}(y_1) = \text{Re}(y_L)$

$$\text{Re}(y_3) = \text{Re}(y_2)$$

“deslocando”  $d \Rightarrow y_2 \leftrightarrow y_1$

linha adaptada  $\Rightarrow y_3 = 1$



$$y_2 = 1 + j b_2$$

$$b_{sB} = -b_2$$

dimensionar stub B a partir de  $y_2$

$$y_1 = \text{Re}\{y_L\} + j b_1$$

$$b_1 = \text{Im}\{y_L\} + b_{sA} \Leftrightarrow b_{sA} = b_1 - \text{Im}\{y_L\}$$

dimensionar stub A a partir de  $y_1$



## Adaptação com *stub* duplo – projecto usando carta de Smith

Como calcular  $y_1$  e  $y_2$  ?

$$y_1 = y_L + y_{sA}$$

$$\text{Re}(y_1) = \text{Re}(y_L)$$

$$\text{"deslocando"} d \Rightarrow y_2 \leftrightarrow y_1$$

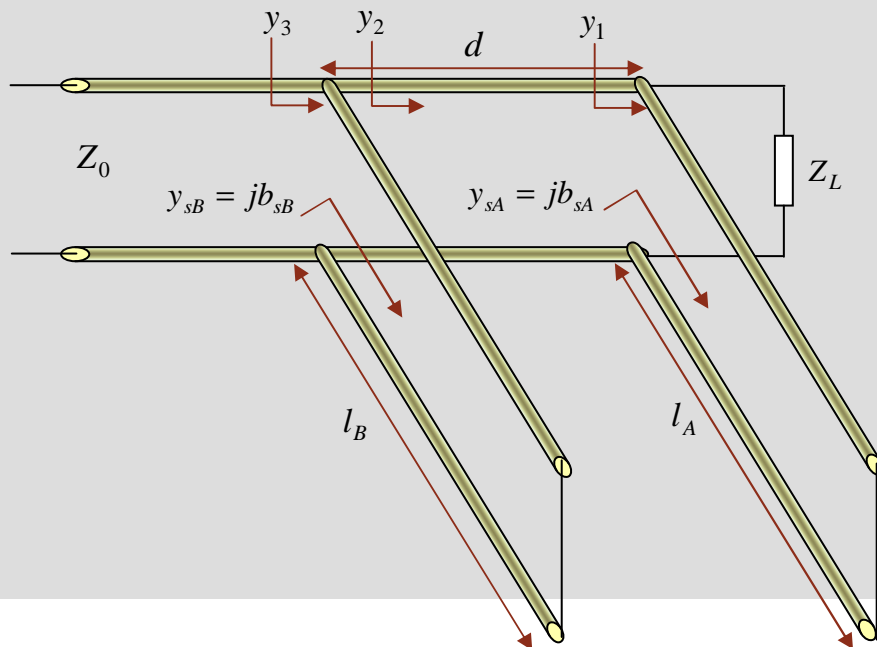
$$y_2 = 1 + jb_2$$

$$b_{sB} = -b_2$$

$\Rightarrow y_1$  na circunferência  $r = \text{Re}\{y_L\}$

$\Rightarrow y_1$  na circunferência  $r = 1$ , rodada de " $d$ " em direcção à carga 

$\Rightarrow y_2$  na circunferência  $r = 1$

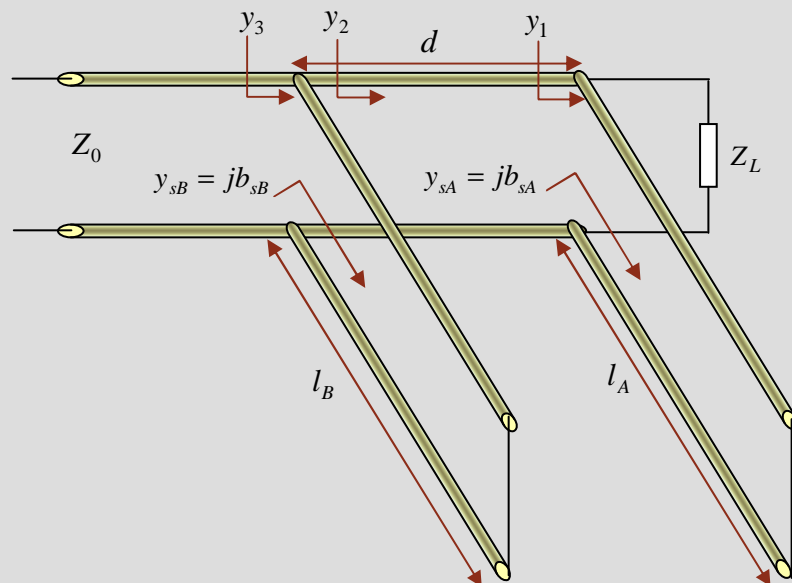


intersecção destas duas circunferências  $\rightarrow y_1$

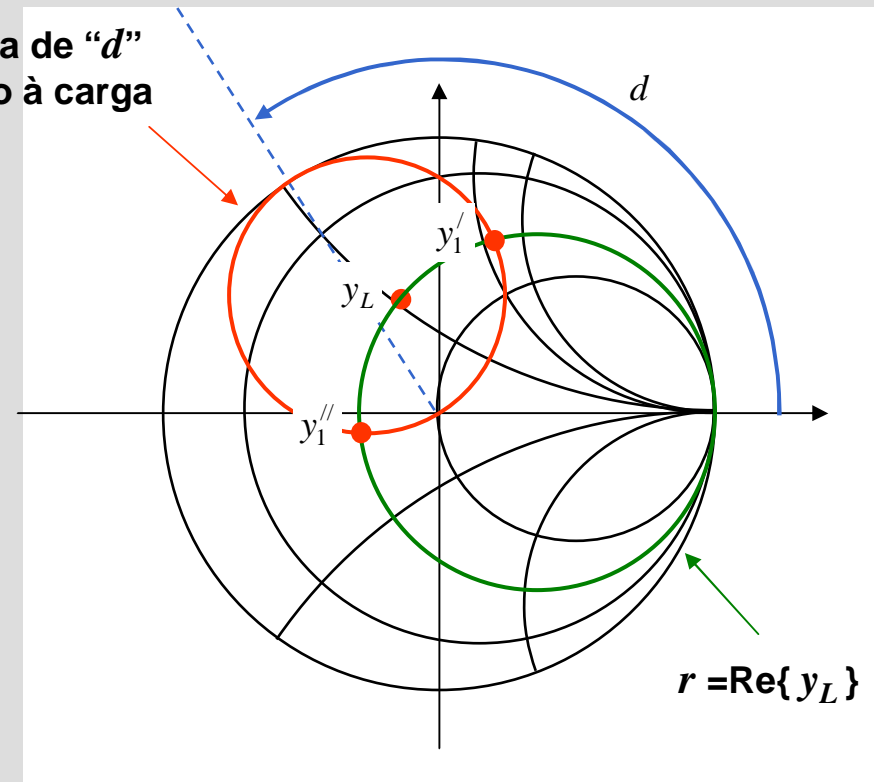
## Adaptação com *stub* duplo – carta de Smith

determinação de  $y_I$  → intersecção de  $r = \text{Re}\{y_L\}$  e  $r = 1$ , rodada de “ $d$ ” em direcção à carga

⇒ **Importante:**  
pode não existir  $y_I$

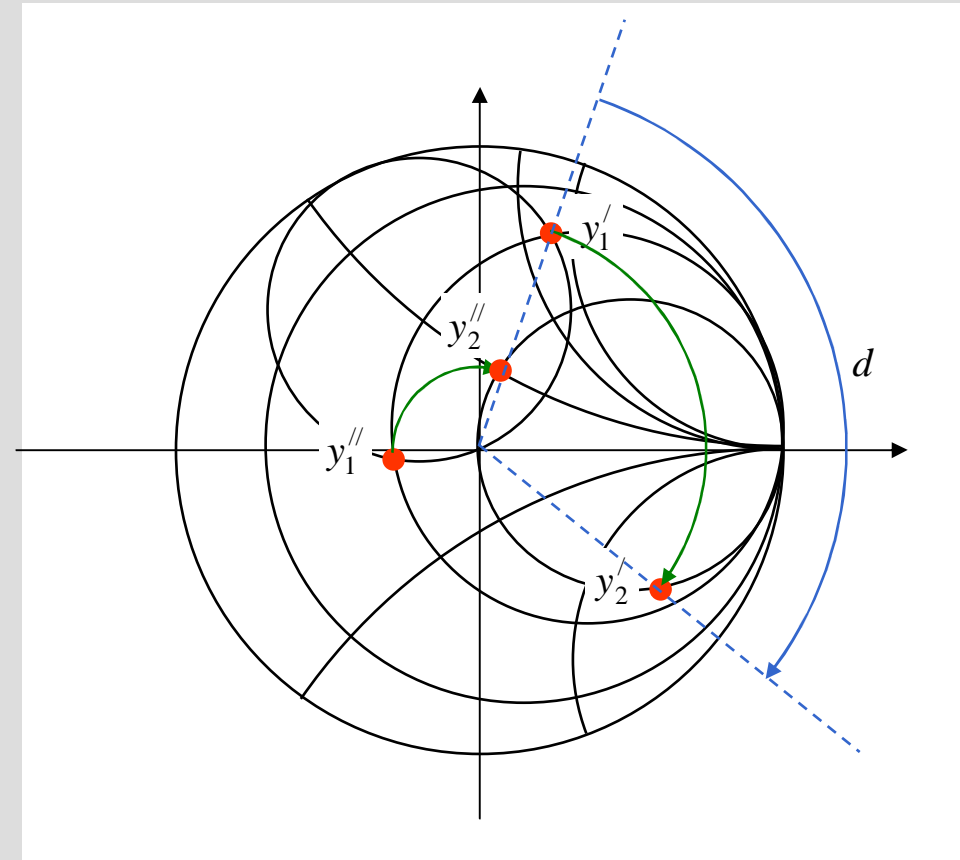
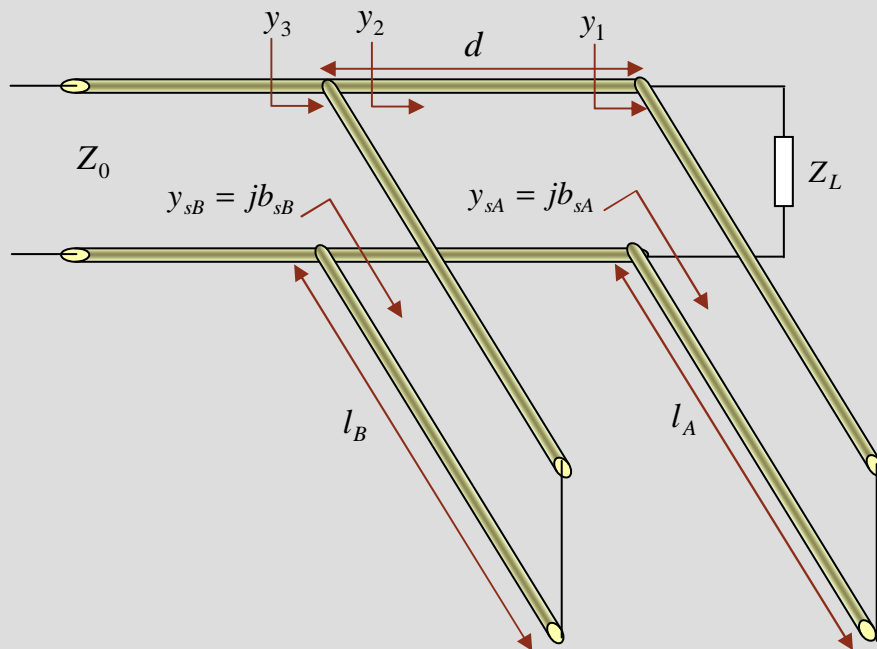


$r = 1$ , rodada de “ $d$ ”  
em direcção à carga



## Adaptação com *stub* duplo – carta de Smith

determinação de  $y_2 \rightarrow$  rodar  $y_1$  de “ $d$ ” em direcção ao gerador



## Adaptação com *stub* duplo – carta de Smith

a partir de  $y_1$  e  $y_2$  → dimensionamento dos stubs

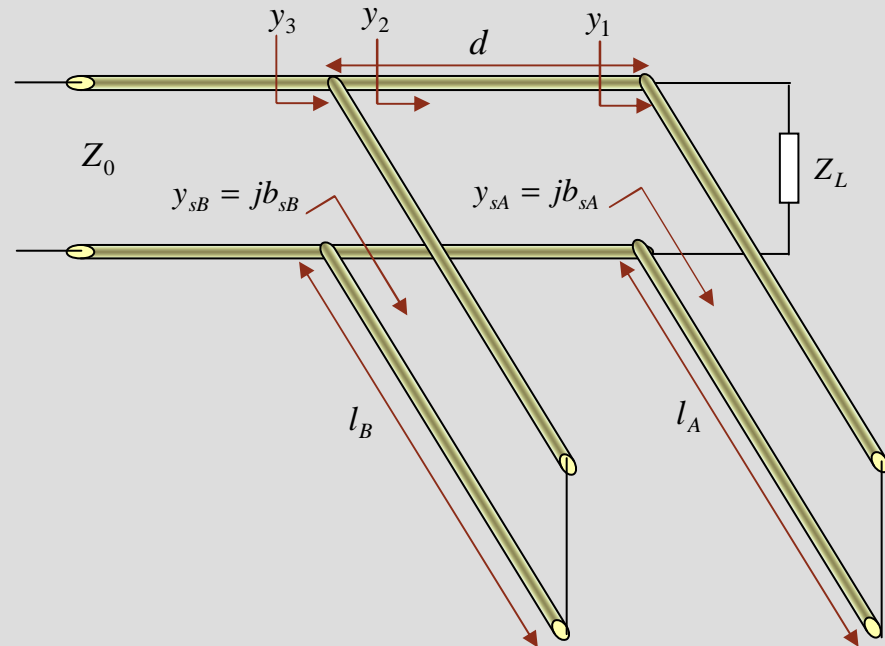
$$y_2 = 1 + jb_2$$

$$y_1 = \text{Re}\{y_L\} + jb_1$$

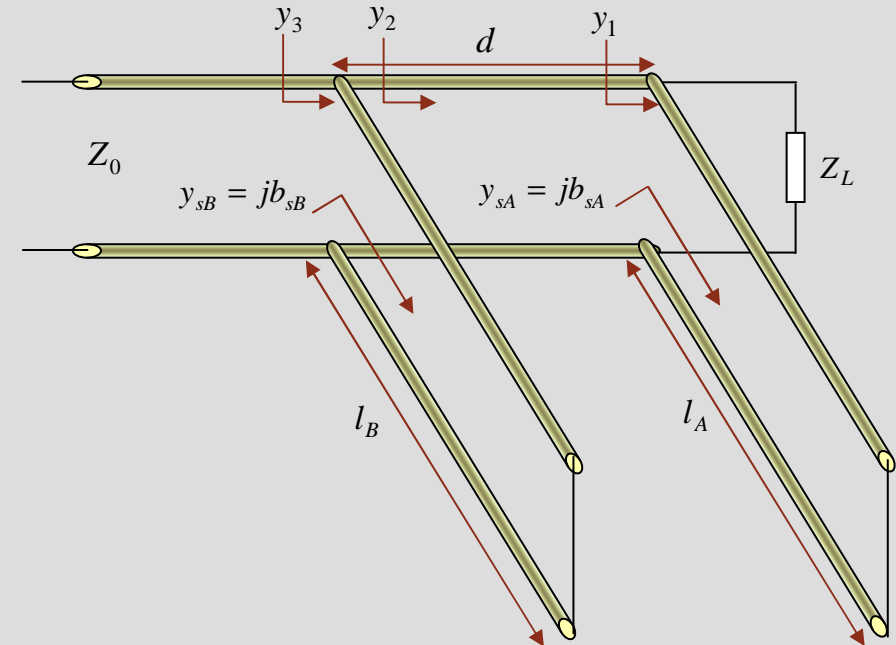
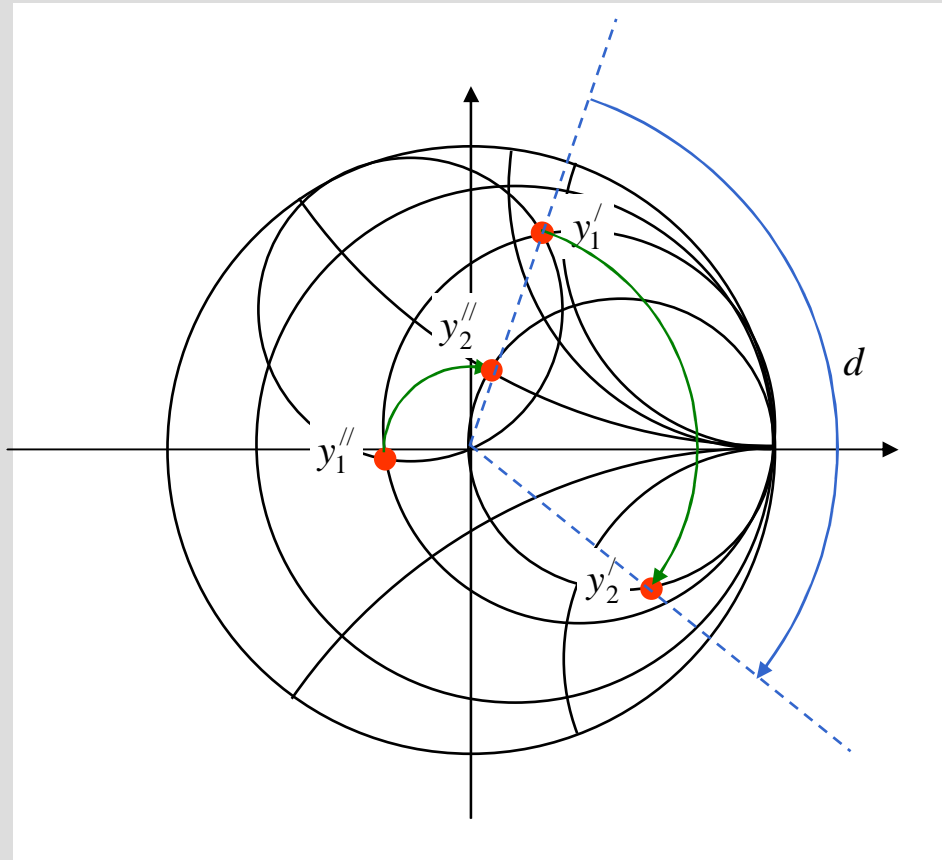
$$b_{sA} = b_1 - \text{Im}\{y_L\}$$

$$b_{sB} = -b_2$$

determinação de  $l_A$  e  $l_B$



## Adaptação com *stub* duplo – projecto



**projecto**

1. determinar  $y_1$
2. determinar  $y_2$
3. determinar comprimento dos stubs

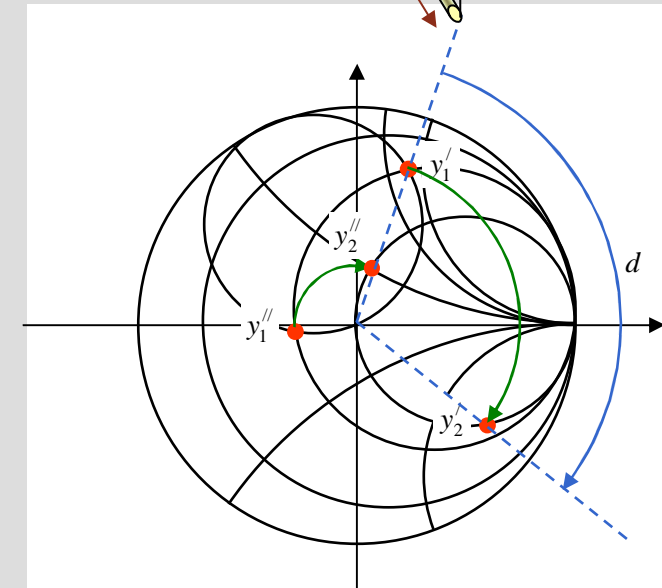
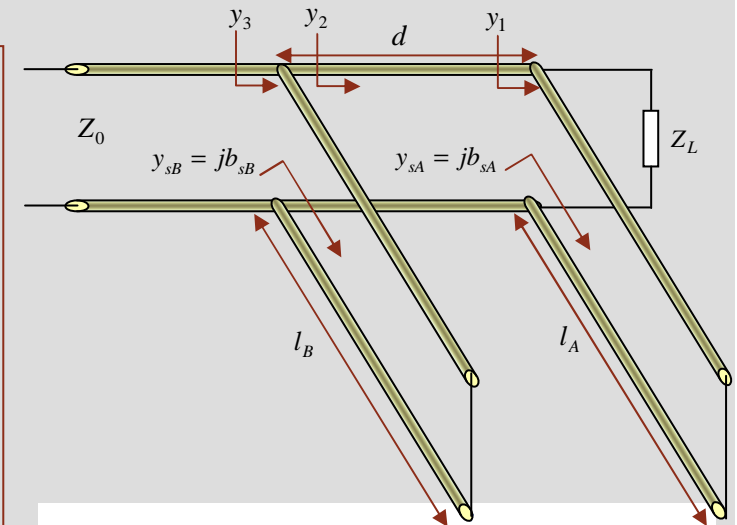
## Adaptação com *stub* duplo – resumo

### projecto

1. marcar  $y_L$
2. desenhar circunferência  $r=1$  rodada  $d$  em direcção à carga
3. determinar  $y_I$  pela intersecção desta circunferência com  $r = \text{Re}\{y_L\}$
4. determinar  $y_2$  rodando  $y_I$  uma distância  $d$  em direcção ao gerador
5. determinar  $l_B$  a partir de  $b_{sB} = -b_2$
6. determinar  $l_A$  a partir de  $b_{sA} = b_1 - \text{Im}\{y_L\}$

### Notas:

- podem existir duas soluções possíveis para  $y_I, y_2, y_{sA}$  e  $y_{sB}$
- para cada *stub* pode ser necessário considerar soluções para terminação em curto-circuito e em circuito-aberto

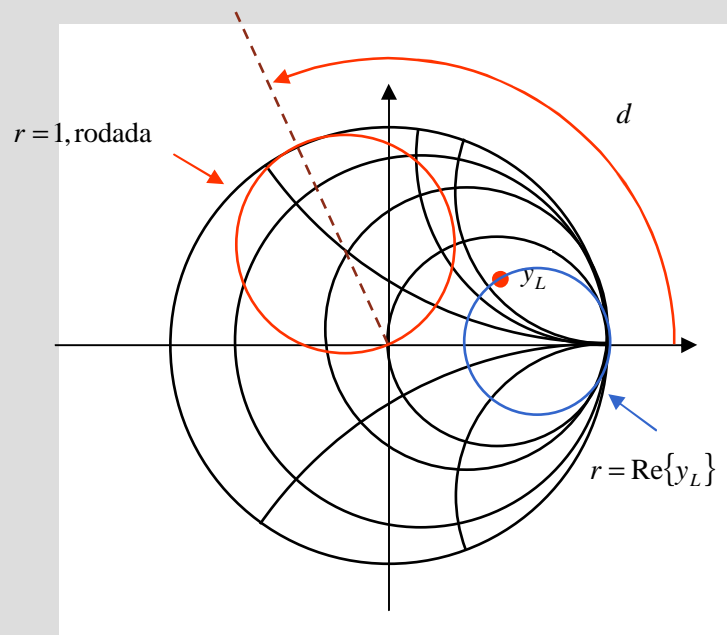


## Adaptação com *stub* duplo – valor da carga vs distância $d$

determinação de  $y_I$   $\rightarrow$  intersecção de  $r = \text{Re}\{y_L\}$  e  $r = 1$  rodada de “ $d$ ” em direcção à carga



esta intersecção pode não ser possível  
para determinadas cargas e distâncias  $d$



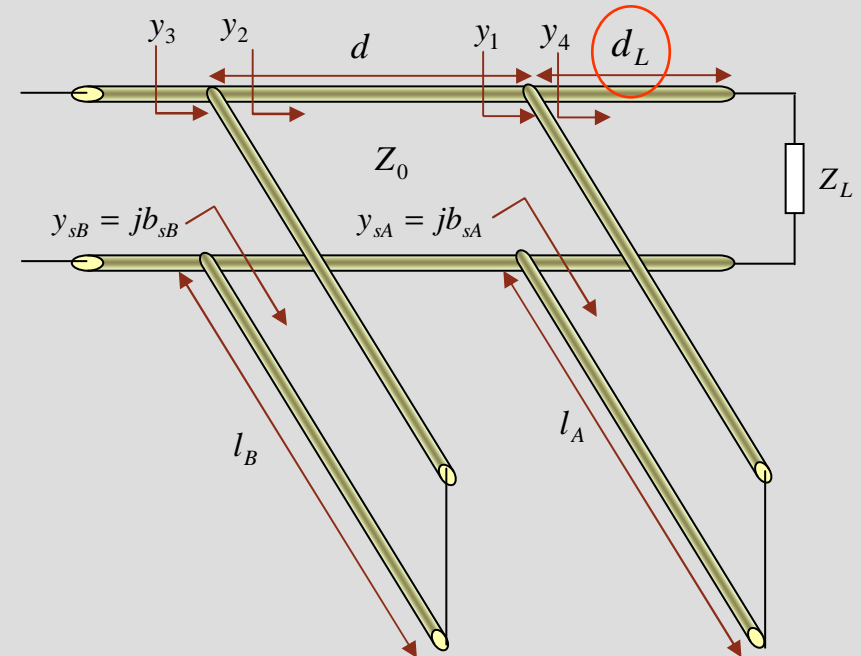
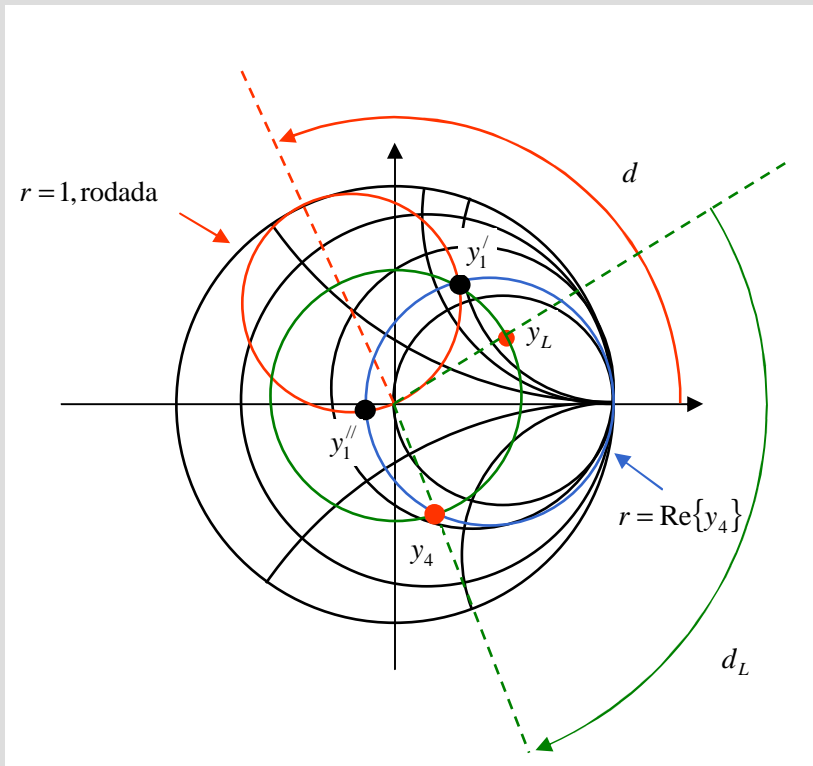
$\rightarrow$  será possível adaptar a linha nesta  
situação usando um *stub* duplo?



não colocar o *stub*  
junto da carga

## Adaptação com *stub* duplo – valor da carga vs distância $d$

colocando o *stub* a uma distância  $d_L$  da carga





## Adaptação de linhas de transmissão – notas finais

- a adaptação só é efectiva à frequência de projecto

- só a linha principal está adaptada

- nem sempre um dado método funciona:

  - linhas  $\lambda/4$**  → nem sempre se encontram linhas com  $Z'_0$  desejado

  - stub simples** → pode não ser possível colocar stub na posição desejada

  - stub duplo** → pode não existir solução para  $y_I$  e não ser possível colocar

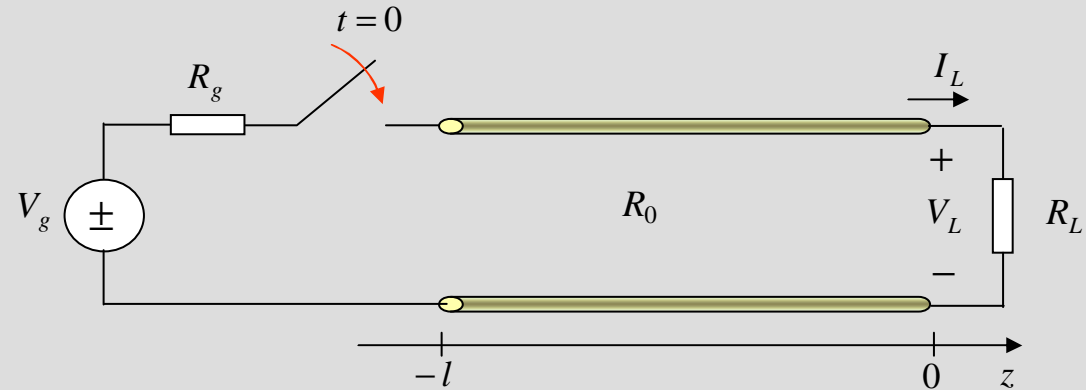
    - stub* a distância  $d_L$  da carga

## Transitórios em linhas sem perdas terminadas resistivamente

linha sem perdas →

$$Z_0 = R_0 = \sqrt{\frac{L}{C}}$$

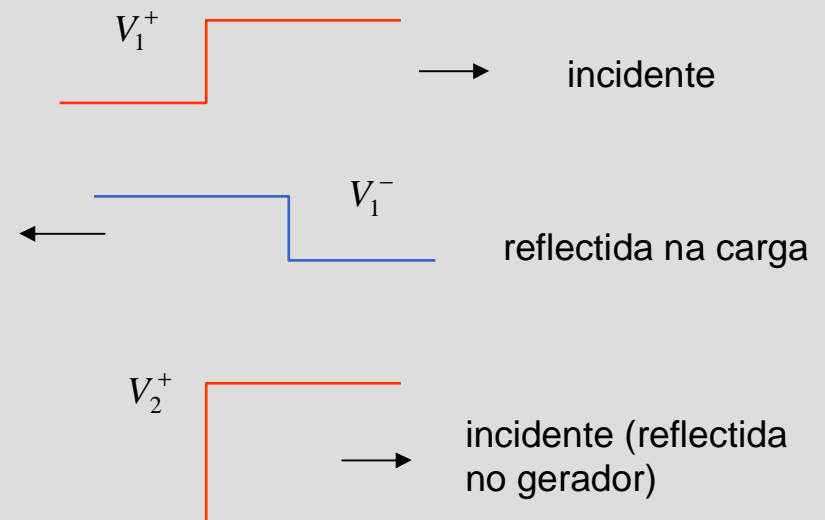
$$v = \frac{1}{\sqrt{LC}}$$



num dado ponto da linha, a tensão (corrente) num dado instante é obtida pela soma de todos os degraus de tensão (corrente) que aí chegaram



é necessário saber a amplitude e a posição dos diferentes degraus em função do tempo



## Amplitude dos degraus de tensão

coeficiente de reflexão na carga  $\rightarrow \Gamma_L = \frac{R_L - R_0}{R_L + R_0}$

coeficiente de reflexão no gerador  $\rightarrow \Gamma_g = \frac{R_g - R_0}{R_g + R_0}$



amplitudes:

1º degrau  $\rightarrow V_1^+ = \frac{R_0}{R_g + R_0} V_g$

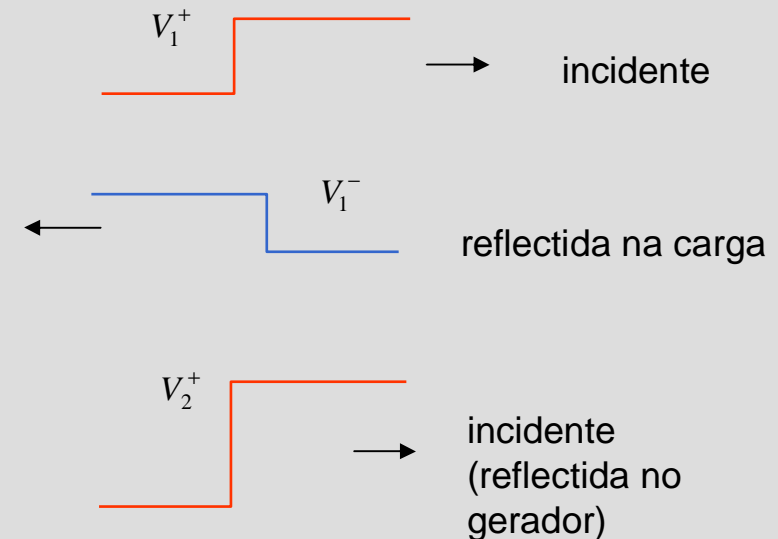
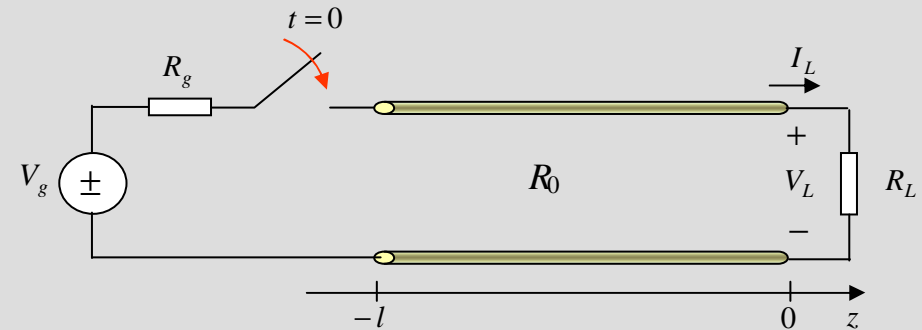
2º degrau  $\rightarrow V_1^- = \Gamma_L V_1^+$

3º degrau  $\rightarrow V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+$

4º degrau  $\rightarrow V_2^- = \Gamma_L V_2^+ = \Gamma_L \Gamma_g \Gamma_L V_1^+$

5º degrau  $\rightarrow V_3^+ = \Gamma_g V_2^- = (\Gamma_g \Gamma_L)^2 V_1^+$

⋮



## Posição dos degraus de tensão

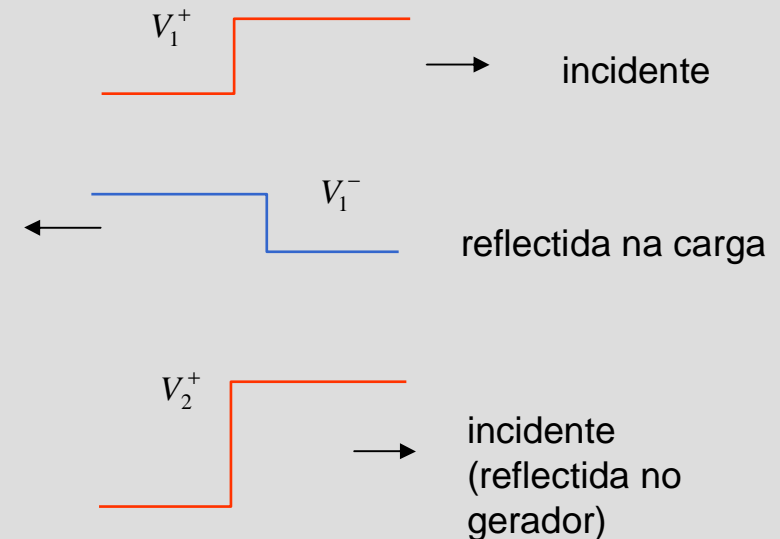
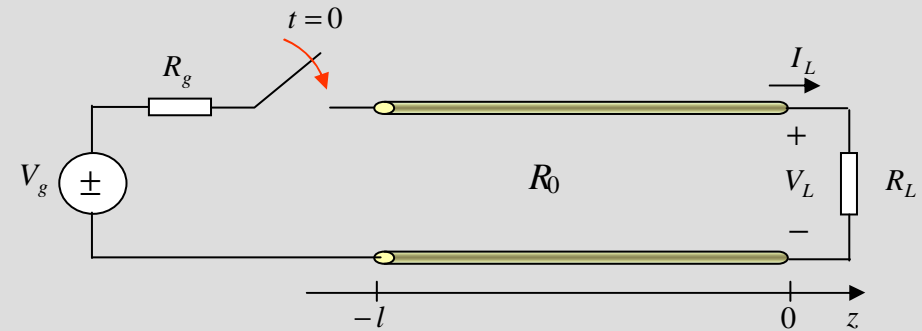
velocidade de propagação  $\rightarrow v$

tempo de propagação ao longo da linha  $\rightarrow T = \frac{l}{v}$



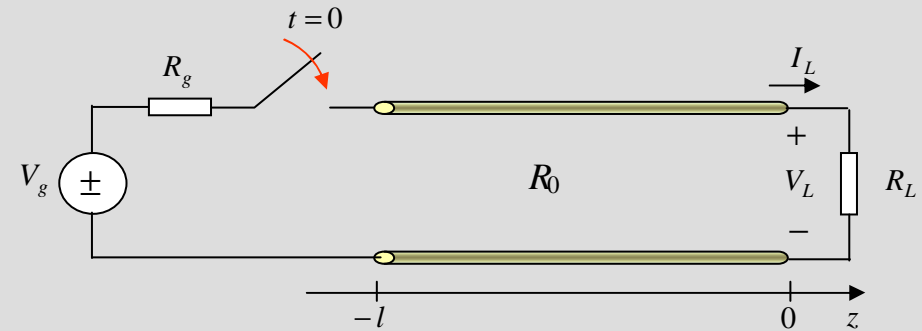
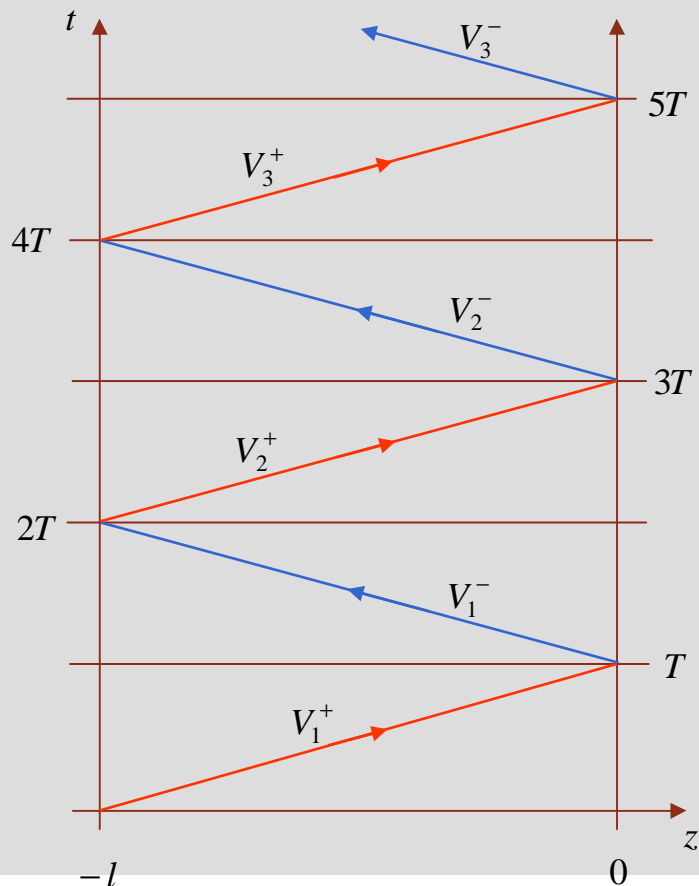
tempos de chegada:

- 1º degrau  $\rightarrow$  carga, em  $t = T$
- 2º degrau  $\rightarrow$  gerador, em  $t = 2T$
- 3º degrau  $\rightarrow$  carga, em  $t = 3T$
- 4º degrau  $\rightarrow$  gerador, em  $t = 4T$
- 5º degrau  $\rightarrow$  carga, em  $t = 5T$
- ⋮



## Diagrama de reflexões de tensão

diagrama de reflexões de tensão → permite representar graficamente degraus de tensão



$$T = \frac{l}{v}$$

$$V_1^+ = \frac{R_0}{R_g + R_0} V_g$$

$$V_1^- = \Gamma_L V_1^+$$

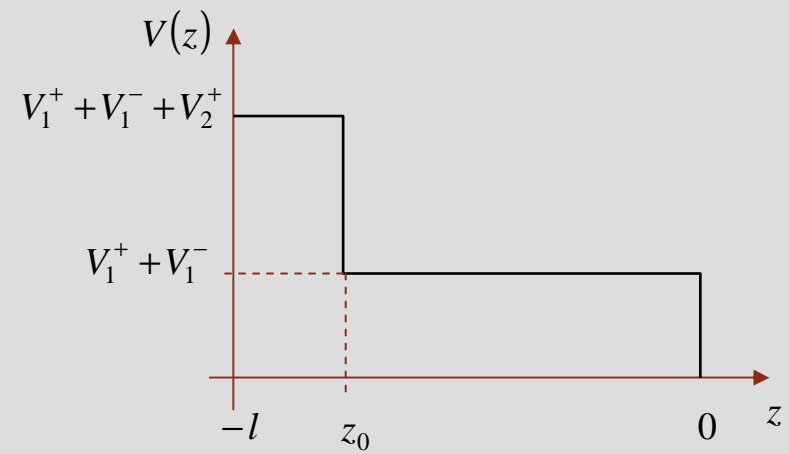
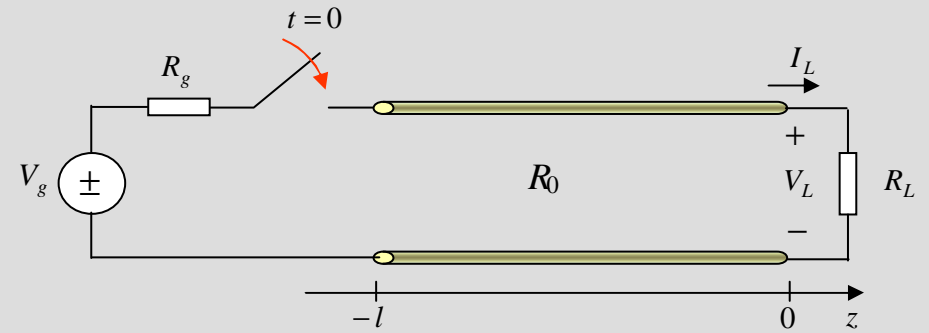
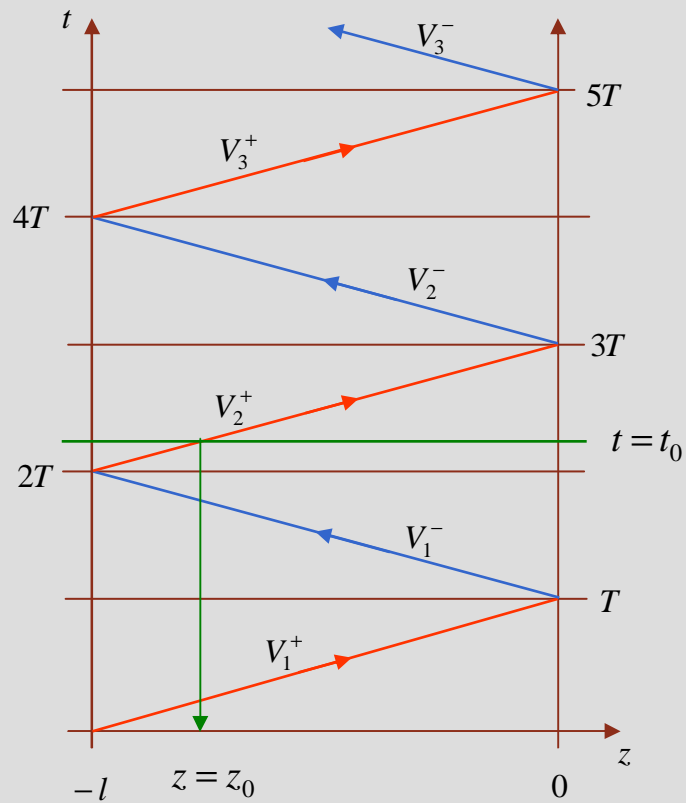
$$V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+$$

$$V_2^- = \Gamma_L V_2^+ = \Gamma_g (\Gamma_L)^2 V_1^+$$

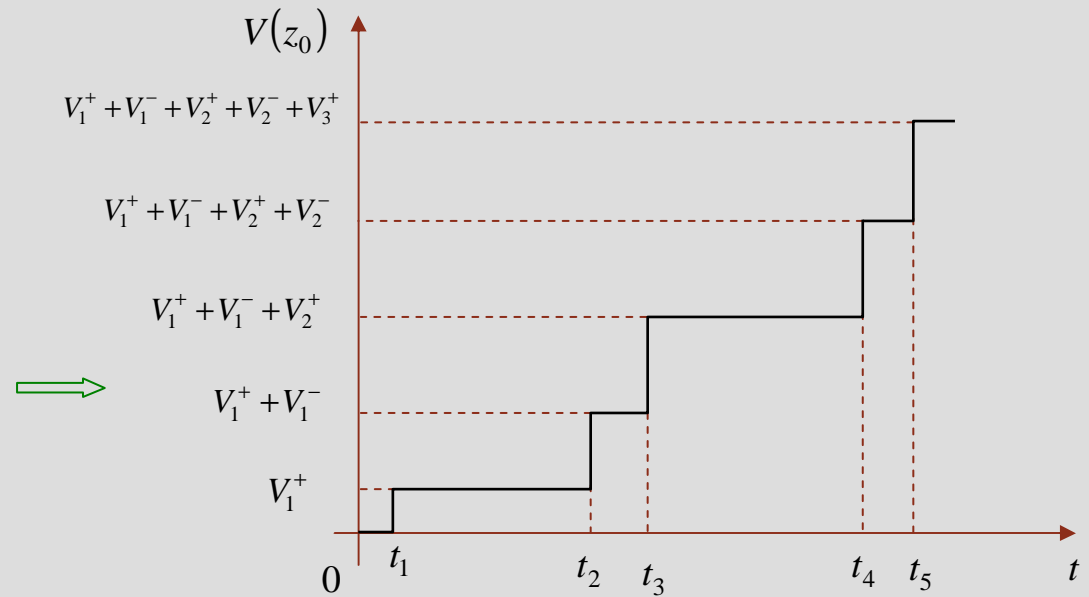
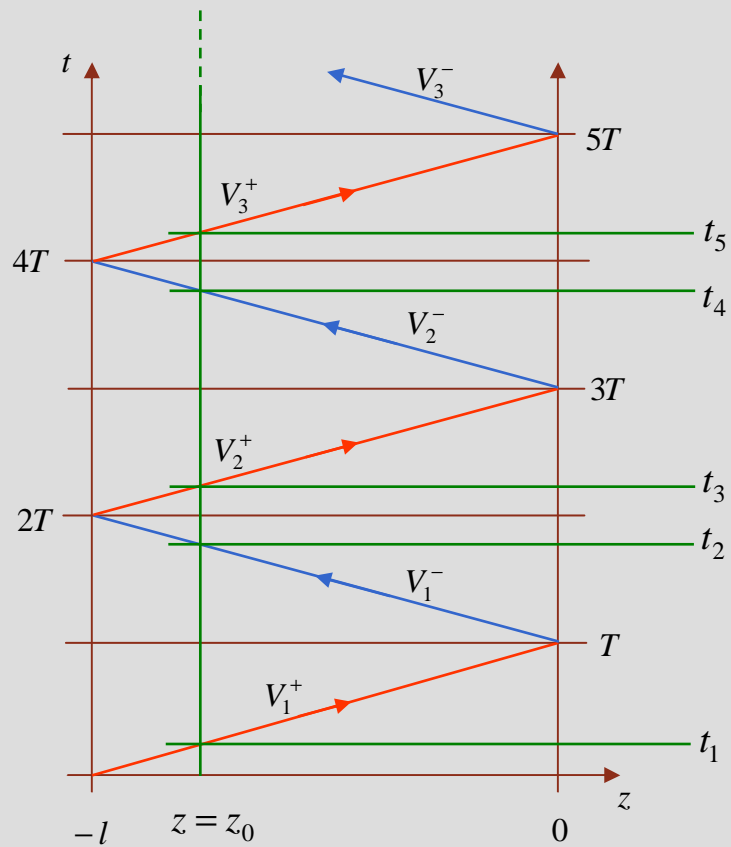
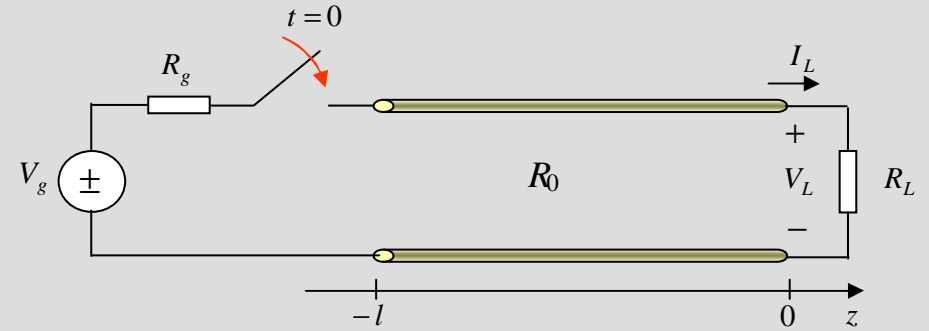
$$V_3^+ = \Gamma_g V_2^- = (\Gamma_g \Gamma_L)^2 V_1^+$$

$$\vdots$$

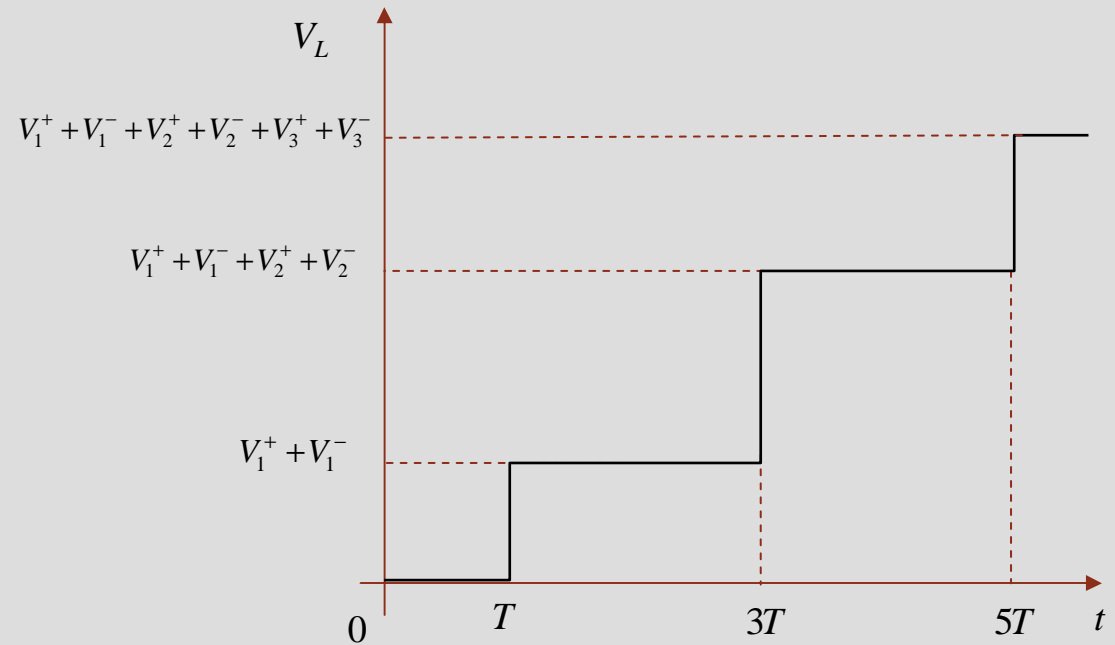
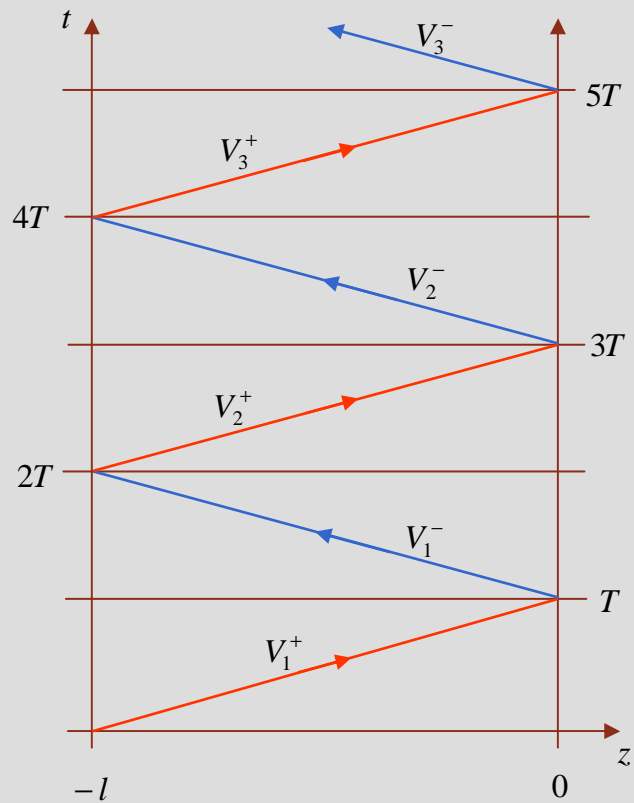
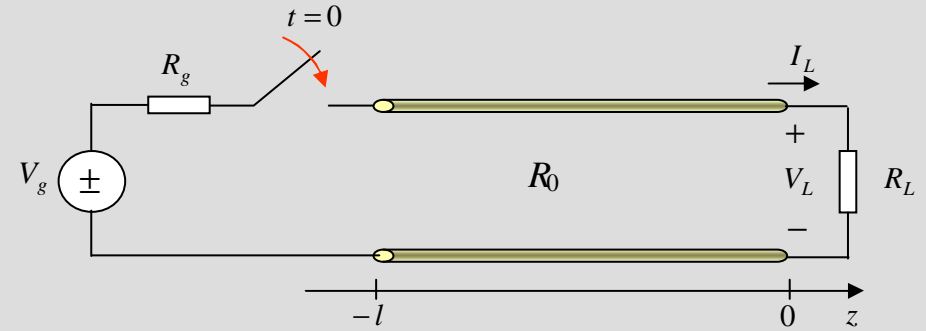
## Tensão ao longo da linha num dado instante



# Tensão num dado ponto em função do tempo



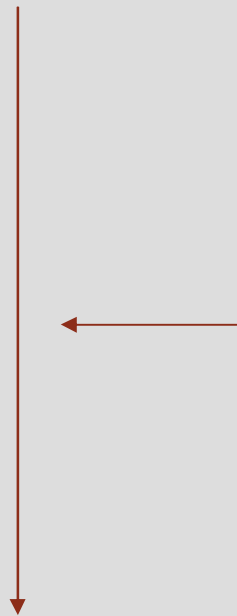
## Tensão na carga em função do tempo





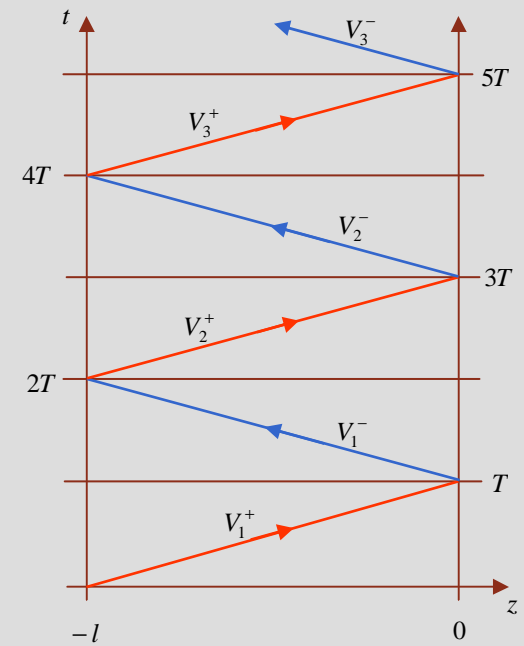
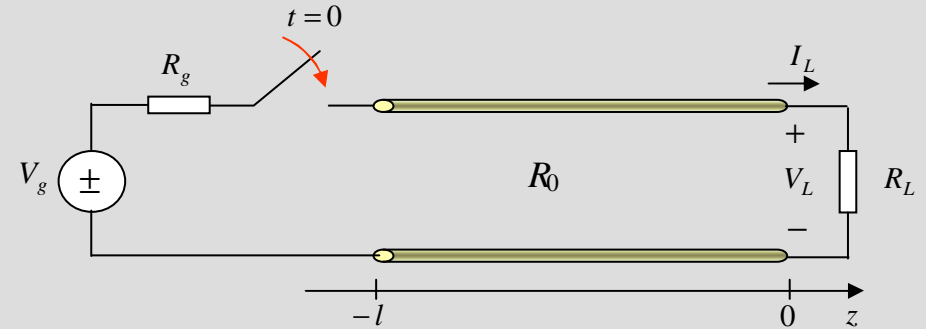
## Valor final da tensão na linha

$$V(t \rightarrow \infty) = V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \dots$$



$$\begin{aligned} V_1^+ &= \frac{R_0}{R_g + R_0} V_g \\ V_1^- &= \Gamma_L V_1^+ \\ V_2^+ &= \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+ \\ V_2^- &= \Gamma_L V_2^+ = \Gamma_g (\Gamma_L)^2 V_1^+ \\ V_3^+ &= \Gamma_g V_2^- = (\Gamma_g \Gamma_L)^2 V_1^+ \\ &\vdots \end{aligned}$$

$$V(t \rightarrow \infty) = V_1^+ (1 + \Gamma_L) [1 + \Gamma_g \Gamma_L + (\Gamma_g \Gamma_L)^2 + (\Gamma_g \Gamma_L)^3 + \dots]$$



## Valor final da tensão na linha

$$V(t \rightarrow \infty) = V_1^+ (1 + \Gamma_L) [1 + \Gamma_g \Gamma_L + (\Gamma_g \Gamma_L)^2 + (\Gamma_g \Gamma_L)^3 + \dots]$$

se  $|\Gamma_g \Gamma_L| < 1$

$$V(t \rightarrow \infty) = V_1^+ \frac{1 + \Gamma_L}{1 - \Gamma_g \Gamma_L}$$

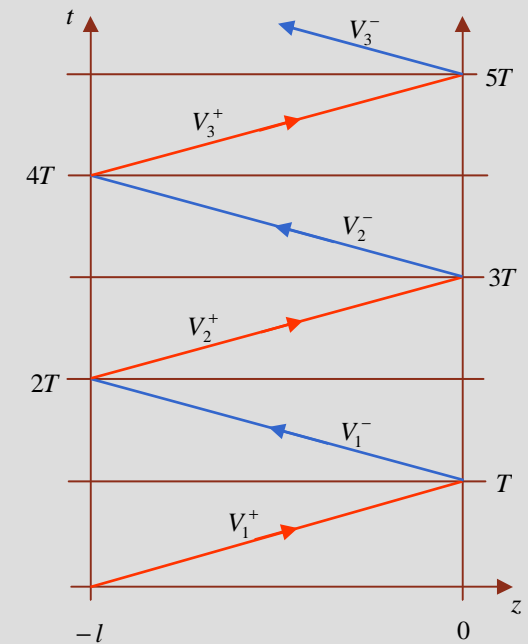
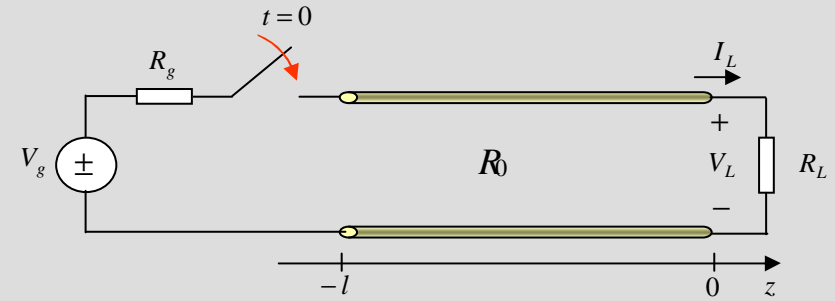
$$V_1^+ = \frac{R_0}{R_0 + R_g} V_g$$

$$\Gamma_g = \frac{R_g - R_0}{R_g + R_0}$$

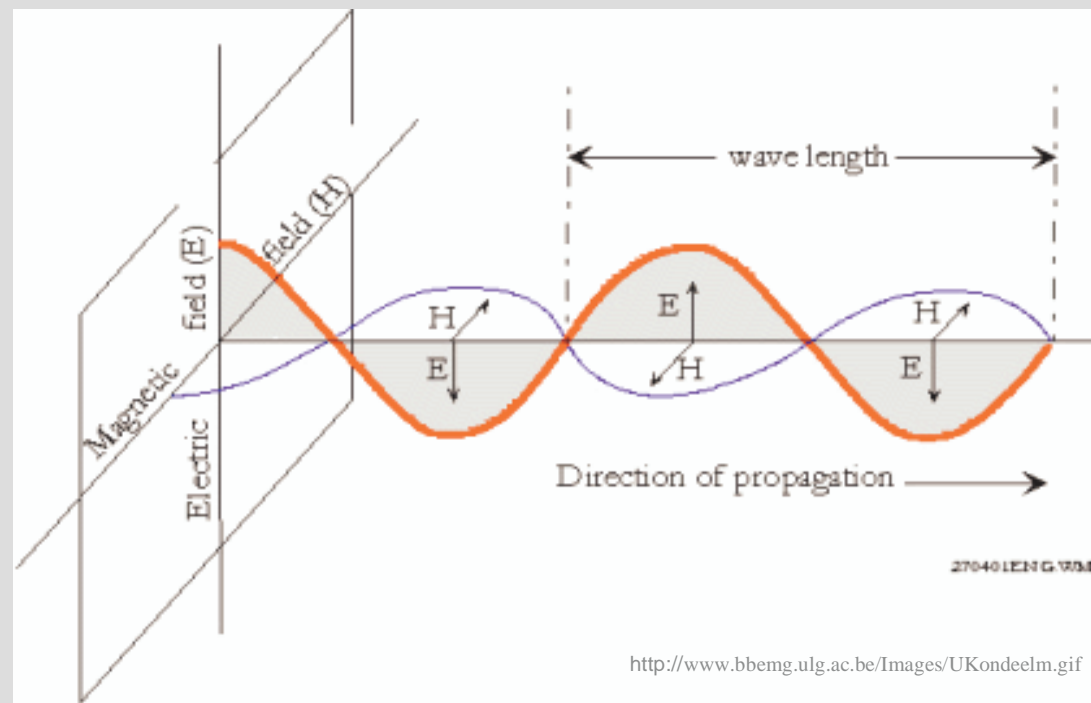
$$\Gamma_L = \frac{R_L - R_0}{R_L + R_0}$$

$$V(t \rightarrow \infty) = \frac{R_L}{R_L + R_g} V_g$$

valor final não depende de  $R_0$



# Ondas electromagnéticas planas



Linhas de transmissão → 6 aulas



**Ondas electromagnéticas planas → 5 aulas**

Guias de onda e cavidades → 7 aulas

Antenas e radiação → 4 aulas

Métodos Numéricos → 2 aulas

- **Equações de Maxwell**
  - formas integral e diferencial
  - meios isotrópicos, homogéneos e lineares
  - campos harmónicos
- **Equação de onda e equação de Helmholtz**
- **Ondas planas e uniformes em meios infinitos sem perdas**
  - relação de dispersão
  - velocidade de fase
  - impedância característica do meio
  - propagação numa direcção arbitrária

# Equações de Maxwell – formas integral e diferencial

	forma integral	forma diferencial
lei de Faraday	$\oint_C \vec{E} \cdot d\vec{l} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$	$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$
lei de Ampère	$\oint_C \vec{H} \cdot d\vec{l} = \int_S \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s}$	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
lei de Gauss	$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho dv$	$\nabla \cdot \vec{D} = \rho$
	$\oint_S \vec{B} \cdot d\vec{s} = 0$	$\nabla \cdot \vec{B} = 0$

# Equações de Maxwell – formas integral e diferencial

forma integral	forma diferencial
$\oint_C \vec{E} \cdot d\vec{l} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$	$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$
$\oint_C \vec{H} \cdot d\vec{l} = \int_S \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s}$	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho dv$	$\nabla \cdot \vec{D} = \rho$
$\oint_S \vec{B} \cdot d\vec{s} = 0$	$\nabla \cdot \vec{B} = 0$

$\vec{E} = \vec{E}(\vec{r}, t)$  → campo eléctrico (V/m)

$\vec{H} = \vec{H}(\vec{r}, t)$  → campo magnético (A/m)

$\vec{B} = \vec{B}(\vec{r}, t)$  → densidade de fluxo magnético (T)

$\vec{D} = \vec{D}(\vec{r}, t)$  → deslocamento eléctrico (C/m<sup>2</sup>)

$\vec{J} = \vec{J}(\vec{r}, t)$  → densidade de corrente eléctrica (A/m<sup>2</sup>)

$\rho = \rho(\vec{r}, t)$  → densidade de carga eléctrica (C/m<sup>3</sup>)

## Meios isotrópicos, homogéneos e lineares

meios:

isotrópicos → propriedades não dependem da direcção

→  $\epsilon$  e  $\mu$  escalares

homogéneos → propriedades não dependem da posição

→  $\epsilon$  e  $\mu$  constantes

lineares → satisfazem o princípio da sobreposição

→  $\vec{D} = \epsilon \vec{E}$  e  $\vec{B} = \mu \vec{H}$

$\epsilon$  → permitividade eléctrica (F/m)

$\mu$  → permeabilidade magnética (H/m)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = \rho / \epsilon$$

$$\nabla \cdot \vec{H} = 0$$

Nota:  
em meios condutores  $\vec{J} = \sigma \vec{E}$



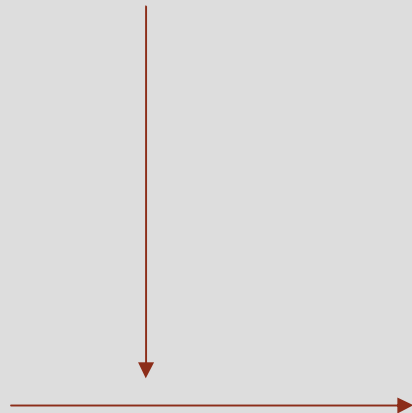
# Equações de Maxwell – campos hamónicos

usando notação fasorial

$$\frac{\partial \vec{X}(\vec{r}, t)}{\partial t} \rightarrow j\omega \vec{X}(\vec{r})$$

$$\begin{aligned} \nabla \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} \\ \nabla \times \vec{H} &= \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} &= \rho / \epsilon \\ \nabla \cdot \vec{H} &= 0 \end{aligned}$$

funções do espaço  
e do tempo



$$\begin{aligned} \nabla \times \vec{E} &= -j\omega\mu\vec{H} \\ \nabla \times \vec{H} &= \vec{J} + j\omega\epsilon\vec{E} \\ \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon} \\ \nabla \cdot \vec{H} &= 0 \end{aligned}$$

funções do espaço

FASORES

# Equação de onda – meios sem cargas e sem perdas

meios em cargas  $\rightarrow \rho = 0$   
 meios em perdas  $\rightarrow \sigma = 0 \Rightarrow \vec{J} = 0$

$$\begin{aligned} \nabla \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} & \nabla \times \vec{H} &= \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} &= \rho / \epsilon & \nabla \cdot \vec{H} &= 0 \end{aligned}$$

$$\begin{aligned} \nabla \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} & \nabla \times \vec{H} &= \epsilon \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} &= 0 & \nabla \cdot \vec{H} &= 0 \end{aligned}$$

$$\nabla \times (\nabla \times \vec{E}) = -\mu \nabla \times \left( \frac{\partial \vec{H}}{\partial t} \right) = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H})$$

$$\nabla \times (\nabla \times \vec{E}) = -\epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\begin{aligned} \nabla \times (\nabla \times \vec{X}) &= \nabla(\nabla \cdot \vec{X}) - \nabla^2 \vec{X} \\ \nabla \cdot \vec{E} &= 0 \end{aligned}$$

# Equação de onda – meios sem cargas e sem perdas

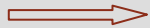
$$\nabla \times (\nabla \times \vec{H}) = \epsilon \nabla \times \left( \frac{\partial \vec{E}}{\partial t} \right) = \epsilon \frac{\partial}{\partial t} (\nabla \times \vec{E}) \quad \leftarrow$$

$$\begin{aligned} \nabla \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} & \nabla \times \vec{H} &= \epsilon \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} &= 0 & \nabla \cdot \vec{H} &= 0 \end{aligned}$$

$$\nabla \times (\nabla \times \vec{H}) = -\epsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\begin{aligned} \nabla \times (\nabla \times \vec{X}) &= \nabla(\nabla \cdot \vec{X}) - \nabla^2 \vec{X} \\ \nabla \cdot \vec{H} &= 0 \end{aligned}$$

$$\nabla^2 \vec{H} = \epsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2}$$



$$\begin{aligned} \nabla^2 \vec{E} &= \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} \\ \nabla^2 \vec{H} &= \epsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2} \end{aligned}$$

**equações de onda**

# Equação de onda – meios sem cargas e sem perdas

em coordenadas cartesianas

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 \vec{X} = \frac{\partial^2 \vec{X}}{\partial x^2} + \frac{\partial^2 \vec{X}}{\partial y^2} + \frac{\partial^2 \vec{X}}{\partial z^2}$$

$$\nabla^2 \vec{E} = \varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \varepsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2}$$

↓

$$\nabla^2 \vec{E} = \varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \varepsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2}$$



$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\frac{\partial^2 \vec{H}}{\partial x^2} + \frac{\partial^2 \vec{H}}{\partial y^2} + \frac{\partial^2 \vec{H}}{\partial z^2} = \varepsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2}$$

# Equação de Helmholtz

em notação fasorial  $\frac{\partial^2 \vec{X}(\vec{r}, t)}{\partial t^2} \rightarrow (j\omega)^2 \vec{X}(\vec{r}) = -\omega^2 \vec{X}(\vec{r})$

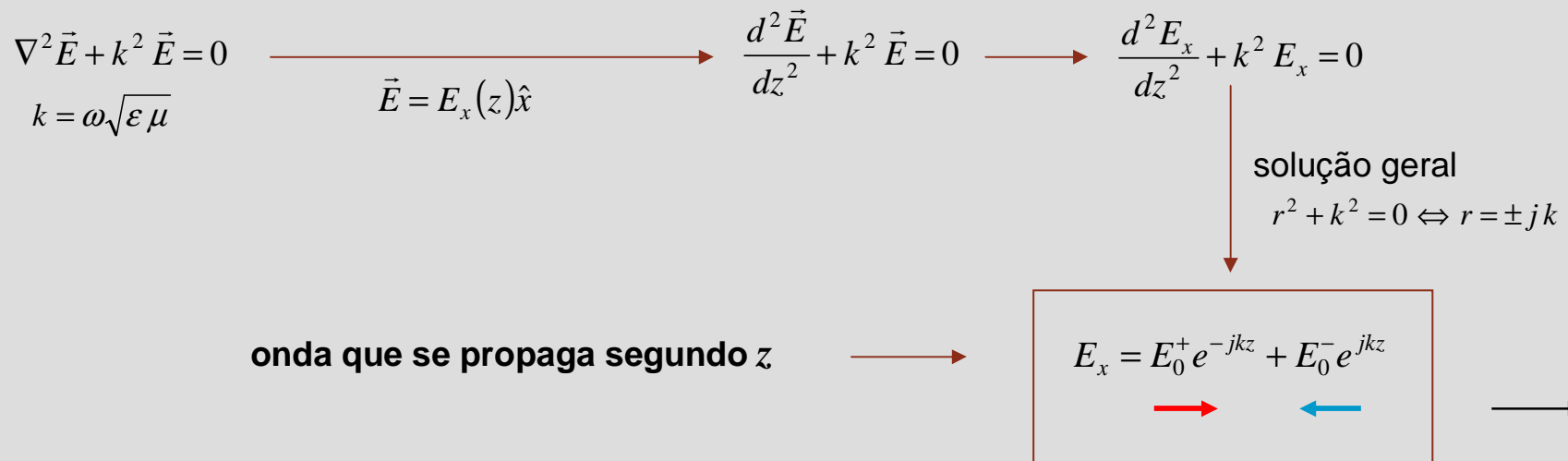
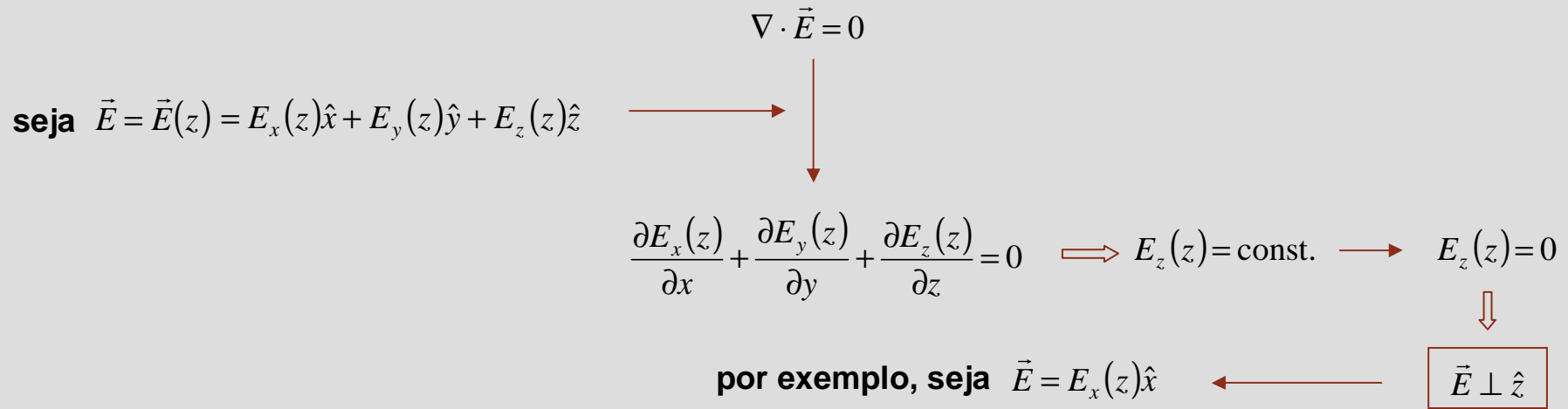
$$\nabla^2 \vec{E} = \varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\nabla^2 \vec{H} = \varepsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2}$$

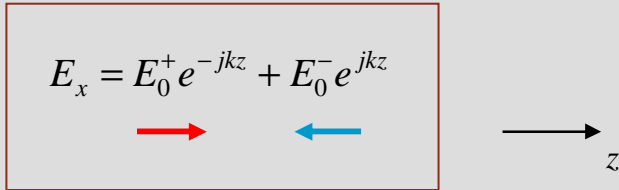
$\Rightarrow$

$$\begin{cases} \nabla^2 \vec{E} + \omega^2 \varepsilon \mu \vec{E} = 0 \\ \nabla^2 \vec{H} + \omega^2 \varepsilon \mu \vec{H} = 0 \end{cases} \iff \begin{cases} \nabla^2 \vec{E} + k^2 \vec{E} = 0 \\ \nabla^2 \vec{H} + k^2 \vec{H} = 0 \end{cases}$$

**equações de Helmholtz**

$$k = \omega \sqrt{\varepsilon \mu} \rightarrow \text{número de onda}$$



$$E_x = E_0^+ e^{-jkz} + E_0^- e^{jkz}$$


fase e amplitude constantes  
nos planos  $z=\text{const.}$



onda plana uniforme que se propaga segundo  $z$

**ondas planas** → fase é constante em planos  
perpendiculares à direcção de propagação

**ondas planas uniformes** → amplitude é constante  
nos planos de fase constante

# Ondas electromagnéticas planas – velocidade de fase

$$E_x = E_0^+ e^{-jkz} + E_0^- e^{jkz} \quad \xrightarrow{v(t) = \text{Re}\{V e^{j\omega t}\}} \quad E_x(z,t) = E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz)$$

$$v_f = \frac{\omega}{k} \quad \xleftarrow{k = \omega \sqrt{\epsilon \mu} \text{ rad/m}} \quad v_f = \frac{1}{\sqrt{\epsilon \mu}} \text{ m/s}$$

↑  
velocidade depende do meio

**no vazio**

$$\epsilon = \epsilon_0 \cong \frac{1}{36\pi} \times 10^{-9} \text{ F/m}$$

$$\mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

$$v_f = c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \cong 3 \times 10^8 \text{ m/s}$$



# Campo magnético

$$E_x = E_0^+ e^{-jkz} + E_0^- e^{jkz}$$

$$\vec{E} = E(z)\hat{x}$$

$$\longrightarrow \vec{H} = ?$$

$$\vec{H} = \frac{j}{\omega\mu} \nabla \times \vec{E}$$

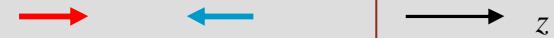
$$\begin{aligned} \nabla \times \vec{E} &= -j\omega\mu\vec{H} \\ \nabla \times \vec{H} &= \vec{J} + j\omega\epsilon\vec{E} \\ \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon} \\ \nabla \cdot \vec{H} &= 0 \end{aligned}$$

$$\vec{H} = \frac{j}{\omega\mu} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0^+ e^{-jkz} + E_0^- e^{jkz} & 0 & 0 \end{vmatrix} = \frac{j}{\omega\mu} \left[ + \frac{\partial}{\partial z} (E_0^+ e^{-jkz} + E_0^- e^{jkz}) \right] \hat{y} = \left( \frac{k}{\omega\mu} E_0^+ e^{-jkz} - \frac{k}{\omega\mu} E_0^- e^{jkz} \right) \hat{y}$$

$$k = \omega\sqrt{\epsilon\mu}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

$$\vec{H} = \left( \frac{1}{\eta} E_0^+ e^{-jkz} - \frac{1}{\eta} E_0^- e^{jkz} \right) \hat{y}$$



$$\vec{E} = (E_0^+ e^{-jkz} + E_0^- e^{jkz}) \hat{x}$$
$$\vec{H} = \left( \frac{1}{\eta} E_0^+ e^{-jkz} - \frac{1}{\eta} E_0^- e^{jkz} \right) \hat{y}$$



$$\eta = \sqrt{\frac{\mu}{\epsilon}} \Omega \quad \text{é a impedância intrínseca do meio}$$

**no vazio**

$$\epsilon = \epsilon_0 \cong \frac{1}{36\pi} \times 10^{-9} \text{ F/m}$$

$$\mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$



$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 120\pi \Omega \cong 377 \Omega$$

## Ondas electromagnéticas transversais

**direccção de propagação:**  $z$

$$\vec{E} = (E_0^+ e^{-jkz} + E_0^- e^{jkz}) \hat{x}$$

$$\vec{H} = \left( \frac{1}{\eta} E_0^+ e^{-jkz} - \frac{1}{\eta} E_0^- e^{jkz} \right) \hat{y}$$

⇒  $\vec{E}$  e  $\vec{H}$  são perpendiculares entre si e ambos são perpendiculares à direcção de propagação



**ondas electromagnéticas transversais**



**ondas TEM**

## Ondas TEM – propagação numa direcção arbitrária

seja  $\vec{E} = E_0 e^{-j(k_x x + k_y y + k_z z)} \hat{p}_e$  ← **versor que indica direcção do vector campo eléctrico**

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

$$k = \omega \sqrt{\mu \epsilon}$$

$$-(k_x^2 + k_y^2 + k_z^2) \vec{E} + k^2 \vec{E} = 0$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \epsilon$$

$k_x, k_y$  e  $k_z \rightarrow$  componentes de um vector com valor absoluto  $k = \omega \sqrt{\mu \epsilon}$

$$\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} = k \hat{a}_n$$

**vector segundo direcção de propagação**

$$\vec{E} = E_0 e^{-j\vec{k} \cdot \vec{r}} \hat{p}_e$$

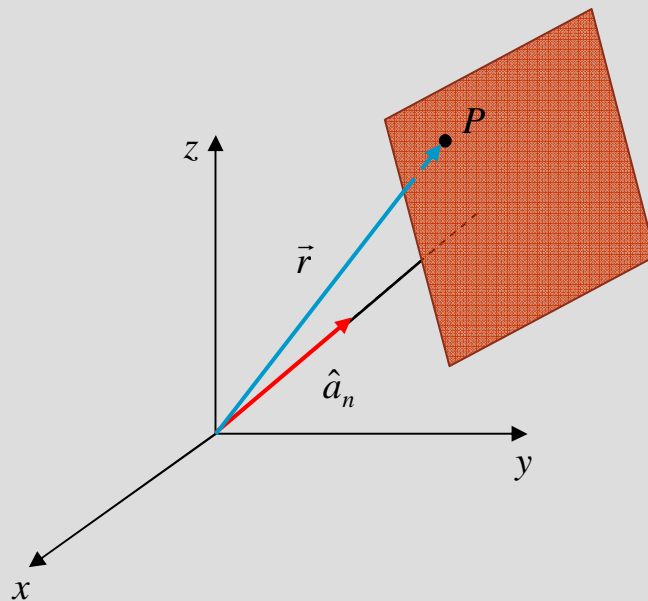
onde  $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$  é o vector de posição

## Ondas TEM – planos de fase constante

$$\vec{E} = E_0 e^{-j\vec{k} \cdot \vec{r}} \hat{p}_e$$

planos de fase constante:  $\vec{k} \cdot \vec{r} = \text{const.}$   $\xrightarrow{\vec{k} = k\hat{a}_n}$   $\hat{a}_n \cdot \vec{r} = \text{const.}$   $\rightarrow$  equação de planos perpendiculares a  $\hat{a}_n$

↑  
projecção de  $\vec{r}$  na direcção de  $\hat{a}_n$



← plano de fase constante e amplitude uniforme

## Ondas TEM – direcção do campo eléctrico

$$\vec{E} = E_0 e^{-j\vec{k}\cdot\vec{r}} \hat{p}_e \xrightarrow{\nabla \cdot \vec{E} = 0} E_0 \nabla \cdot (e^{-j\vec{k}\cdot\vec{r}} \hat{p}_e) = 0 \xrightarrow{\nabla \cdot (f \vec{X}) = f \nabla \cdot \vec{X} + \vec{X} \cdot \nabla f} E_0 \nabla (e^{-j\vec{k}\cdot\vec{r}}) \cdot \hat{p}_e = 0$$

$$\begin{aligned} \nabla(e^{-j\vec{k}\cdot\vec{r}}) &= \left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) e^{-j(k_x x + k_y y + k_z z)} \\ &= -j(k_x \hat{x} + k_y \hat{y} + k_z \hat{z}) e^{-j\vec{k}\cdot\vec{r}} \\ &= -j\vec{k} e^{-j\vec{k}\cdot\vec{r}} \end{aligned}$$

$\vec{E}$  é perpendicular à direcção de propagação!

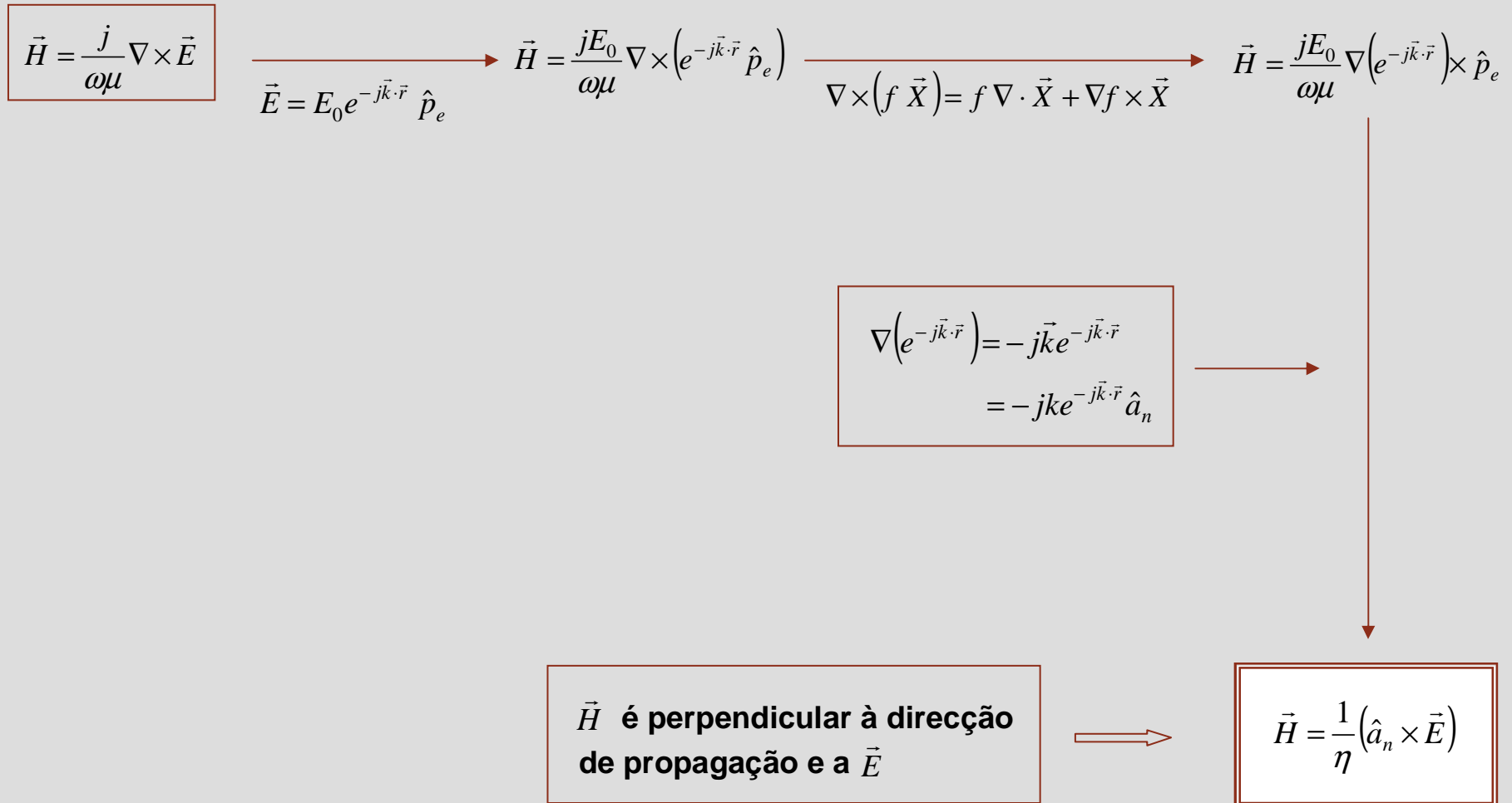


$$\hat{a}_n \cdot \hat{p}_e = 0$$



$$-jkE_0 e^{-j\vec{k}\cdot\vec{r}} \hat{a}_n \cdot \hat{p}_e = 0$$

## Ondas TEM – campo magnético



direcção de  $\vec{E}$   $\rightarrow$  indica a **POLARIZAÇÃO** da onda

se  $\vec{E} = E_0 e^{-jkz} \hat{x} \implies$  onda polarizada **LINEARMENTE** segundo  $\hat{x}$   
 $\downarrow$   
 $\vec{E}(z,t) = E_0 \cos(\omega t - kz) \hat{x} \longrightarrow$  direcção de polarização fixa  
 $\uparrow$

**CASO GERAL**  
 para ondas TEM que se propagam segundo  $+z \implies \vec{E} \perp \hat{z} \implies \vec{E} = E_x \hat{x} + E_y \hat{y}$   
 $\longrightarrow \vec{E} = E_{x0} e^{-jkz} \hat{x} + E_{y0} e^{-jkz} \hat{y}$  onde  $E_{x0}, E_{y0}$  são complexos  $\longrightarrow \begin{matrix} E_{x0} = A_1 e^{j\phi_1} \\ E_{y0} = A_2 e^{j\phi_2} \end{matrix}$   
 $\downarrow$   
 $\vec{E} = A_1 e^{j(\phi_1 - kz)} \hat{x} + A_2 e^{j(\phi_2 - kz)} \hat{y}$

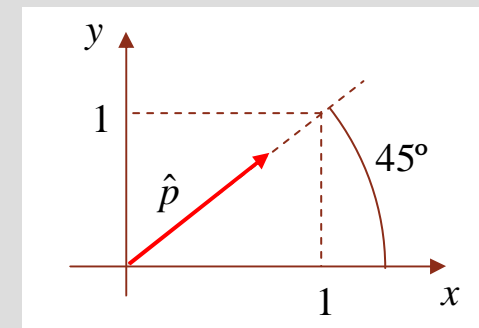


$$\vec{E} = A_1 e^{j(\phi_1 - kz)} \hat{x} + A_2 e^{j(\phi_2 - kz)} \hat{y} \xrightarrow{v(t) = \text{Re}\{V e^{j\omega t}\}} \boxed{\vec{E}(z, t) = A_1 \cos(\omega t - kz + \phi_1) \hat{x} + A_2 \cos(\omega t - kz + \phi_2) \hat{y}}$$

## casos particulares

1.  $A_2 = 0$   $\longrightarrow$   $\vec{E}(z, t) = A_1 \cos(\omega t - kz + \phi_1) \hat{x}$   $\longrightarrow$  polarização linear segundo  $\hat{x}$
2.  $A_1 = 0$   $\longrightarrow$   $\vec{E}(z, t) = A_2 \cos(\omega t - kz + \phi_2) \hat{y}$   $\longrightarrow$  polarização linear segundo  $\hat{y}$
3.  $\phi_1 = \phi_2 = \phi$   $\longrightarrow$   $\vec{E}(z, t) = A \cos(\omega t - kz + \phi) (\hat{x} + \hat{y})$   
 $A_1 = A_2 = A$   
 $= A_0 \cos(\omega t - kz + \phi) \hat{p}$   $\longrightarrow$  polarização linear segundo  $\hat{p}$

onde  $A_0 = \sqrt{2}$  e  $\hat{p} = \frac{\hat{x} + \hat{y}}{\sqrt{2}}$



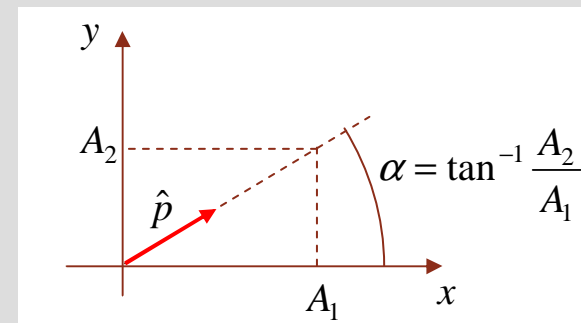
$$\vec{E}(z,t) = A_1 \cos(\omega t - k z + \phi_1) \hat{x} + A_2 \cos(\omega t - k z + \phi_2) \hat{y}$$

## casos particulares

4.  $\phi_1 = \phi_2 = \phi$   $\longrightarrow$   $\vec{E}(z,t) = \cos(\omega t - k z + \phi)(A_1 \hat{x} + A_2 \hat{y})$   
 $A_1 \neq A_2$   $\longrightarrow$   $= A_0 \cos(\omega t - k z + \phi) \hat{p}$   $\longrightarrow$  polarização linear segundo  $\hat{p}$

onde  $A_0 = \sqrt{A_1^2 + A_2^2}$

$$\hat{p} = \frac{A_1 \hat{x} + A_2 \hat{y}}{\sqrt{A_1^2 + A_2^2}}$$



# Polarização de ondas planas – polarização circular direita

$$\vec{E}(z,t) = A_1 \cos(\omega t - k z + \phi_1) \hat{x} + A_2 \cos(\omega t - k z + \phi_2) \hat{y}$$

## casos particulares

5.  $\phi_1 = 0$   $\longrightarrow$   $\vec{E}(z,t) = A \cos(\omega t - k z) \hat{x} + A \cos\left(\omega t - k z - \frac{\pi}{2}\right) \hat{y}$   
 $\phi_2 = -\frac{\pi}{2}$   
 $A_1 = A_2 = A$

$$= A \cos(\omega t - k z) \hat{x} + A \sin(\omega t - k z) \hat{y}$$

↓

$$\vec{E}(0,t) = A \cos(\omega t) \hat{x} + A \sin(\omega t) \hat{y}$$

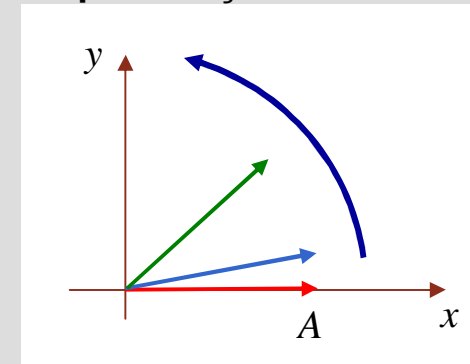
↓

$$\vec{E}(0,0) = A \hat{x}$$

$$\vec{E}(0,t_1) = A \cos(\omega t_1) \hat{x} + A \sin(\omega t_1) \hat{y}$$

$$\vec{E}(0,t_2) = A \cos(\omega t_2) \hat{x} + A \sin(\omega t_2) \hat{y}$$

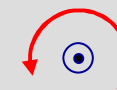
### polarização circular



regra da **mão direita**  $\rightarrow$  **polegar aponta no sentido de propagação**  
**dedos indicam direcção de**  $\vec{E}(0,t)$



**polarização circular direita**



$$\vec{E}(z,t) = A_1 \cos(\omega t - k z + \phi_1) \hat{x} + A_2 \cos(\omega t - k z + \phi_2) \hat{y}$$

## casos particulares

6.  $\phi_1 = 0$   $\longrightarrow$   $\vec{E}(z,t) = A \cos(\omega t - k z) \hat{x} + A \cos\left(\omega t - k z + \frac{\pi}{2}\right) \hat{y}$   
 $\phi_2 = +\frac{\pi}{2}$   
 $A_1 = A_2 = A$

$$= A \cos(\omega t - k z) \hat{x} - A \sin(\omega t - k z) \hat{y}$$

↓

$$\vec{E}(0,t) = A \cos(\omega t) \hat{x} - A \sin(\omega t) \hat{y}$$

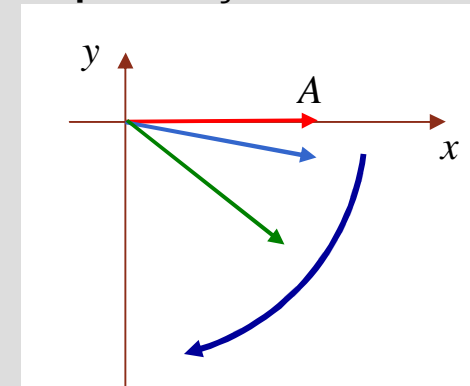
↓

$$\vec{E}(0,0) = A \hat{x}$$

$$\vec{E}(0,t_1) = A \cos(\omega t_1) \hat{x} - A \sin(\omega t_1) \hat{y}$$

$$\vec{E}(0,t_2) = A \cos(\omega t_2) \hat{x} - A \sin(\omega t_2) \hat{y}$$

### polarização circular



regra da mão “esquerda”  $\rightarrow$  polegar aponta no sentido de propagação  $\implies$  polarização circular esquerda  
 dedos indicam direcção de  $\vec{E}(0,t)$



$$\vec{E}(z,t) = A_1 \cos(\omega t - k z + \phi_1) \hat{x} + A_2 \cos(\omega t - k z + \phi_2) \hat{y}$$

## casos particulares

7.  $\phi_1 = 0$   $\longrightarrow$   $\vec{E}(z,t) = A_1 \cos(\omega t - k z) \hat{x} + A_2 \cos\left(\omega t - k z \pm \frac{\pi}{2}\right) \hat{y}$   
 $\phi_2 = \pm \frac{\pi}{2}$   
 $A_1 \neq A_2$

$$= A_1 \cos(\omega t - k z) \hat{x} \mp A_2 \sin(\omega t - k z) \hat{y}$$



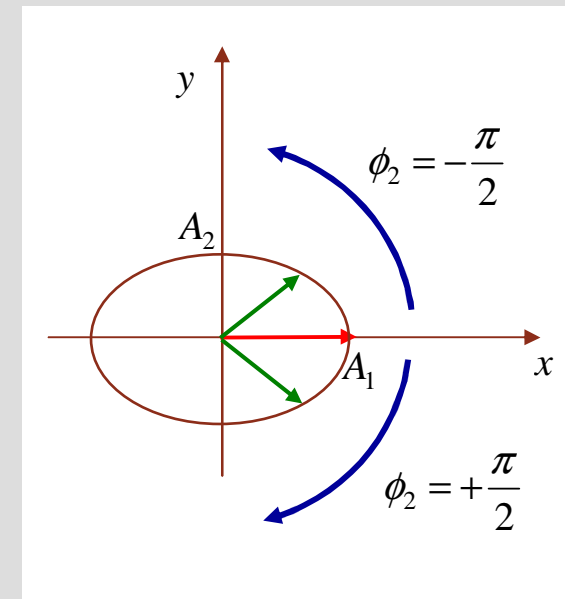
$$\vec{E}(0,t) = A_1 \cos(\omega t) \hat{x} \mp A_2 \sin(\omega t) \hat{y}$$



$$\vec{E}(0,0) = A_1 \hat{x}$$

$$\vec{E}(0,t_1) = A_1 \cos(\omega t_1) \hat{x} \mp A_2 \sin(\omega t_1) \hat{y}$$

## polarização elíptica



## Polarização de ondas planas – resumo

seja  $\vec{E} = E_x \hat{x} + E_y \hat{y}$  (soma de duas ondas linearmente polarizadas em quadratura no espaço)

**se ondas em fase  $\longrightarrow$  onda resultante tem polarização linear**

direcção do versor de polarização depende da relação entre amplitudes das duas ondas

**se diferença de fase =  $90^\circ$   $\longrightarrow$  onda resultante tem polarização circular ou elíptica**

circular  $\rightarrow$  amplitudes iguais

elíptica  $\rightarrow$  amplitudes diferentes

**se diferença de fase arbitrária  $\longrightarrow$  onda resultante tem polarização elíptica**

eixos da elipse não coincidem com  $x$  e  $y$

## Polarização de ondas planas – aplicações

**ondas AM** → emitidas em polarização linear, com  $\vec{E}$  orientado perpendicularmente ao solo  
antena de recepção deve ser paralela a  $\vec{E}$

**ondas TV** → emitidas em polarização linear, com  $\vec{E}$  orientado paralelamente ao solo  
antena de recepção deve ser paralela a  $\vec{E}$



antenas nos telhados são horizontais

**ondas FM** → emitidas em polarização circular  
antena de recepção deve estar num plano normal à direcção de propagação

## Ondas planas em meios com perdas – permitividade complexa

$$\begin{aligned} \nabla \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} & \nabla \times \vec{H} &= \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} &= \rho / \epsilon & \nabla \cdot \vec{H} &= 0 \end{aligned}$$

**Equações de Maxwell para campos harmónicos em meios LHI com perdas e sem cargas:**

$$\begin{aligned} \vec{D} &= \epsilon \vec{E} & \vec{J} &= \sigma \vec{E} & \rho &= 0 \\ \vec{B} &= \mu \vec{H} \end{aligned}$$

$$\begin{aligned} \Rightarrow \begin{cases} \nabla \times \vec{E} = -j\omega\mu\vec{H} \\ \nabla \times \vec{H} = \sigma\vec{E} + j\omega\epsilon\vec{E} \\ \nabla \cdot \vec{E} = 0 \\ \nabla \cdot \vec{H} = 0 \end{cases} & \longrightarrow \nabla \times \vec{H} = (\sigma + j\omega\epsilon)\vec{E} = j\omega\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)\vec{E} = j\omega\epsilon_c\vec{E} \end{aligned}$$

$$\text{permitividade complexa } \epsilon_c = \epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)$$



## Ondas planas em meios com perdas – tangente de perdas

tangente de perdas  $\rightarrow \tan \delta_c = \frac{\sigma}{\omega \epsilon}$

$$\epsilon_c = \epsilon \left( 1 - j \frac{\sigma}{\omega \epsilon} \right)$$

bom condutor  $\rightarrow \sigma \gg \omega \epsilon$

bom isolador  $\rightarrow \sigma \ll \omega \epsilon$

$\rightarrow$  comportamento de um dado material varia com a frequência

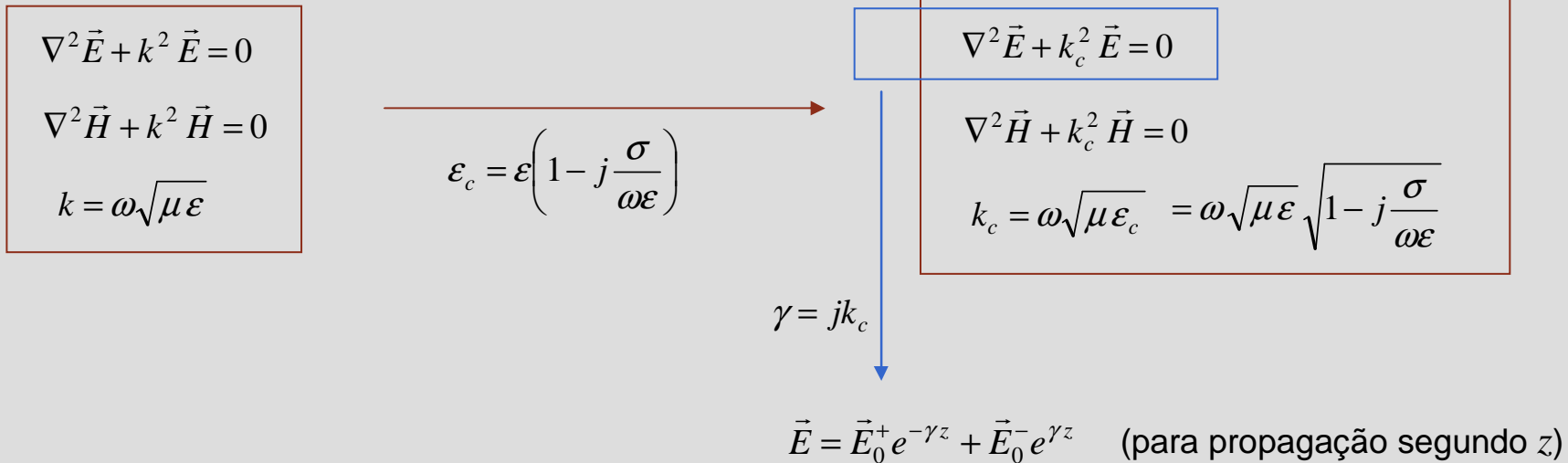
**Ex:**

água do mar  $\rightarrow \sigma = 4 \text{ S/m}$   
 $\epsilon = 72 \epsilon_0$

$f = 50 \text{ Hz} \rightarrow \omega \epsilon = 2 \times 10^{-7} \ll \sigma \rightarrow$  bom condutor

$f = 1 \text{ GHz} \rightarrow \omega \epsilon = 4 = \sigma \rightarrow$  condutor

## Ondas planas em meios com perdas – constante de propagação



**constante de propagação**  $\rightarrow \gamma = \alpha + j\beta = j\omega \sqrt{\mu \epsilon} \sqrt{1 - j \frac{\sigma}{\omega \epsilon}}$

**constante de atenuação**  $\rightarrow \alpha = \text{Re} \left\{ j\omega \sqrt{\mu \epsilon} \sqrt{1 - j \frac{\sigma}{\omega \epsilon}} \right\}$

**constante de fase**  $\rightarrow \beta = \text{Im} \left\{ j\omega \sqrt{\mu \epsilon} \sqrt{1 - j \frac{\sigma}{\omega \epsilon}} \right\}$

**Nota:**  $\beta = \frac{2\pi}{\lambda}$   
 $v_f = \frac{\omega}{\beta}$

## Ondas planas em meios com perdas – impedância complexa

$$\vec{H} = \frac{1}{\eta} (\hat{a}_n \times \vec{E})$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \Omega$$

$\xrightarrow{\epsilon_c = \epsilon \left(1 - j \frac{\sigma}{\omega \epsilon}\right)}$

$$\vec{H} = \frac{1}{\eta_c} (\hat{a}_n \times \vec{E})$$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} \Omega$$

impedância complexa  $\rightarrow$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\sqrt{1 - j \frac{\sigma}{\omega \epsilon}}}$$

em meios com perdas,  $\vec{E}$  e  $\vec{H}$  não estão em fase

## Bons condutores – constante de propagação

bom condutor  $\rightarrow \sigma \gg \omega \epsilon$

$$\sqrt{1 - j \frac{\sigma}{\omega \epsilon}} \cong \sqrt{-j \frac{\sigma}{\omega \epsilon}} = \sqrt{\frac{\sigma}{\omega \epsilon}} \sqrt{-j} = \sqrt{\frac{\sigma}{\omega \epsilon}} \sqrt{\frac{1}{j}}$$

$$\gamma \cong j \omega \sqrt{\mu \epsilon} \sqrt{\frac{\sigma}{\omega \epsilon}} \sqrt{\frac{1}{j}} = \sqrt{\omega \mu \sigma} \sqrt{j} = \sqrt{\frac{\omega \mu \sigma}{2}} (1 + j) = \sqrt{\pi f \mu \sigma} (1 + j)$$

$$\Rightarrow \alpha = \beta = \sqrt{\pi f \mu \sigma}$$

$$v_f = \sqrt{\frac{2\omega}{\mu \sigma}}$$

$$\lambda = 2 \sqrt{\frac{\pi}{f \mu \sigma}} = \frac{v_f}{f}$$

$$\gamma = \alpha + j\beta = j\omega \sqrt{\mu \epsilon} \sqrt{1 - j \frac{\sigma}{\omega \epsilon}}$$

$$v_f = \frac{\omega}{\beta} \quad \lambda = \frac{2\pi}{\beta}$$

$$\nabla \times \vec{H} = (\sigma + j\omega \epsilon) \vec{E}$$

## Bons condutores – impedância intrínseca

bom condutor  $\rightarrow \sigma \gg \omega\epsilon \longrightarrow \sqrt{1 - j\frac{\sigma}{\omega\epsilon}} \cong \sqrt{\frac{\sigma}{\omega\epsilon}} \sqrt{\frac{1}{j}}$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\sqrt{1 - j\frac{\sigma}{\omega\epsilon}}}$$

$\eta_c = \sqrt{\frac{\pi f \mu}{\sigma}} (1 + j) \implies \angle \eta_c = 45^\circ \longrightarrow \vec{H}$  atrasado  $45^\circ$  em relação a  $\vec{E}$

$$\vec{H} = \frac{1}{\eta_c} (\hat{a}_n \times \vec{E})$$

**Nota:**

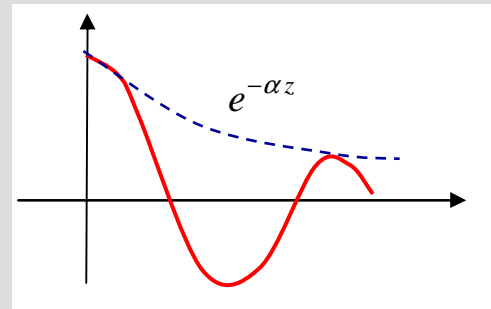
se  $\angle \eta_c = 45^\circ \implies \sigma \gg \omega\epsilon \implies$  **bom condutor**

## Bons condutores – efeito pelicular

bom condutor  $\rightarrow \sigma \gg \omega\epsilon \longrightarrow \alpha = \beta = \sqrt{\pi f \mu \sigma}$

para propagação segundo +z:  $\vec{E} = \vec{E}_0^+ e^{-\gamma z} = \vec{E}_0^+ e^{-\alpha z} e^{-j\beta z}$

factor de atenuação:  $e^{-\alpha z}$



frequências elevadas  $\rightarrow \alpha$  elevado  $\rightarrow$  onda sofre atenuação considerável

propagação apenas numa pequena película

**efeito pelicular**

profundidade de penetração  $\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}$  (m)

## Bons condutores – efeito pelicular

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \text{ (m)}$$

### profundidade de penetração

material	$\sigma$ (S/m)	$f = 60$ (Hz)	$f = 1$ (MHz)	$f = 1$ (GHz)
prata	$6.17 \times 10^7$	8.27 mm	0.064 mm	0.002 mm
cobre	$5.8 \times 10^7$	8.53 mm	0.066 mm	0.0021 mm
ouro	$4.1 \times 10^7$	10.14 mm	0.079 mm	0.0025 mm
alumínio	$3.54 \times 10^7$	10.92 mm	0.084 mm	0.0027 mm
água do mar	4	32 m	0.25 m	(já não é bom condutor a esta frequência)

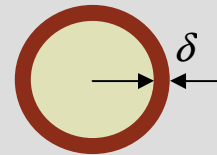
## Bons condutores – efeito pelicular

profundidade de penetração  $\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}$  (m)

**efeito pelicular** → propagação apenas numa pequena película para altas frequências



**condutor cilíndrico** → a altas frequências, a corrente circula numa coroa cilíndrica exterior de espessura  $\delta$



são usados tubos cilíndricos ocos em condutores para altas frequências (ex. antenas)



## Bons condutores – efeito pelicular

profundidade de penetração  $\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}$  (m)

**efeito pelicular** → circulação de corrente numa pequena película para altas frequências



**variação da resistência com a frequência**

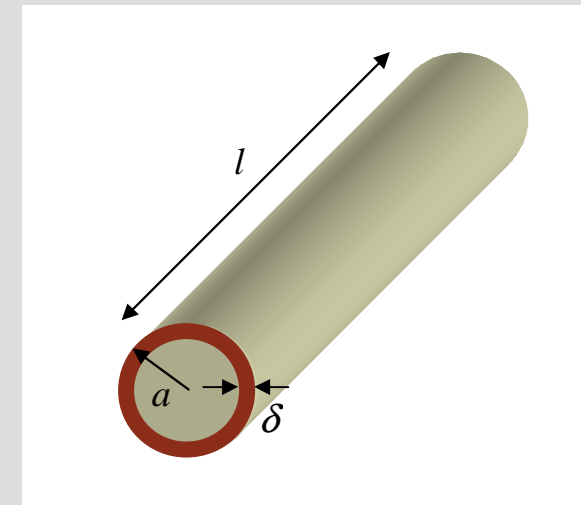
resistência DC →

$$R_{DC} = \frac{l}{\sigma \pi a^2}$$

resistência AC →

$$R_{AC} = \frac{l}{\sigma \delta 2\pi a}$$

→  $\frac{R_{AC}}{R_{DC}} = \frac{a}{2\delta}$



a altas frequências →  $\delta \ll a$  ⇒  $R_{AC} \gg R_{DC}$

## Velocidade de grupo – dispersão

velocidade de fase → velocidade de propagação da frente de onda de fase constante

$$\longrightarrow v_f = \frac{\omega}{\beta}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon} \sqrt{1 - j\frac{\sigma}{\omega\epsilon}}$$

$$\beta = \text{Im} \left\{ j\omega\sqrt{\mu\epsilon} \sqrt{1 - j\frac{\sigma}{\omega\epsilon}} \right\}$$

meios sem perdas →  $\beta = \omega\sqrt{\mu\epsilon}$  →  $v_f = \frac{1}{\sqrt{\mu\epsilon}}$  é constante

meios com perdas →  $\beta$  não é função linear de  $\omega$  →  $v_f$  depende da frequência

**DISPERSÃO**

em sinais que consistem numa dada banda de frequências, as componentes a diferentes frequências propagam-se a velocidades de fase diferentes → distorção do sinal

## Velocidade de grupo – relação de dispersão; meios dispersivos

**relação de dispersão** → equação que relaciona  $\beta$  com  $\omega$

Ex: meios sem perdas →  $\beta = \omega\sqrt{\mu\epsilon}$

**meios dispersivos** → meios para os quais a velocidade de fase depende da frequência

meios sem perdas são meios não dispersivos  
meios com perdas são meios dispersivos

## Velocidade de grupo

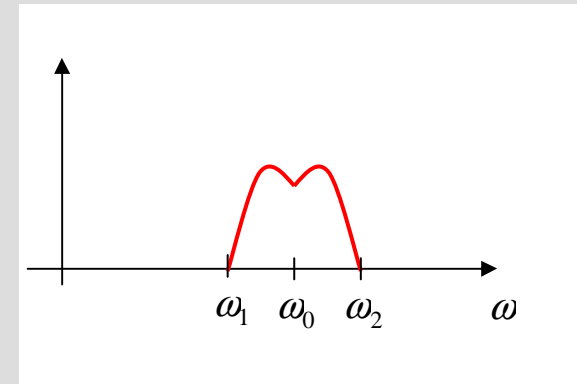
no caso geral  $\beta = \beta(\omega)$  ← relação de dispersão

$$v_f = v_f(\omega)$$

sinal de largura de banda  $2\Delta\omega$  centrada numa portadora  $\omega_0$

$$\begin{aligned} \rightarrow \quad \omega_1 &= \omega_0 - \Delta\omega \\ \omega_2 &= \omega_0 + \Delta\omega \end{aligned} \quad \omega_0 \gg \Delta\omega$$

$$\begin{aligned} \downarrow \\ \beta_1 &= \beta_0 - \Delta\beta \\ \beta_2 &= \beta_0 + \Delta\beta \end{aligned}$$



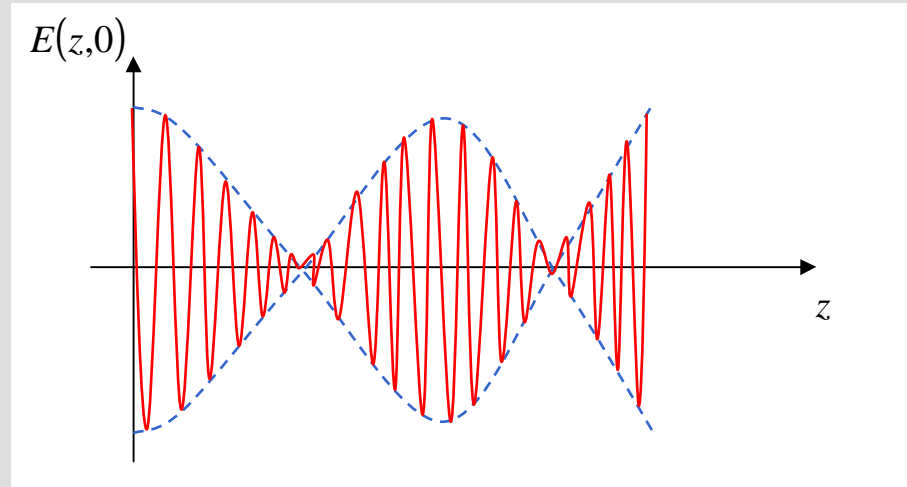
ondas planas correspondentes a  $\omega_1$  e  $\omega_2$ :  
(propagação segundo  $+z$ )

$$\begin{aligned} E(z,t) &= E_0 \{ \cos[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z] + \cos[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z] \} \\ &= 2E_0 \cos(\Delta\omega t - \Delta\beta z) \cos(\omega_0 t - \beta_0 z) \end{aligned}$$

## Velocidade de grupo

$$E(z,t) = 2E_0 \cos(\Delta\omega t - \Delta\beta z) \cos(\omega_0 t - \beta_0 z)$$

↓ envolvente      ↓ portadora



portadora → propaga-se à velocidade  $v_f = \frac{\omega_0}{\beta_0}$

envolvente → propaga-se à velocidade  $v_g = \frac{\Delta\omega}{\Delta\beta}$

$\xrightarrow{\lim \Delta\omega \rightarrow 0}$

$$v_g = \frac{1}{d\beta/d\omega} \text{ (m/s)}$$

**velocidade de grupo**

## Velocidade de grupo – dispersão normal e anómala

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega} \left( \frac{\omega}{v_f} \right) = \frac{v_f - \omega \frac{dv_f}{d\omega}}{v_f^2}$$



$$v_g = \frac{v_f}{1 - \frac{\omega}{v_f} \frac{dv_f}{d\omega}}$$

$$v_g = \frac{1}{d\beta/d\omega} \quad \leftarrow \text{velocidade de grupo}$$

$$v_f = \frac{\omega}{\beta} \quad \leftarrow \text{velocidade de fase}$$

### casos particulares

1.  $\frac{dv_f}{d\omega} = 0 \quad \longrightarrow \quad v_g = v_f \quad \longleftarrow \quad \text{sem dispersão} \quad (v_f \text{ constante})$
2.  $\frac{dv_f}{d\omega} < 0 \quad \longrightarrow \quad v_g < v_f \quad \longleftarrow \quad \text{dispersão normal} \quad (v_f \text{ diminui com } \omega)$
3.  $\frac{dv_f}{d\omega} > 0 \quad \longrightarrow \quad v_g > v_f \quad \longleftarrow \quad \text{dispersão anómala} \quad (v_f \text{ aumenta com } \omega)$

## Energia transportada por uma onda

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) \quad (\text{igualdade vectorial})$$

W/m<sup>2</sup>

$$\begin{aligned} \nabla \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} \\ \nabla \times \vec{H} &= \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot \left( -\mu \frac{\partial \vec{H}}{\partial t} \right) - \vec{E} \cdot \left( \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \right) \longrightarrow \nabla \cdot (\vec{E} \times \vec{H}) = -\mu \frac{1}{2} \frac{\partial}{\partial t} (\vec{H} \cdot \vec{H}) - \epsilon \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E}) - \vec{E} \cdot \vec{J}$$

$$\vec{A} \cdot \frac{\partial \vec{A}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{A} \cdot \vec{A})$$

$$\vec{J} = \sigma \vec{E}$$

$$\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_V \left( \frac{\mu}{2} |\vec{H}|^2 + \frac{\epsilon}{2} |\vec{E}|^2 \right) dv - \int_V \sigma |\vec{E}|^2 dv$$

$$\nabla \cdot (\vec{E} \times \vec{H}) = -\frac{\partial}{\partial t} \left( \frac{\mu}{2} |\vec{H}|^2 + \frac{\epsilon}{2} |\vec{E}|^2 \right) - \sigma |\vec{E}|^2$$

$$\oint_S \vec{A} \cdot d\vec{s} = \int_V (\nabla \cdot \vec{A}) dv$$

**Nota: expressões instantâneas**

## Teorema de Poynting

$$\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_V \left( \frac{\mu}{2} |\vec{H}|^2 + \frac{\epsilon}{2} |\vec{E}|^2 \right) dv - \int_V \sigma |\vec{E}|^2 dv$$

→ **conservação de energia**

↓  
potência que atravessa  $S$

↓  
diminuição da energia armazenada no campo EM por unidade de tempo

↓  
potência dissipada por condução

**Nota: expressões instantâneas**

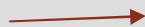


## Vector de Poynting

$$\text{vector de Poynting} \rightarrow \vec{S} = \vec{E} \times \vec{H} \quad (\text{W/m}^2)$$



representa a **densidade de potência instantânea** transportada pela onda electromagnética



$$\oint_S \vec{S} \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_V (w_m + w_e) dv - \int_V p_\sigma dv$$

$$\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_V \left( \frac{\mu}{2} |\vec{H}|^2 + \frac{\epsilon}{2} |\vec{E}|^2 \right) dv - \int_V \sigma |\vec{E}|^2 dv$$

$$w_m = \frac{1}{2} \mu |\vec{H}|^2$$

$$w_e = \frac{1}{2} \epsilon |\vec{E}|^2$$

$$p_\sigma = \sigma |\vec{E}|^2$$

Nota: expressões instantâneas

## Vector de Poynting – campos harmónicos

$$\vec{E}(\vec{r}, t) = \text{Re}\{\vec{E}(\vec{r})e^{j\omega t}\}$$

$$\vec{H}(\vec{r}, t) = \text{Re}\{\vec{H}(\vec{r})e^{j\omega t}\}$$

$$\vec{S}(\vec{r}, t) = \vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t) = \text{Re}\{\vec{E}(\vec{r})e^{j\omega t}\} \times \text{Re}\{\vec{H}(\vec{r})e^{j\omega t}\}$$

$$\vec{S}(\vec{r}, t) = \frac{1}{2} \text{Re}\{\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r}) + \vec{E}(\vec{r}) \times \vec{H}(\vec{r})e^{j2\omega t}\}$$

valor instantâneo

fasores

$$\text{Re}\{\vec{X}\} = \frac{1}{2} \{\vec{X} + \vec{X}^*\}$$

$$\begin{aligned} \text{Re}\{\vec{A}\} \times \text{Re}\{\vec{B}\} &= \frac{1}{2} \{\vec{A} + \vec{A}^*\} \times \frac{1}{2} \{\vec{B} + \vec{B}^*\} \\ &= \frac{1}{4} \{\vec{A} \times \vec{B}^* + \vec{A}^* \times \vec{B} + \vec{A} \times \vec{B} + \vec{A}^* \times \vec{B}^*\} \\ &= \frac{1}{2} \text{Re}\{\vec{A} \times \vec{B}^* + \vec{A} \times \vec{B}\} \end{aligned}$$

## Vector de Poynting médio

Densidade de potência média  $\vec{S}_{\text{med}}(\vec{r}) = \frac{1}{T} \int_T \vec{S}(\vec{r}, t) dt$

$$\vec{S}(\vec{r}, t) = \frac{1}{2} \text{Re} \left\{ \vec{E}(\vec{r}) \times \vec{H}^*(\vec{r}) + \vec{E}(\vec{r}) \times \vec{H}(\vec{r}) e^{j2\omega t} \right\}$$

$$\vec{S}_{\text{med}}(\vec{r}) = \frac{1}{2} \text{Re} \left\{ \vec{E}(\vec{r}) \times \vec{H}^*(\vec{r}) \right\} \quad (\text{W/m}^2)$$

### Ondas TEM

$$\vec{H} = \frac{1}{\eta} \hat{a}_n \times \vec{E}$$

$$\vec{E} \perp \vec{H} \perp \hat{a}_n$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - \vec{C} (\vec{A} \cdot \vec{B})$$

$$\vec{E} \times \vec{H}^* = \frac{1}{\eta^*} \vec{E} \times (\hat{a}_n \times \vec{E}^*) = \frac{1}{\eta^*} \left\{ (\vec{E} \cdot \vec{E}^*) \hat{a}_n - \vec{E}^* (\vec{E} \cdot \hat{a}_n) \right\} = \frac{1}{\eta^*} |\vec{E}|^2 \hat{a}_n$$

$$\vec{S}_{\text{med}} = \frac{1}{2} \text{Re} \left\{ \vec{E} \times \vec{H}^* \right\} = \frac{1}{2} |\vec{E}|^2 \text{Re} \left\{ \frac{1}{\eta^*} \right\} \hat{a}_n$$

**vector de Poynting médio aponta na direcção e sentido de propagação da onda**

## Vector de Poynting médio – casos particulares

### 1. Onda TEM com polarização linear

$$\vec{E} = E_0 e^{-(\alpha + j\beta)z} \hat{x}$$

$$\vec{H} = \frac{1}{\eta} E_0 e^{-(\alpha + j\beta)z} \hat{y}$$

$$\vec{S}_{\text{med}} = \frac{1}{2|\eta|} E_0^2 e^{-2\alpha z} \cos \theta_\eta \hat{z}$$

### 2. Onda TEM com polarização circular

$$\vec{E} = E_0 e^{-(\alpha + j\beta)z} (\hat{x} + j\hat{y})$$

$$\vec{H} = \frac{1}{\eta} E_0 e^{-(\alpha + j\beta)z} (\hat{y} - j\hat{x})$$

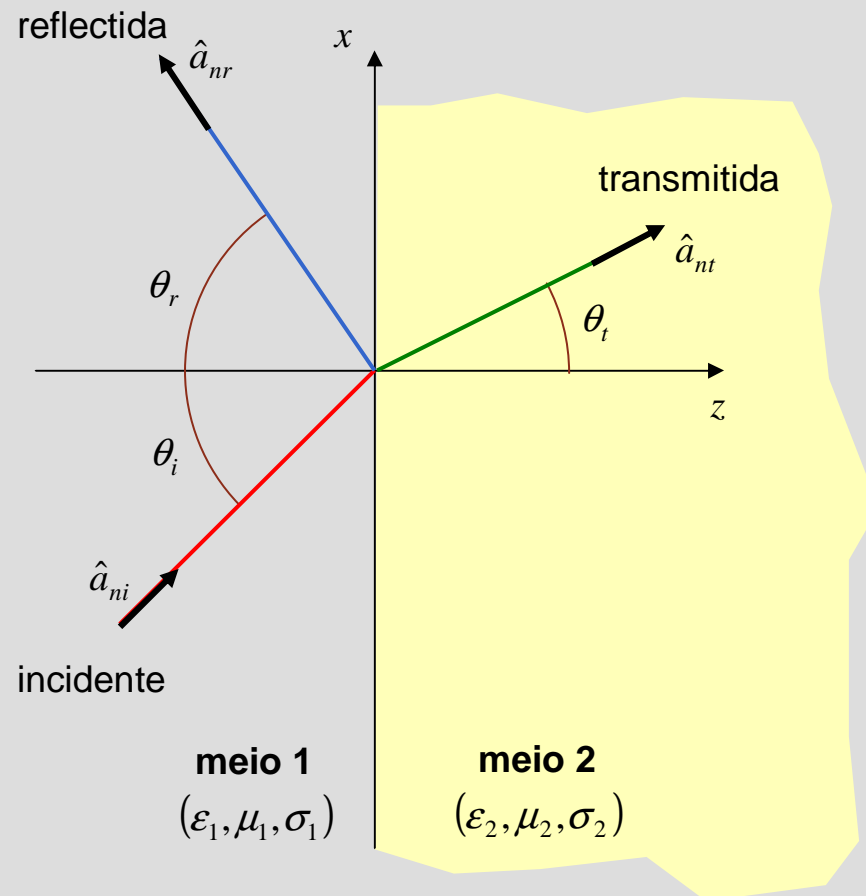
$$\vec{S}_{\text{med}} = \frac{1}{|\eta|} E_0^2 e^{-2\alpha z} \cos \theta_\eta \hat{z}$$

$$\vec{S}_{\text{med}} = \frac{1}{2} \text{Re}\{\vec{E} \times \vec{H}^*\} = \frac{1}{2} |\vec{E}|^2 \text{Re}\left\{\frac{1}{\eta^*}\right\} \hat{a}_n$$

$$\left. \begin{array}{l} \gamma = \alpha + j\beta \\ \eta = |\eta| e^{j\theta_\eta} \end{array} \right\} \text{caso geral: meio com perdas}$$

$$\text{Re}\left\{\frac{1}{\eta^*}\right\} = \text{Re}\left\{\frac{\eta}{|\eta|^2}\right\} = \frac{\cos \theta_\eta}{|\eta|}$$

## Incidência de uma onda TEM numa interface plana



ângulo de incidência  $\rightarrow \theta_i$

plano de incidência  $\rightarrow$  plano  $xz$



plano formado pela normal à interface e pela direcção de propagação da onda incidente

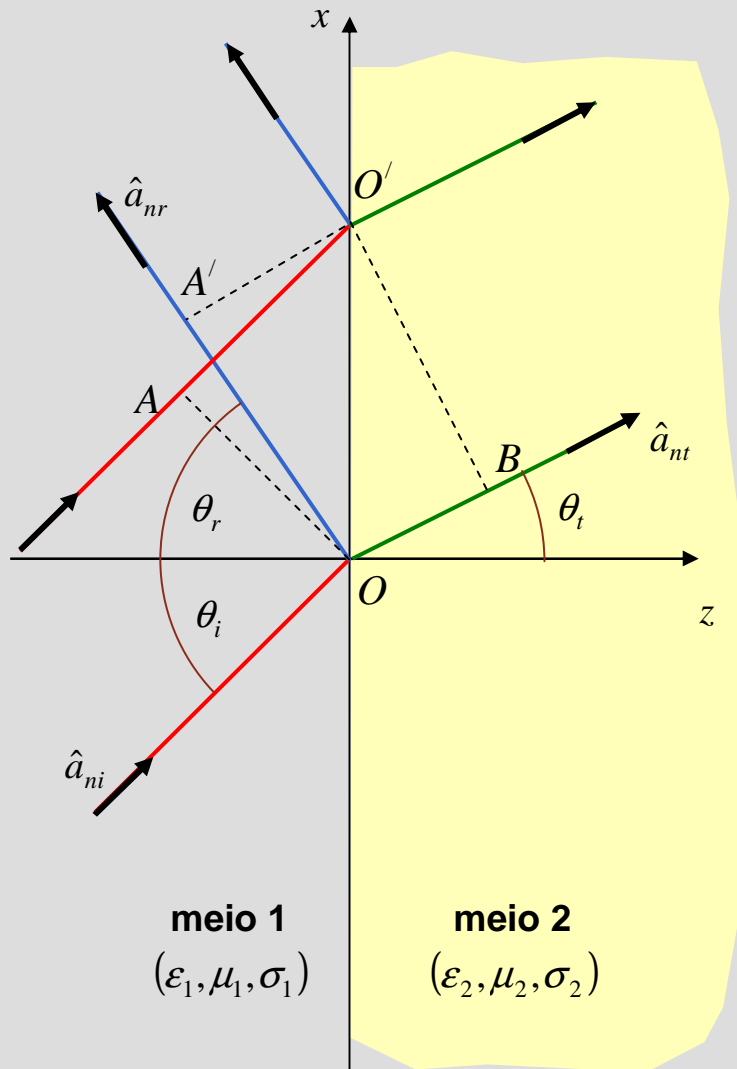
direcções de propagação:

$$\hat{a}_{ni} = \sin \theta_i \hat{x} + \cos \theta_i \hat{z}$$

$$\hat{a}_{nr} = \sin \theta_r \hat{x} - \cos \theta_r \hat{z}$$

$$\hat{a}_{nt} = \sin \theta_t \hat{x} + \cos \theta_t \hat{z}$$

## Leis de Snell – lei da reflexão



frente de onda → mesma fase

ondas planas → frentes de onda são planos normais a  $\hat{a}_n$



pontos  $O$  e  $A$  têm mesma fase

pontos  $O'$  e  $A'$  têm mesma fase



fase =  $\beta \cdot \text{dist.}$



$$\beta_1 \overline{AO'} = \beta_1 \overline{OA'}$$

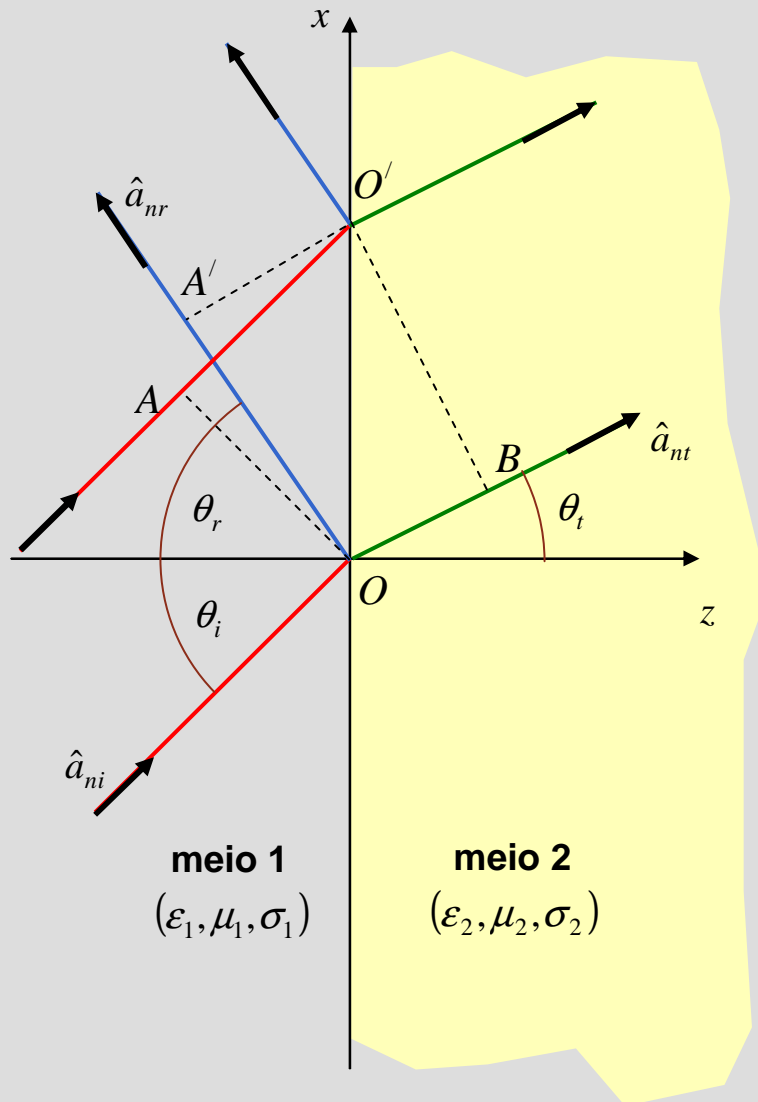


$$\overline{OO'} \sin \theta_i = \overline{OO'} \sin \theta_r$$



$$\theta_i = \theta_r$$

## Leis de Snell – lei da refração



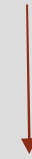
ondas planas → frentes de onda são planos normais a  $\hat{a}_n$



pontos  $O$  e  $A$  têm mesma fase

pontos  $O'$  e  $B$  têm mesma fase

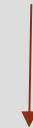
fase =  $\beta \cdot \text{dist.}$



$$\beta_1 \overline{AO'} = \beta_2 \overline{OB}$$



$$\beta_1 \overline{OO'} \sin \theta_i = \beta_2 \overline{OO'} \sin \theta_t$$

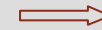


$$\frac{\sin \theta_t}{\sin \theta_i} = \frac{\beta_1}{\beta_2} = \frac{v_{f2}}{v_{f1}}$$

$$\beta = \frac{\omega}{v_f}$$

# Índice de refração

Índice de refração → quociente entre velocidades de propagação no vácuo e no meio



$$n \geq 1$$

$n$  elevado → velocidade baixa

$$n = \frac{c}{v_f}$$

Ex: meio sem perdas  $v_f = \frac{1}{\sqrt{\mu \epsilon}}$

$$n = \frac{\sqrt{\mu \epsilon}}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{\mu_r \epsilon_r}$$

$$\frac{\sin \theta_t}{\sin \theta_i} = \frac{v_{f2}}{v_{f1}}$$

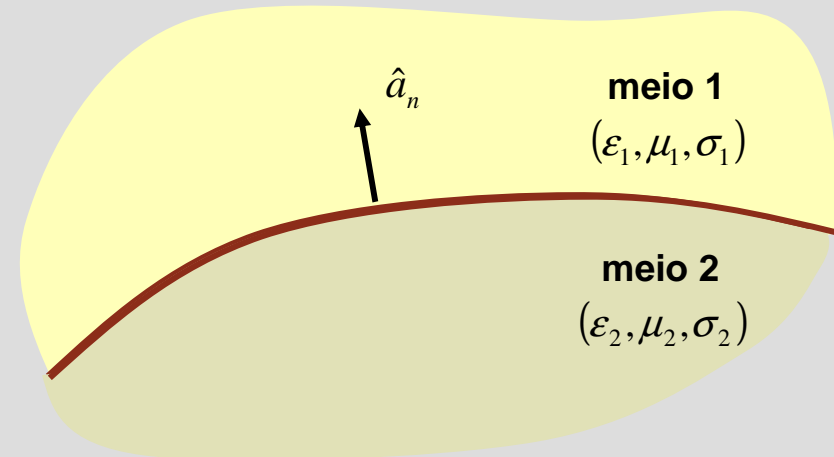
$$\frac{\sin \theta_t}{\sin \theta_i} = \frac{n_1}{n_2}$$

← lei de Snell da refração



## Condições de fronteira

seja agora  $\hat{a}_n$  o versor normal à interface que aponta do meio 2 para o meio 1



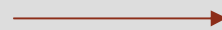
### equações de Maxwell

$$\oint_C \vec{E} \cdot d\vec{l} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s}$$

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho dv$$

$$\oint_S \vec{B} \cdot d\vec{s} = 0$$



$$\hat{a}_n \times (\vec{E}_1 - \vec{E}_2) = 0$$

$$\hat{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$$

$$\hat{a}_n \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$$

$$\hat{a}_n \cdot (\vec{B}_1 - \vec{B}_2) = 0$$

← densidade superficial de corrente (A/m)

← densidade superficial de carga (C/m<sup>3</sup>)

## Condições de fronteira

$$\hat{a}_n \times (\vec{E}_1 - \vec{E}_2) = 0$$

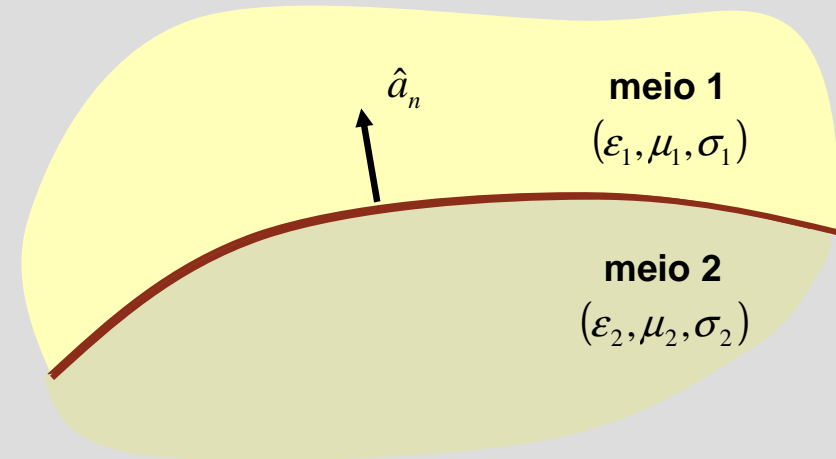
$$\rightarrow E_{1,\text{tan}} = E_{2,\text{tan}}$$

$$\hat{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_S$$

$$\hat{a}_n \cdot (\vec{D}_1 - \vec{D}_2) = \rho_S$$

$$\hat{a}_n \cdot (\vec{B}_1 - \vec{B}_2) = 0$$

$$\rightarrow B_{1,\text{norm}} = B_{2,\text{norm}}$$



$E_{\text{tan}}$  contínuo

$H_{\text{tan}}$  contínuo se  $\vec{J}_S = 0$

$D_{\text{norm}}$  contínuo se  $\rho_S = 0$

$B_{\text{norm}}$  contínuo

**Nota:**

$\vec{J}_S \neq 0$  e  $\rho_S \neq 0$  apenas em condutores perfeitos

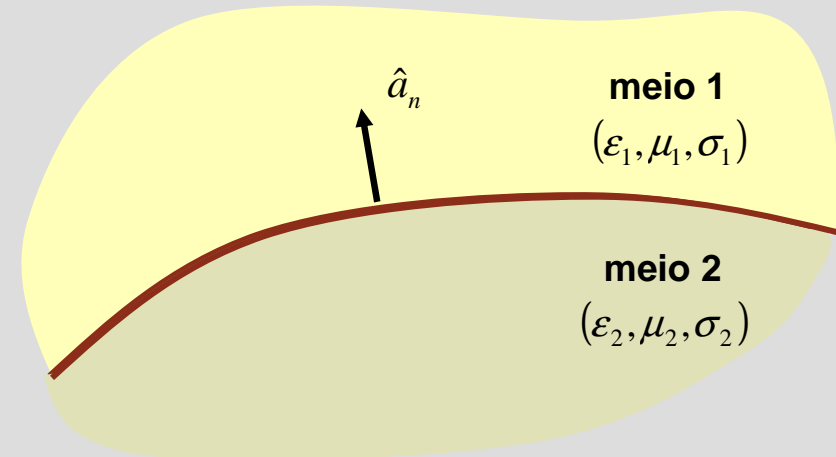
## Condições de fronteira – condutores perfeitos

condutores perfeitos  $\rightarrow \sigma = \infty$

$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}} = 0$$

$$\begin{aligned} \vec{E}_{\text{cond}} &= 0 & \vec{H}_{\text{cond}} &= 0 \\ \vec{D}_{\text{cond}} &= 0 & \vec{B}_{\text{cond}} &= 0 \end{aligned}$$

$\vec{J}_s \neq 0$  e  $\rho_s \neq 0$



**Ex:**  $\sigma_2 = \infty$

$$\vec{J}_s = \hat{a}_n \times (\vec{H}_1 - \vec{H}_2) = \hat{a}_n \times \vec{H}_1 = H_{1,\text{tan}} \hat{a}_t$$

$$\rho_s = \hat{a}_n \cdot (\vec{D}_1 - \vec{D}_2) = \hat{a}_n \cdot \vec{D}_1 = D_{1,\text{norm}} \hat{a}_n$$

## Incidência normal

incidência normal  $\rightarrow \theta_i = 0 \rightarrow \theta_r = \theta_t = 0$

$$\theta_i = \theta_r$$

$$\frac{\sin \theta_t}{\sin \theta_i} = \frac{n_1}{n_2}$$

incidente  $\rightarrow$

$$\vec{E}_i = E_{i0} e^{-\gamma_1 z} \hat{x}$$

$$\vec{H}_i = \frac{E_{i0}}{\eta_1} e^{-\gamma_1 z} \hat{y}$$

reflectida  $\rightarrow$

$$\vec{E}_r = E_{r0} e^{+\gamma_1 z} \hat{x}$$

$$\vec{H}_r = -\frac{E_{r0}}{\eta_1} e^{+\gamma_1 z} \hat{y}$$

transmitida  $\rightarrow$

$$\vec{E}_t = E_{t0} e^{-\gamma_2 z} \hat{x}$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} \hat{y}$$

meio 2

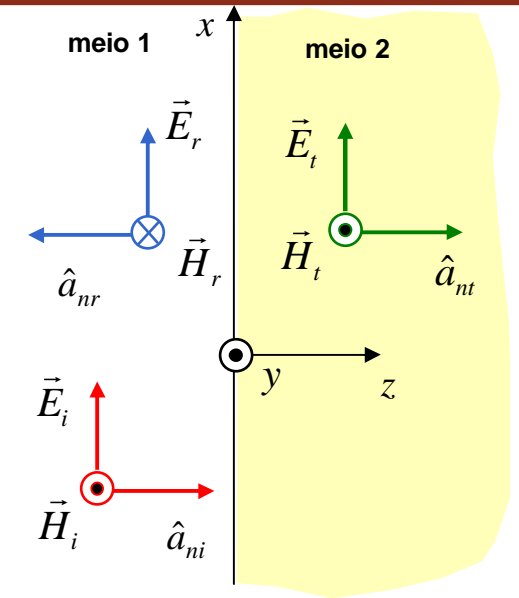
meio 1

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r$$

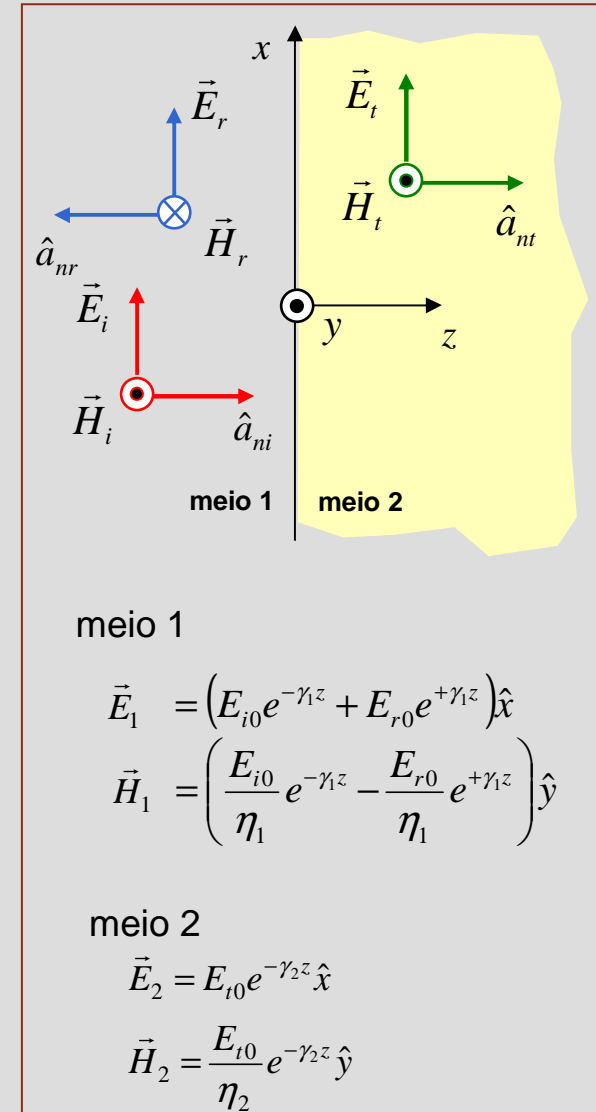
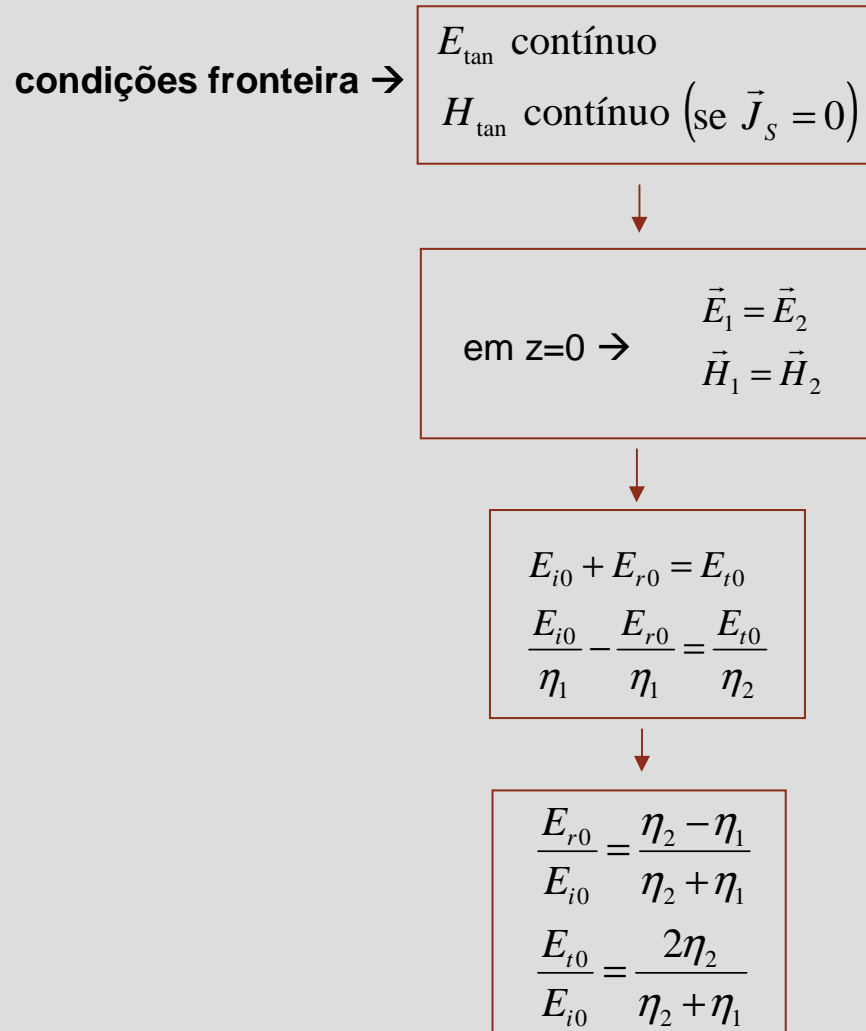
$$= (E_{i0} e^{-\gamma_1 z} + E_{r0} e^{+\gamma_1 z}) \hat{x}$$

$$\vec{H}_1 = \vec{H}_i + \vec{H}_r$$

$$= \left( \frac{E_{i0}}{\eta_1} e^{-\gamma_1 z} - \frac{E_{r0}}{\eta_1} e^{+\gamma_1 z} \right) \hat{y}$$



## Incidência normal – coeficientes de reflexão e transmissão



## Incidência normal – coeficientes de reflexão e transmissão

$$\frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \rightarrow \quad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

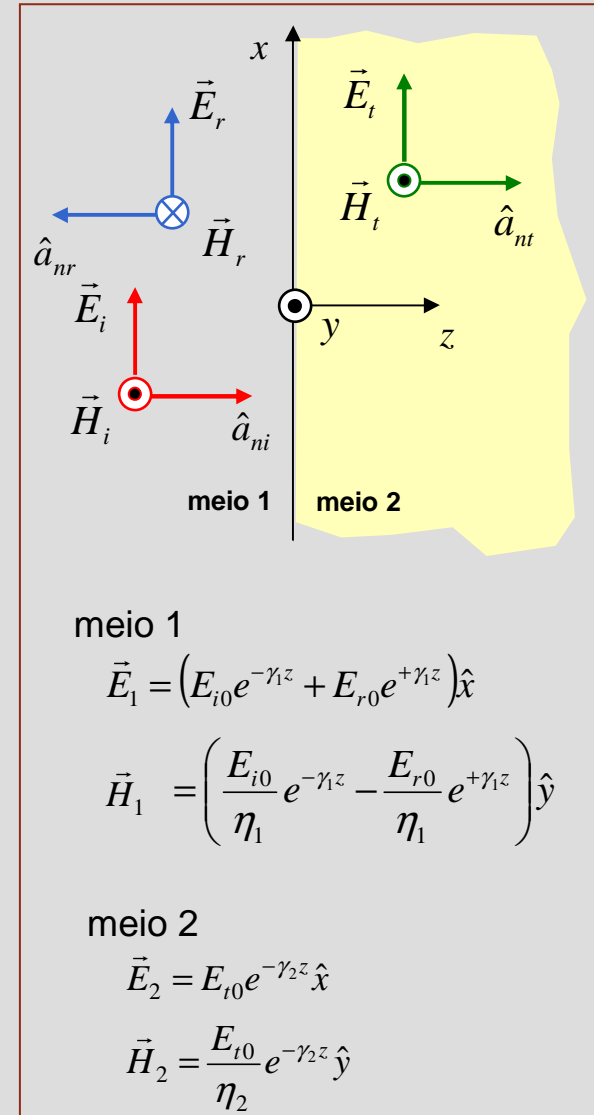
$$\frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1} \quad \rightarrow \quad \tau = \frac{2\eta_2}{\eta_2 + \eta_1}$$

coeficiente de reflexão

coeficiente de transmissão

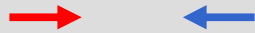
### Notas

1.  $1 + \Gamma = \tau$
2.  $|\Gamma| \leq 1$
3.  $\tau \geq 0$
4.  $\vec{E}_1 = (E_{i0}e^{-\gamma_1 z} + \Gamma E_{i0}e^{+\gamma_1 z})\hat{x}$   
 $\vec{E}_2 = \tau E_{i0}e^{-\gamma_2 z}\hat{x}$



## Incidência normal – onda estacionária

$$\vec{E}_1 = (E_{i0}e^{-\gamma_1 z} + \Gamma E_{i0}e^{+\gamma_1 z})\hat{x} \xrightarrow{1+\Gamma=\tau} \vec{E}_1 = E_{i0}[(\tau-\Gamma)e^{-\gamma_1 z} + \Gamma e^{+\gamma_1 z}]\hat{x}$$



$$\vec{E}_1 = \tau E_{i0}e^{-\gamma_1 z}\hat{x} + \Gamma E_{i0}(e^{+\gamma_1 z} - e^{-\gamma_1 z})\hat{x}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

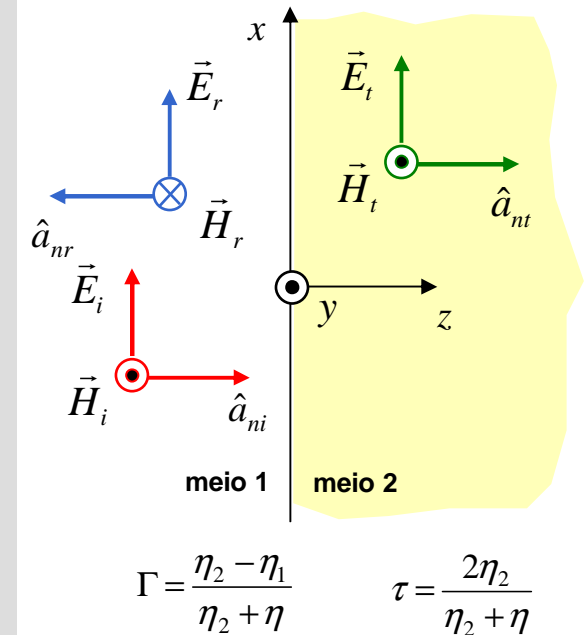
$$\vec{E}_1 = \tau E_{i0}e^{-\gamma_1 z}\hat{x} + 2\Gamma E_{i0} \sinh(\gamma_1 z)\hat{x}$$

$$\gamma_1 = j\beta_1 \quad (\text{meio 1 sem perdas})$$

$$\vec{E}_1 = \tau E_{i0}e^{-\beta_1 z}\hat{x} + 2j\Gamma E_{i0} \sin(\beta_1 z)\hat{x}$$

onda em propagação

onda estacionária



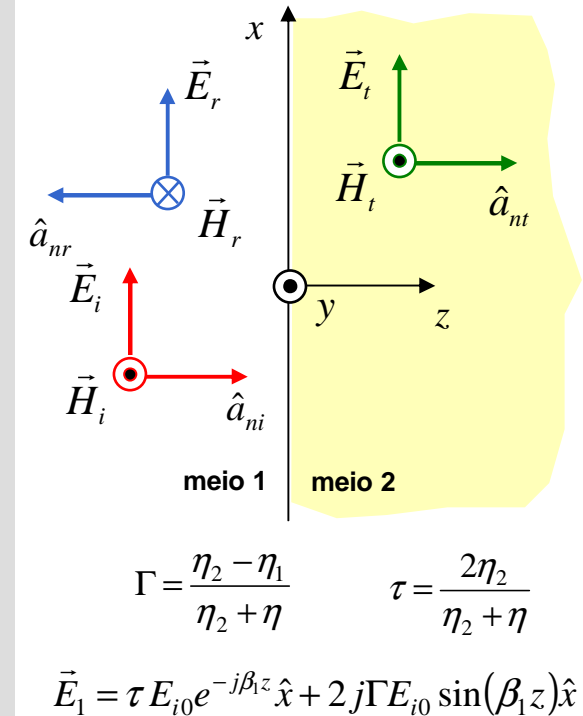
## Incidência normal – máximos e mínimos

meio 1 sem perdas

$$\vec{E}_1 = (E_{i0}e^{-j\beta_1 z} + \Gamma E_{i0}e^{+j\beta_1 z})\hat{x} = E_{i0}e^{-j\beta_1 z}(1 + \Gamma e^{+j2\beta_1 z})\hat{x}$$

$$\Gamma = |\Gamma|e^{j\theta_\Gamma}$$

$$\begin{aligned} |\vec{E}_1| &= |E_{i0}| \sqrt{[1 + |\Gamma| \cos(\theta_\Gamma + 2\beta_1 z)]^2 + (\Gamma \sin(\theta_\Gamma + 2\beta_1 z))^2} \\ &= |E_{i0}| \sqrt{1 + |\Gamma|^2 + 2|\Gamma| \cos(\theta_\Gamma + 2\beta_1 z)} \end{aligned}$$



**máximos:**  $\cos(\theta_\Gamma + 2\beta_1 z) = +1 \rightarrow z_{MAX} = \frac{1}{2\beta_1}(-\theta_\Gamma + 2n\pi) \rightarrow |\vec{E}_1|_{MAX} = |E_{i0}|(1 + |\Gamma|)$

**mínimos:**  $\cos(\theta_\Gamma + 2\beta_1 z) = -1 \rightarrow z_{min} = \frac{1}{2\beta_1}[-\theta_\Gamma + (2n+1)\pi] \rightarrow |\vec{E}_1|_{min} = |E_{i0}|(1 - |\Gamma|)$



## Incidência normal – incidência num condutor ideal

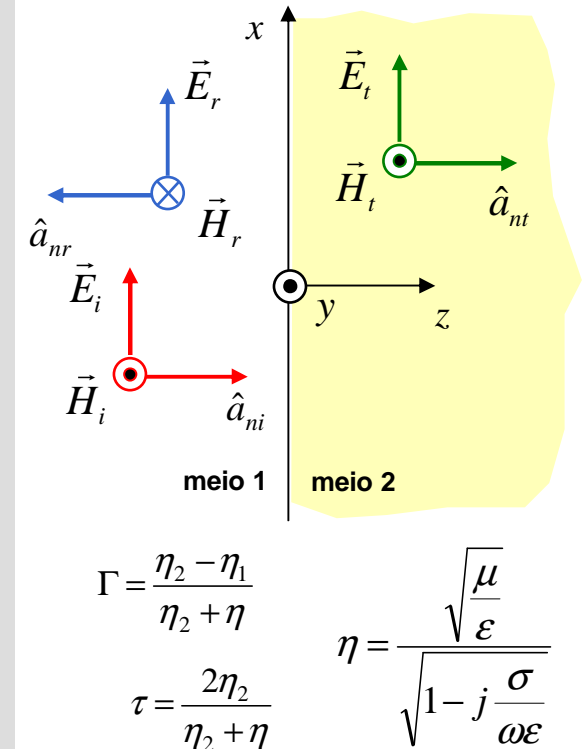
meio 1 sem perdas ( $\sigma_1 = 0$ )  $\longrightarrow \vec{E}_1 = (E_{i0}e^{-j\beta_1 z} + \Gamma E_{i0}e^{+j\beta_1 z})\hat{x}$

meio 2 condutor ideal ( $\sigma_2 = \infty$ )  $\longrightarrow \eta_2 = 0 \longrightarrow \Gamma = -1$   
 $\tau = 0$

$\vec{E}_2 = 0$

não há onda móvel, apenas  
onda estacionária

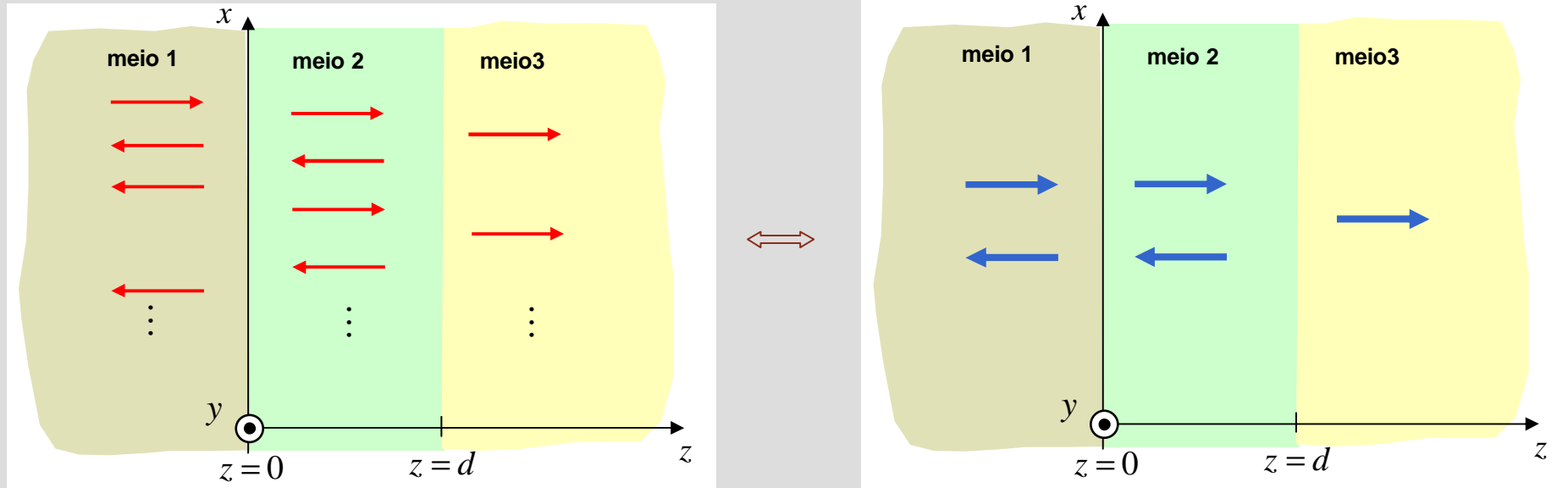
$\vec{E}_1 = E_{i0}(e^{-j\beta_1 z} - e^{+j\beta_1 z})\hat{x}$   
 $= -2jE_{i0} \sin(\beta_1 z)\hat{x}$



**máximos:**  $z_{MAX} = \frac{1}{2\beta_1}(2n+1)\pi$  e  $|\vec{E}_1|_{MAX} = 2|E_{i0}|$

**mínimos:**  $z_{min} = \frac{n\pi}{\beta_1}$  e  $|\vec{E}_1|_{min} = 0$

# Incidência normal – múltiplas interfaces



### meio 1

$$\vec{E}_1 = (E_1^+ e^{-\gamma_1 z} + E_1^- e^{\gamma_1 z}) \hat{x}$$

$$\vec{H}_1 = \left( \frac{E_1^+}{\eta_1} e^{-\gamma_1 z} - \frac{E_1^-}{\eta_1} e^{\gamma_1 z} \right) \hat{y}$$

### meio 2

$$\vec{E}_2 = (E_2^+ e^{-\gamma_2 z} + E_2^- e^{\gamma_2 z}) \hat{x}$$

$$\vec{H}_2 = \left( \frac{E_2^+}{\eta_2} e^{-\gamma_2 z} - \frac{E_2^-}{\eta_2} e^{\gamma_2 z} \right) \hat{y}$$

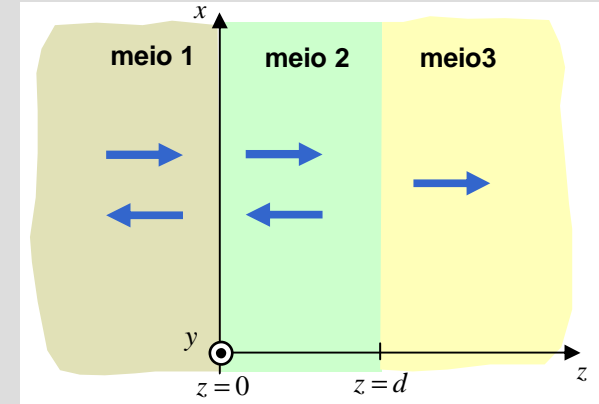
### meio 3

$$\vec{E}_3 = E_2^+ e^{-\gamma_2 z} \hat{x}$$

$$\vec{H}_3 = \frac{E_2^+}{\eta_2} e^{-\gamma_2 z} \hat{y}$$

# Múltiplas interfaces – condições fronteira

meios dieléctricos  $\rightarrow \vec{J}_s = 0 \rightarrow$  
 $E_{\text{tan}}$  contínuo  
 $H_{\text{tan}}$  contínuo



$z = 0 \rightarrow$

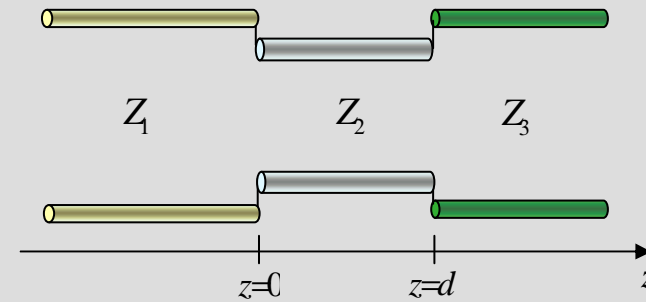
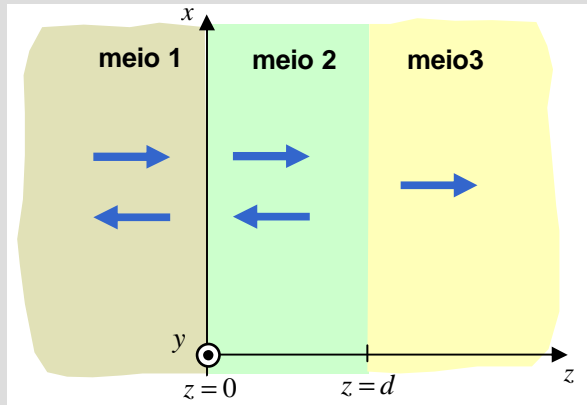
$$\begin{cases} E_1^+ + E_1^- = E_2^+ + E_2^- \\ \frac{E_1^+ - E_1^-}{\eta_1} = \frac{E_2^+ - E_2^-}{\eta_2} \end{cases}$$

$z = d \rightarrow$

$$\begin{cases} E_2^+ e^{-\gamma_2 d} + E_2^- e^{\gamma_2 d} = E_3^+ e^{-\gamma_3 d} + E_3^- e^{\gamma_3 d} \\ \frac{E_2^+ e^{-\gamma_2 d} - E_2^- e^{\gamma_2 d}}{\eta_2} = \frac{E_3^+ e^{-\gamma_3 d} - E_3^- e^{\gamma_3 d}}{\eta_3} \end{cases}$$

4 equações  
4 incógnitas  $\rightarrow (E_1^-, E_2^+, E_2^-, E_3^+)$   
se  $E_1^+$  conhecido

# Múltiplas interfaces – analogia com linhas de transmissão



meio 1

$$\vec{E}_1 = (E_1^+ e^{-\gamma_1 z} + E_1^- e^{\gamma_1 z}) \hat{x}$$

$$\vec{H}_1 = \left( \frac{E_1^+}{\eta_1} e^{-\gamma_1 z} - \frac{E_1^-}{\eta_1} e^{\gamma_1 z} \right) \hat{y}$$

meio 2

$$\vec{E}_2 = (E_2^+ e^{-\gamma_2 z} + E_2^- e^{\gamma_2 z}) \hat{x}$$

$$\vec{H}_2 = \left( \frac{E_2^+}{\eta_2} e^{-\gamma_2 z} - \frac{E_2^-}{\eta_2} e^{\gamma_2 z} \right) \hat{y}$$

meio 3

$$\vec{E}_3 = E_3^+ e^{-\gamma_3 z} \hat{x}$$

$$\vec{H}_3 = \frac{E_3^+}{\eta_3} e^{-\gamma_3 z} \hat{y}$$

linha 1

$$V_1 = V_1^+ e^{-\gamma_1 z} + V_1^- e^{\gamma_1 z}$$

$$I_1 = \frac{V_1^+}{Z_1} e^{-\gamma_1 z} - \frac{V_1^-}{Z_1} e^{\gamma_1 z}$$

linha 2

$$V_2 = V_2^+ e^{-\gamma_2 z} + V_2^- e^{\gamma_2 z}$$

$$I_2 = \frac{V_2^+}{Z_2} e^{-\gamma_2 z} - \frac{V_2^-}{Z_2} e^{\gamma_2 z}$$

linha 3

$$V_3 = V_3^+ e^{-\gamma_3 z}$$

$$I_3 = \frac{V_3^+}{Z_3} e^{-\gamma_3 z}$$

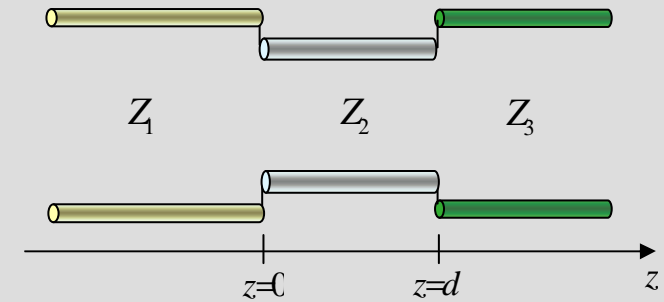
# Múltiplas interfaces – analogia com linhas de transmissão

Impedância ao longo da linha →

$$Z(z) = \frac{V(z)}{I(z)}$$

linha 3 infinita →  $Z(z \geq d) = Z_3$

linha 2 finita →  $Z(0 \leq z \leq d) = Z_2 \frac{Z_3 + Z_2 \tanh(\gamma_2(d-z))}{Z_2 + Z_3 \tanh(\gamma_2(d-z))}$



Coefficientes de reflexão

$$\Gamma_{23,\text{eff}} = \frac{Z(z=d) - Z_2}{Z(z=d) + Z_2} = \frac{Z_3 - Z_2}{Z_3 + Z_2} \rightarrow$$

$$\begin{aligned} V_2^- e^{\gamma_2 d} &= \Gamma_{23,\text{eff}} V_2^+ e^{-\gamma_2 d} \\ V_3^+ e^{-\gamma_3 d} &= (1 - \Gamma_{23,\text{eff}}) V_2^+ e^{-\gamma_2 d} \end{aligned}$$

$$\Gamma_{12,\text{eff}} = \frac{Z(z=0) - Z_1}{Z(z=0) + Z_1} \rightarrow$$

$$\begin{aligned} V_1^- &= \Gamma_{12,\text{eff}} V_1^+ \\ V_2^+ &= (1 - \Gamma_{12,\text{eff}}) V_1^+ \end{aligned}$$

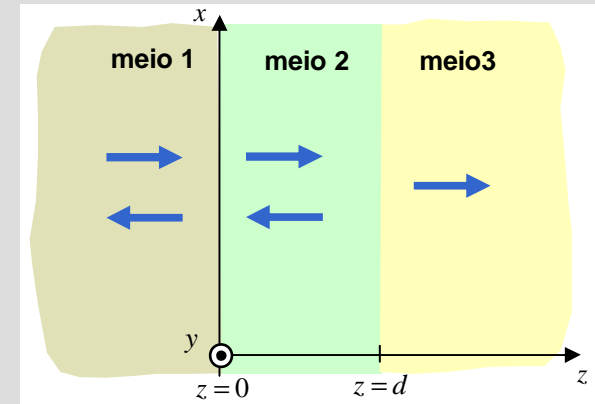
# Múltiplas interfaces – analogia com linhas de transmissão

Impedância de onda

$$Z(z) = \frac{E_x(z)}{H_y(z)}$$

meio 3 infinito  $\rightarrow Z(z \geq d) = \eta_3$

meio 2 finito  $\rightarrow Z(0 \leq z \leq d) = \eta_2 \frac{\eta_3 + \eta_2 \tanh(\gamma_2(d-z))}{\eta_2 + \eta_3 \tanh(\gamma_2(d-z))}$



Coeficientes de reflexão

$$\Gamma_{23,\text{eff}} = \frac{Z(z=d) - \eta_2}{Z(z=d) + \eta_2} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \rightarrow \begin{cases} E_2^- e^{\gamma_2 d} = \Gamma_{23,\text{eff}} E_2^+ e^{-\gamma_2 d} \\ E_3^+ e^{-\gamma_3 d} = (1 - \Gamma_{23,\text{eff}}) E_2^+ e^{-\gamma_2 d} \end{cases}$$

$$\Gamma_{12,\text{eff}} = \frac{Z(z=0) - \eta_1}{Z(z=0) + \eta_1} \rightarrow \begin{cases} E_1^- = \Gamma_{12,\text{eff}} E_1^+ \\ E_2^+ = (1 - \Gamma_{12,\text{eff}}) E_1^+ \end{cases}$$

4 equações

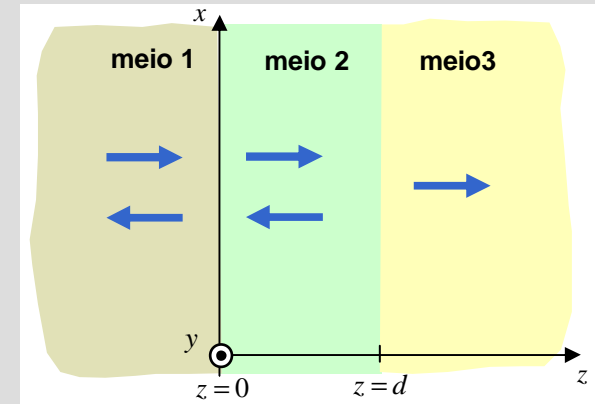
4 incógnitas

$\downarrow$  se  $E_1^+$  conhecido  
 $(E_1^-, E_2^+, E_2^-, E_3^+)$

**Eliminação de reflexões na interface 1 → 3 através da inserção do meio 2**

$$\Gamma_{12,\text{eff}} = 0 \longrightarrow Z(z=0) = \eta_1$$

Nota: meio 2 pode ser visto como adaptador de  $\lambda/4$



## Aplicações práticas

- eliminação de reflexos em lentes
- atenuação de ecos de radar (aviões invisíveis)
- ...

## Incidência oblíqua de uma onda TEM numa interface plana

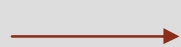
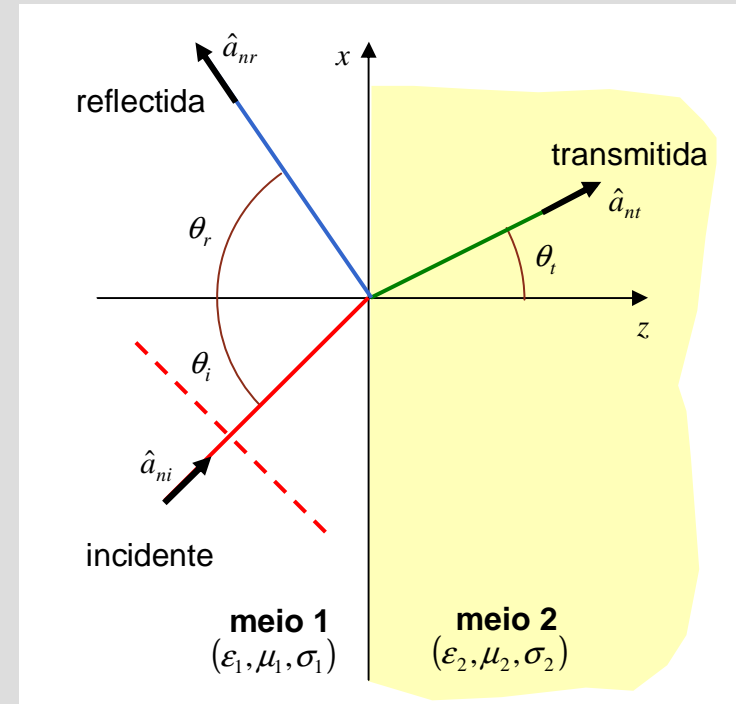
**onda TEM**  $\rightarrow \vec{E} \perp \vec{H} \perp \hat{a}_n \rightarrow \vec{E}_i$  e  $\vec{H}_i$  estão no plano  $\perp \hat{a}_n$

1.  $\vec{E}_i$  **perpendicular** ao plano de incidência

$$\vec{E}_i = E_{i0} e^{-jk_1 \hat{a}_{ni} \cdot \vec{r}} \hat{y} \rightarrow \text{polarização perpendicular}$$

2.  $\vec{E}_i$  **paralelo** ao plano de incidência

$$\vec{E}_i = E_{i0} e^{-jk_1 \hat{a}_{ni} \cdot \vec{r}} (\cos \theta_i \hat{x} - \sin \theta_i \hat{z}) \rightarrow \text{polarização paralela}$$



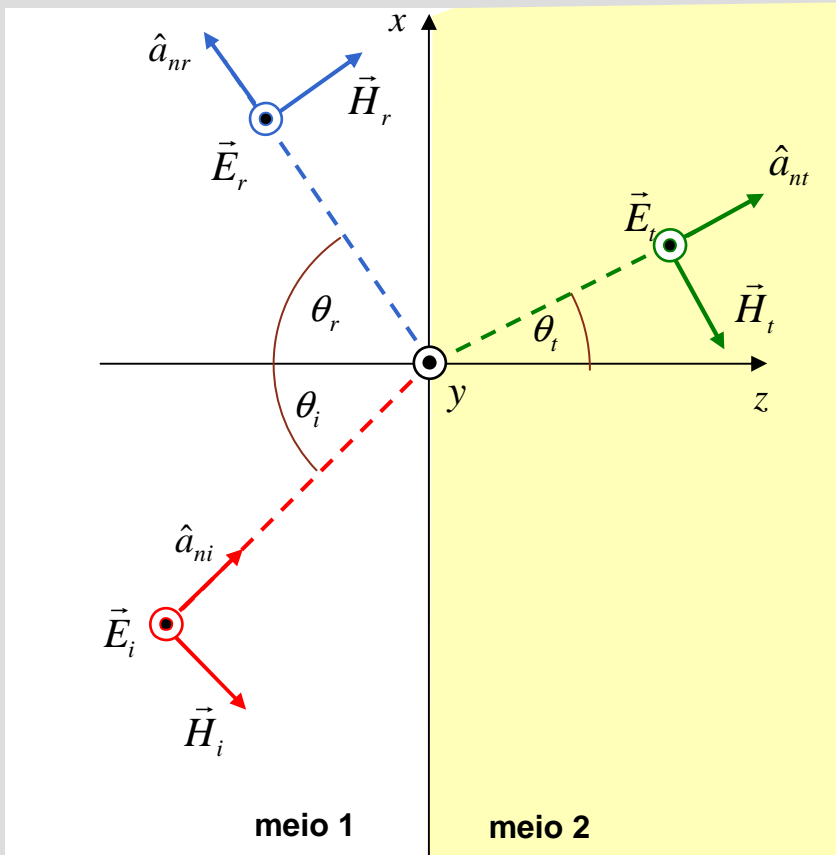
**Caso geral:**  $\vec{E}_i = E_{i0,1} e^{-jk_1 \hat{a}_{ni} \cdot \vec{r}} (\cos \theta_i \hat{x} - \sin \theta_i \hat{z}) + E_{i0,2} e^{-jk_1 \hat{a}_{ni} \cdot \vec{r}} \hat{y}$

$\downarrow$   
polarização paralela
 $\downarrow$   
polarização perpendicular

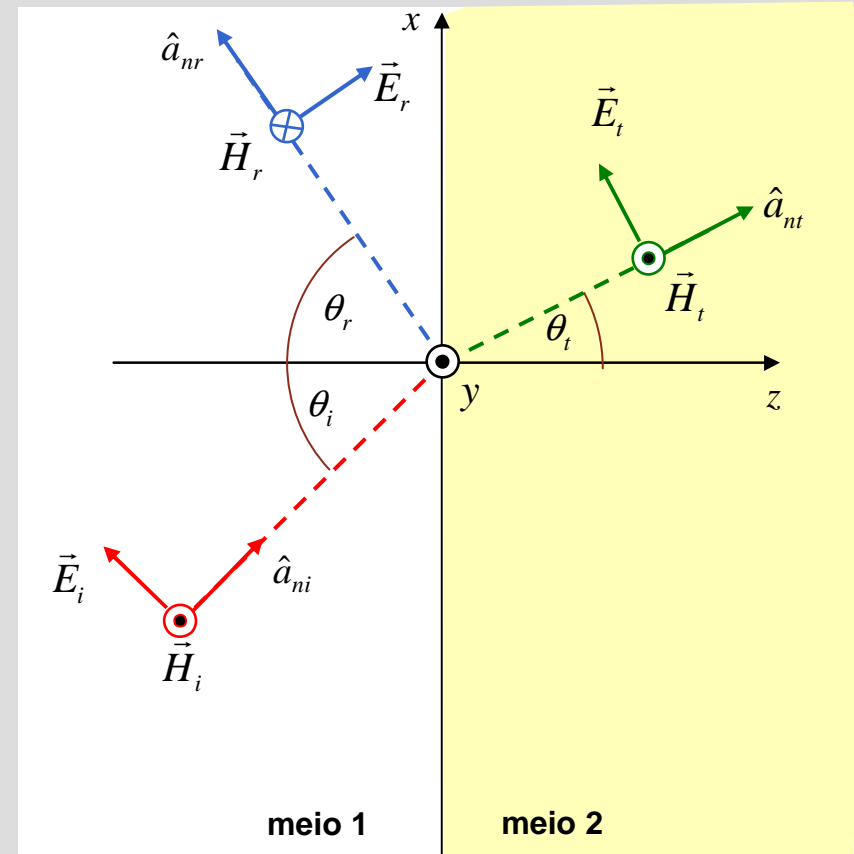


## Polarizações perpendicular e paralela – convenção

polarização perpendicular



polarização paralela



componentes de  $\vec{E}_i$ ,  $\vec{E}_r$  e  $\vec{E}_t$  tangentes à interface mantêm o sentido

## Polarização perpendicular – campos eléctrico e magnético

### incidente

$$\hat{a}_{ni} = \sin \theta_i \hat{x} + \cos \theta_i \hat{z}$$

$$\vec{E}_i = E_{i0} e^{-\gamma_1 \hat{a}_{ni} \cdot \vec{r}} \hat{y}$$

$$\vec{H}_i = \frac{1}{\eta_1} \hat{a}_{ni} \times \vec{E}_i = \frac{E_{i0}}{\eta_1} e^{-\gamma_1 \hat{a}_{ni} \cdot \vec{r}} (\sin \theta_i \hat{z} - \cos \theta_i \hat{x})$$

### reflectida

$$\hat{a}_{nr} = \sin \theta_r \hat{x} - \cos \theta_r \hat{z} = \sin \theta_i \hat{x} - \cos \theta_i \hat{z}$$

$$\vec{E}_r = E_{r0} e^{-\gamma_1 \hat{a}_{nr} \cdot \vec{r}} \hat{y}$$

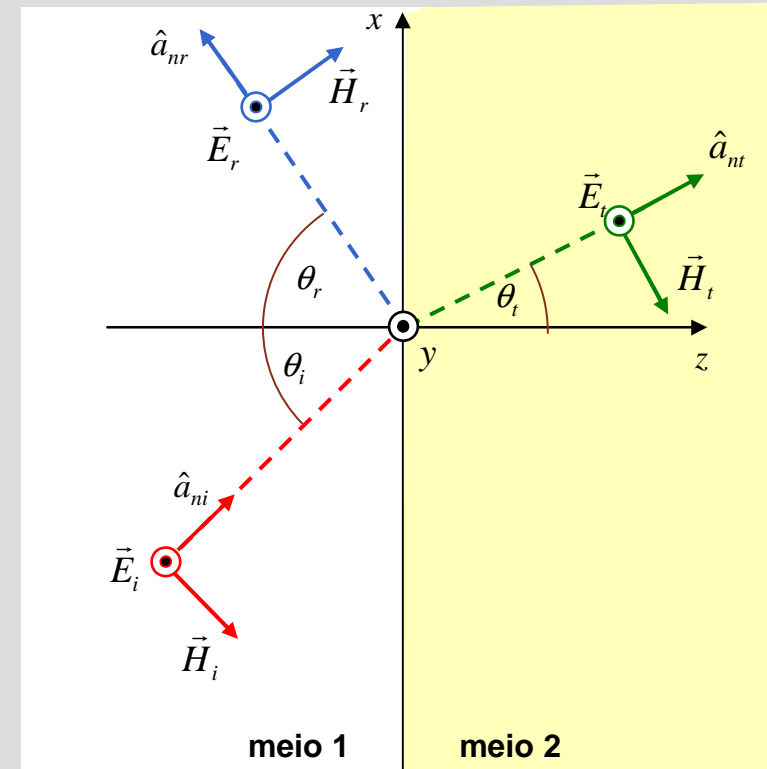
$$\vec{H}_r = \frac{1}{\eta_1} \hat{a}_{nr} \times \vec{E}_r = \frac{E_{r0}}{\eta_1} e^{-\gamma_1 \hat{a}_{nr} \cdot \vec{r}} (\sin \theta_i \hat{z} + \cos \theta_i \hat{x})$$

### transmitida

$$\hat{a}_{nt} = \sin \theta_t \hat{x} + \cos \theta_t \hat{z}$$

$$\vec{E}_t = E_{t0} e^{-\gamma_2 \hat{a}_{nt} \cdot \vec{r}} \hat{y}$$

$$\vec{H}_t = \frac{1}{\eta_2} \hat{a}_{nt} \times \vec{E}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 \hat{a}_{nt} \cdot \vec{r}} (\sin \theta_t \hat{z} - \cos \theta_t \hat{x})$$



→ relações entre  $E_{i0}$ ,  $E_{r0}$  e  $E_{t0}$  obtidas a partir das condições fronteira

## Polarização perpendicular – campos eléctrico e magnético

meios sem perdas  $\rightarrow \sigma_1 = \sigma_2 = 0 \rightarrow \gamma_1 = j\beta_1$   
 $\gamma_2 = j\beta_2$

condições fronteira  $\rightarrow E_{\tan}$  contínuo  
 $H_{\tan}$  contínuo (se  $\vec{J}_S = 0$ )

em  $z = 0 \rightarrow \vec{E}_i + \vec{E}_r = \vec{E}_t$   
 $H_{ix} + H_{rx} = H_{tx}$

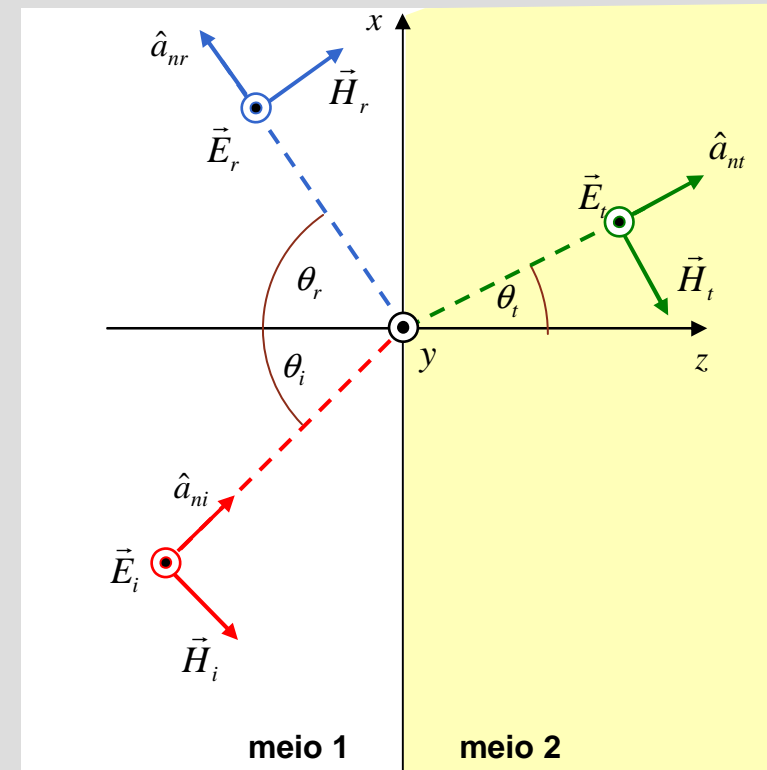
$$E_{i0}e^{-j\beta_1 \sin \theta_i x} + E_{r0}e^{-j\beta_1 \sin \theta_i x} = E_{t0}e^{-j\beta_2 \sin \theta_t x}$$

$$\frac{-E_{i0} \cos \theta_i e^{-j\beta_1 \sin \theta_i x} + E_{r0} \cos \theta_i e^{-j\beta_1 \sin \theta_i x}}{\eta_1} = -\frac{E_{t0} \cos \theta_t e^{-j\beta_2 \sin \theta_t x}}{\eta_2}$$

$$\beta_1 \sin \theta_i = \beta_2 \sin \theta_t$$

$$E_{i0} + E_{r0} = E_{t0}$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0}) \cos \theta_i = \frac{E_{t0}}{\eta_2} \cos \theta_t$$



## Polarização perpendicular – coeficientes de reflexão e transmissão

$$E_{i0} + E_{r0} = E_{t0}$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0})\cos\theta_i = \frac{E_{t0}}{\eta_2}\cos\theta_t$$

$$\frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

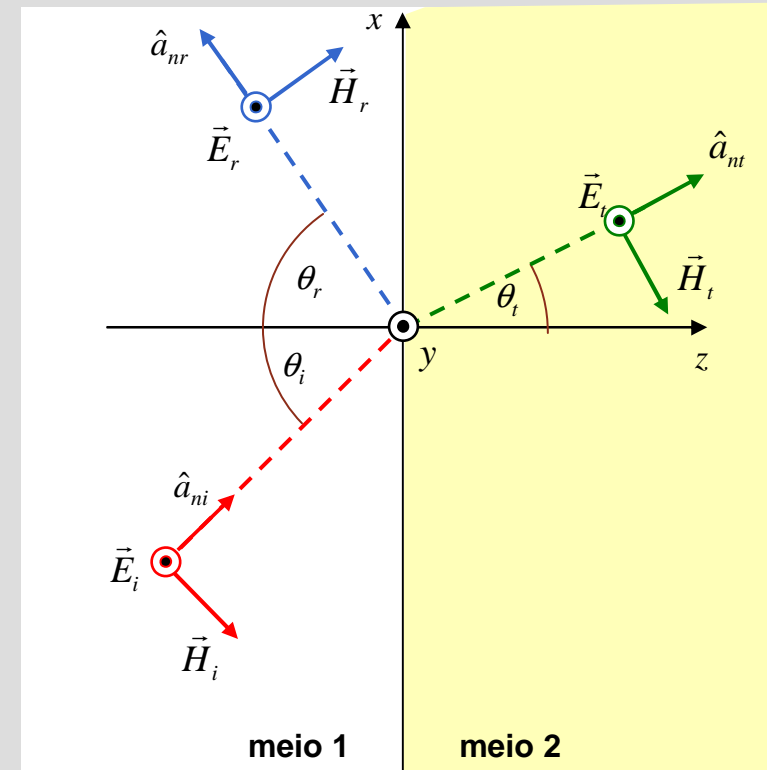
$$\frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$\Gamma_{\perp} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$\tau_{\perp} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

coeficiente de reflexão

coeficiente de transmissão



## Polarização perpendicular – coeficientes de reflexão e transmissão

$$\Gamma_{\perp} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$$

coeficiente de reflexão

$$\tau_{\perp} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$$

coeficiente de transmissão

### notas

1.  $1 + \Gamma_{\perp} = \tau_{\perp}$  (tal como para incidência normal)

2. é possível que  $\Gamma_{\perp} = 0 \rightarrow \eta_2 \cos \theta_i = \eta_1 \cos \theta_t$

(ângulo de Brewster)  $\theta_i = \theta_{B\perp}$

$$n_1 \sin \theta_i = n_2 \sin \theta_t$$

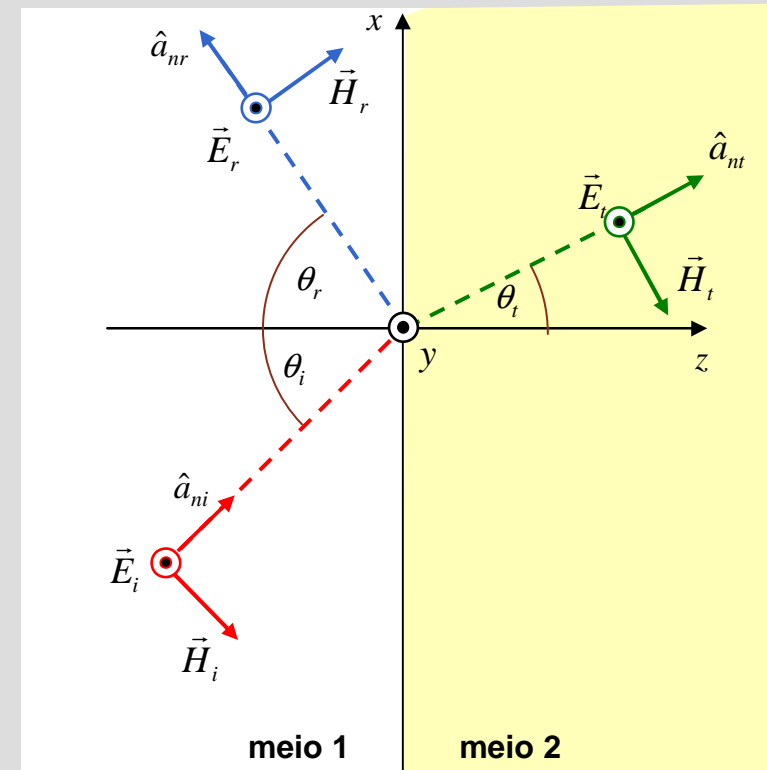
$$\sin^2 \theta_{B\perp} = \frac{1 - \mu_1 \epsilon_2 / \mu_2 \epsilon_1}{1 - (\mu_1 / \mu_2)^2}$$

→ só possível quando  $\mu_1 \neq \mu_2$

3. se meio 2 for condutor perfeito,  $\eta_2 = 0 \rightarrow$

$$\Gamma_{\perp} = -1$$

$$\tau_{\perp} = 0$$



## Polarização paralela – campos eléctrico e magnético

### incidente

$$\hat{a}_{ni} = \sin \theta_i \hat{x} + \cos \theta_i \hat{z}$$

$$\vec{E}_i = E_{i0} e^{-\gamma_1 \hat{a}_{ni} \cdot \vec{r}} (\cos \theta_i \hat{x} - \sin \theta_i \hat{z})$$

$$\vec{H}_i = \frac{E_{i0}}{\eta_1} e^{-\gamma_1 \hat{a}_{ni} \cdot \vec{r}} \hat{y}$$

### reflectida

$$\hat{a}_{nr} = \sin \theta_r \hat{x} - \cos \theta_r \hat{z} = \sin \theta_i \hat{x} - \cos \theta_i \hat{z}$$

$$\vec{E}_r = E_{r0} e^{-\gamma_1 \hat{a}_{nr} \cdot \vec{r}} (\cos \theta_i \hat{x} + \sin \theta_i \hat{z})$$

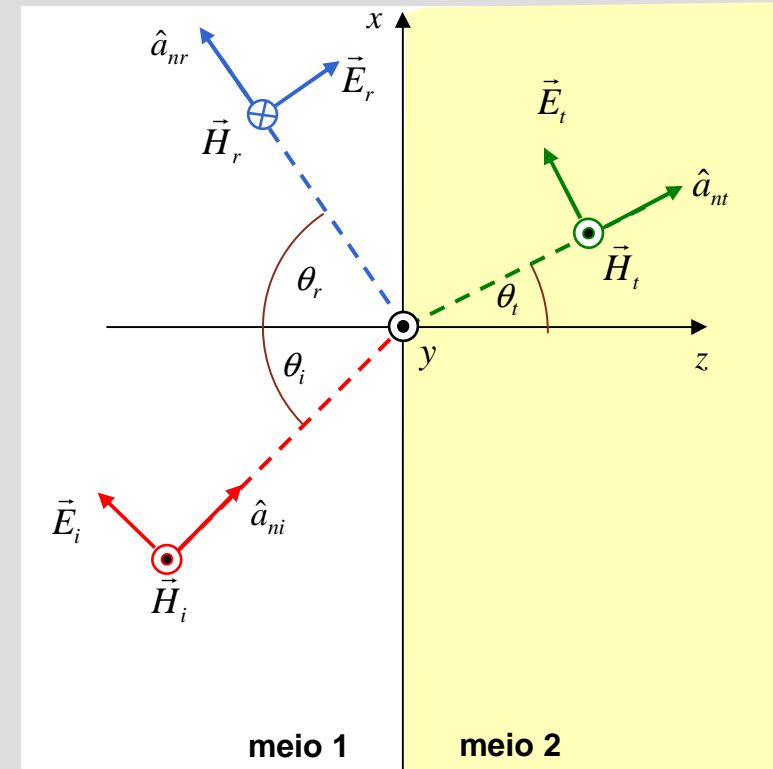
$$\vec{H}_r = -\frac{E_{r0}}{\eta_1} e^{-\gamma_1 \hat{a}_{nr} \cdot \vec{r}} \hat{y}$$

### transmitida

$$\hat{a}_{nt} = \sin \theta_t \hat{x} + \cos \theta_t \hat{z}$$

$$\vec{E}_t = E_{t0} e^{-\gamma_2 \hat{a}_{nt} \cdot \vec{r}} (\cos \theta_t \hat{x} - \sin \theta_t \hat{z})$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 \hat{a}_{nt} \cdot \vec{r}} \hat{y}$$



relações entre  $E_{i0}$ ,  $E_{r0}$  e  $E_{t0}$  obtidas a partir das condições fronteira

## Polarização paralela – campos eléctrico e magnético

meios sem perdas  $\rightarrow \sigma_1 = \sigma_2 = 0 \rightarrow \gamma_1 = j\beta_1$   
 $\gamma_2 = j\beta_2$

condições fronteira  $\rightarrow E_{\tan}$  contínuo  
 $H_{\tan}$  contínuo (se  $\vec{J}_S = 0$ )

em  $z = 0 \rightarrow E_{ix} + E_{rx} = E_{tx}$

$$\vec{H}_i + \vec{H}_r = \vec{H}_t$$

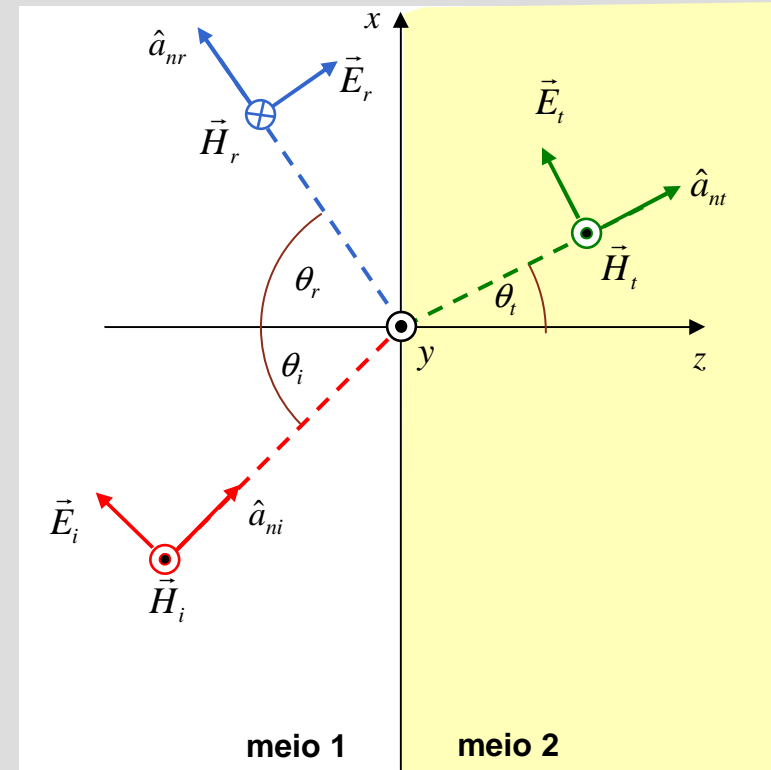
$$E_{i0} \cos \theta_i e^{-j\beta_1 \sin \theta_i x} + E_{r0} \cos \theta_i e^{-j\beta_1 \sin \theta_i x} = E_{t0} \cos \theta_t e^{-j\beta_2 \sin \theta_t x}$$

$$\frac{E_{i0} e^{-j\beta_1 \sin \theta_i x} - E_{r0} e^{-j\beta_1 \sin \theta_i x}}{\eta_1} = \frac{E_{t0} e^{-j\beta_2 \sin \theta_t x}}{\eta_2}$$

$$\beta_1 \sin \theta_i = \beta_2 \sin \theta_t$$

$$(E_{i0} + E_{r0}) \cos \theta_i = E_{t0} \cos \theta_t$$

$$\frac{1}{\eta_1} (E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}$$



## Polarização paralela – coeficientes de reflexão e transmissão

$$(E_{i0} + E_{r0}) \cos \theta_i = E_{t0} \cos \theta_t$$

$$\frac{1}{\eta_1} (E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}$$

$$\frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

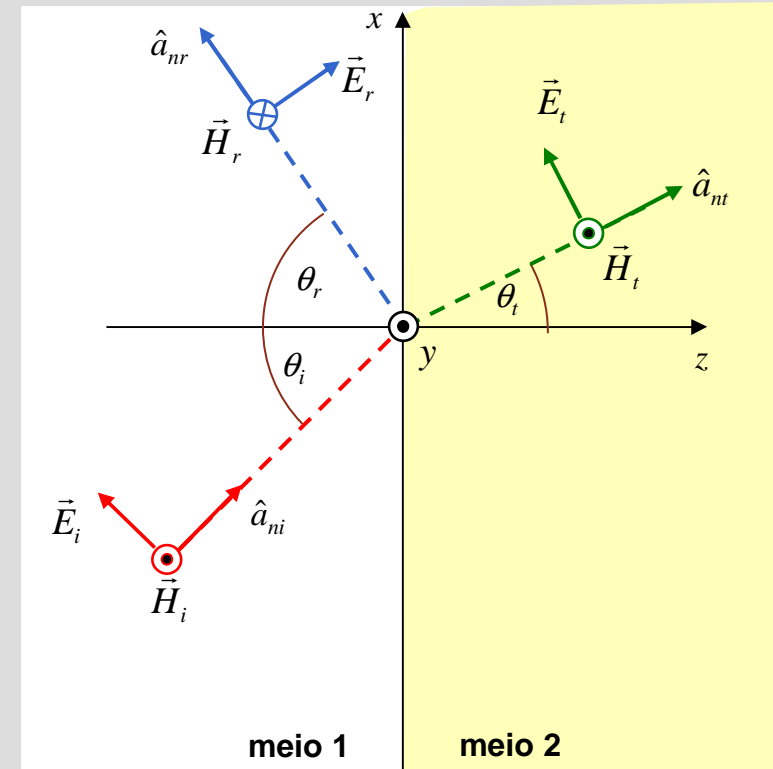
$$\frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

$$\Gamma_{\parallel} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

$$\tau_{\parallel} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

coeficiente de reflexão

coeficiente de transmissão





## Polarização paralela – coeficientes de reflexão e transmissão

$$\Gamma_{\parallel} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

coeficiente de reflexão

$$\tau_{\parallel} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

coeficiente de transmissão

notas

1.  $1 + \Gamma_{\parallel} = \tau_{\parallel} \left( \frac{\cos \theta_t}{\cos \theta_i} \right)$

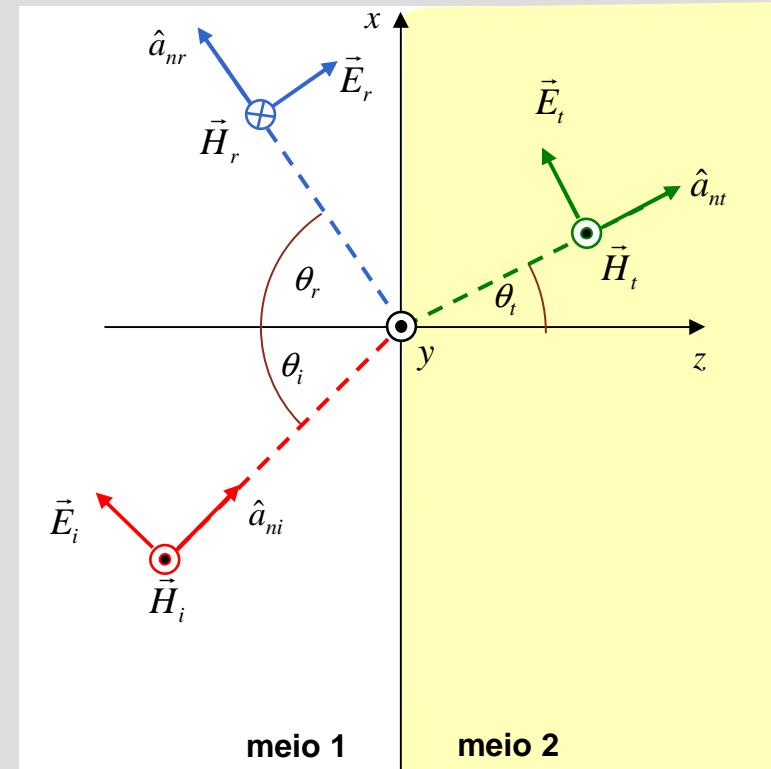
2. é possível que  $\Gamma_{\parallel} = 0 \rightarrow \eta_2 \cos \theta_t = \eta_1 \cos \theta_i$

(ângulo de Brewster)  $\theta_i = \theta_{B\parallel}$

$$n_1 \sin \theta_i = n_2 \sin \theta_t$$

$$\sin^2 \theta_{B\parallel} = \frac{1 - \mu_2 \epsilon_1 / \mu_1 \epsilon_2}{1 - (\epsilon_1 / \epsilon_2)^2} \rightarrow \text{quando } \mu_1 = \mu_2 \rightarrow \sin \theta_{B\parallel} = \frac{1}{\sqrt{1 + (\epsilon_1 / \epsilon_2)}}$$

3. se meio 2 for condutor perfeito,  $\eta_2 = 0 \rightarrow \Gamma_{\parallel} = -1$   
 $\tau_{\parallel} = 0$



## Campo eléctrico no meio 1 – polarização perpendicular

meio 1 sem perdas  $\rightarrow \gamma_1 = j\beta_1$

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r = E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} \hat{y} + E_{r0} e^{-j\beta_1 \hat{a}_{nr} \cdot \vec{r}} \hat{y}$$

$$1 + \Gamma_{\perp} = \tau_{\perp}$$

$$\hat{a}_{ni} \cdot \vec{r} = \sin \theta_i x + \cos \theta_i z$$

$$\hat{a}_{nr} \cdot \vec{r} = \sin \theta_i x - \cos \theta_i z$$

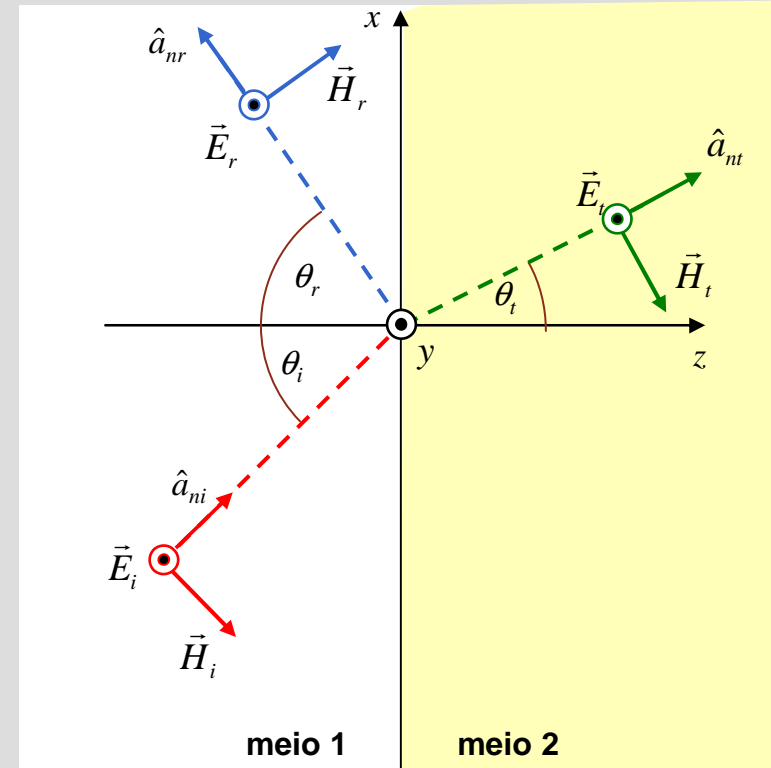
$$\vec{E}_1 = E_{i0} \left[ (\tau_{\perp} - \Gamma_{\perp}) e^{-j\beta_1 \cos \theta_i z} + \Gamma_{\perp} e^{j\beta_1 \cos \theta_i z} \right] e^{-j\beta_1 \sin \theta_i x} \hat{y}$$

$$= \tau_{\perp} E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} \hat{y} + \Gamma_{\perp} E_{i0} \left( e^{j\beta_1 \cos \theta_i z} - e^{-j\beta_1 \cos \theta_i z} \right) e^{-j\beta_1 \sin \theta_i x} \hat{y}$$

$$= \tau_{\perp} E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} \hat{y} + j2\Gamma_{\perp} E_{i0} \sin(\beta_1 \cos \theta_i z) e^{-j\beta_1 \sin \theta_i x} \hat{y}$$

onda em propagação  
segundo  $\hat{a}_{ni}$

onda em propagação segundo  $x$ ,  
com amplitude dependente de  $z$



## Máximos e mínimos no meio 1 – polarização perpendicular

$$\vec{E}_1 = E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} \hat{y} + E_{r0} e^{-j\beta_1 \hat{a}_{nr} \cdot \vec{r}} \hat{y}$$

$$\hat{a}_{ni} \cdot \vec{r} = \sin \theta_i x + \cos \theta_i z$$

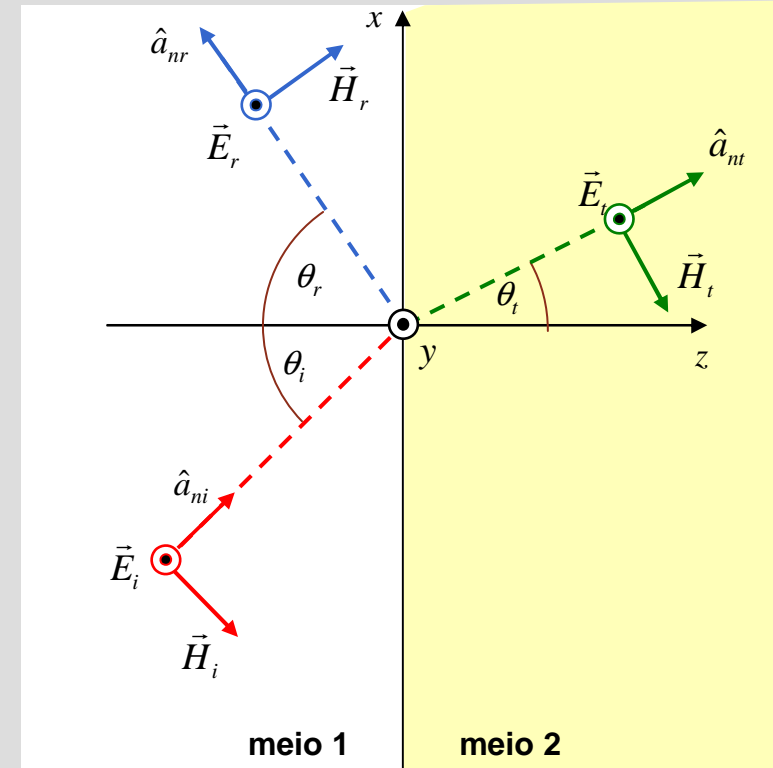
$$\hat{a}_{nr} \cdot \vec{r} = \sin \theta_i x - \cos \theta_i z$$

$$\vec{E}_1 = E_{i0} e^{-j\beta_1 (\sin \theta_i x + \beta_1 \cos \theta_i z)} (1 + \Gamma_{\perp} e^{+j2\beta_1 \cos \theta_i z}) \hat{y}$$

$$\Gamma_{\perp} = |\Gamma_{\perp}| e^{j\theta_r}$$

$$|\vec{E}_1| = |E_{i0}| \sqrt{[1 + |\Gamma_{\perp}| \cos(\theta_r + 2\beta_1 \cos \theta_i z)]^2 + [|\Gamma_{\perp}| \sin(\theta_r + 2\beta_1 \cos \theta_i z)]^2}$$

$$= |E_{i0}| \sqrt{1 + |\Gamma_{\perp}|^2 + 2|\Gamma_{\perp}| \cos(\theta_r + 2\beta_1 \cos \theta_i z)}$$



**máximos:**  $\cos(\theta_r + 2\beta_1 \cos \theta_i z) = +1 \rightarrow z_{MAX} = \frac{1}{2\beta_1 \cos \theta_i} (-\theta_r + 2n\pi) \rightarrow |\vec{E}_1|_{MAX} = |E_{i0}| (1 + |\Gamma_{\perp}|)$

**mínimos:**  $\cos(\theta_r + 2\beta_1 \cos \theta_i z) = -1 \rightarrow z_{min} = \frac{1}{2\beta_1 \cos \theta_i} [-\theta_r + (2n+1)\pi] \rightarrow |\vec{E}_1|_{min} = |E_{i0}| (1 - |\Gamma_{\perp}|)$

## Incidência num condutor ideal – polarização perpendicular

meio 2 condutor ideal ( $\sigma_2 = \infty$ )  $\longrightarrow$   $\eta_2 = 0$   $\longrightarrow$   $\Gamma_{\perp} = -1$   
 $\tau_{\perp} = 0$

$$\vec{E}_2 = 0$$

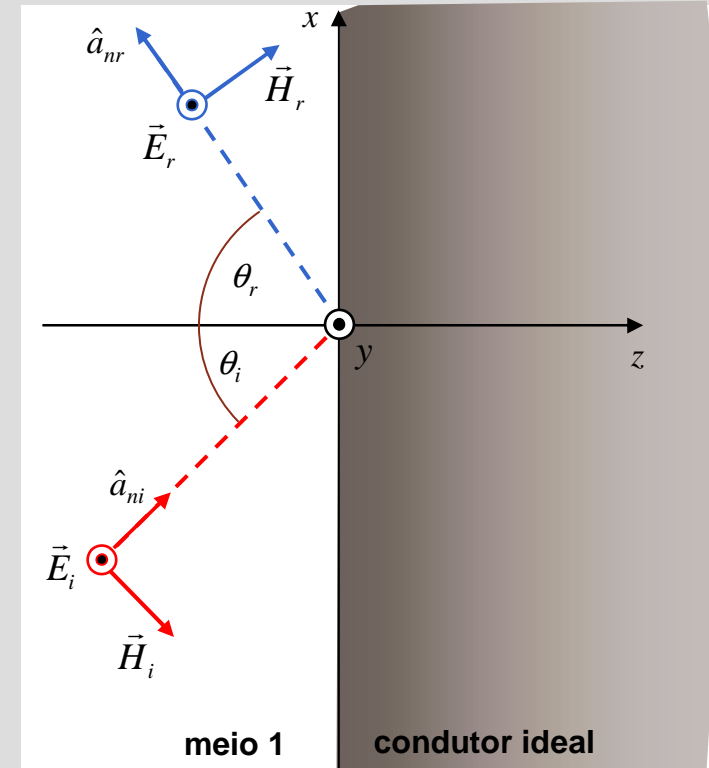
$$\vec{E}_1 = E_{i0} e^{-j\beta_1 \sin \theta_i x} \left( e^{-j\beta_1 \cos \theta_i z} - e^{j\beta_1 \cos \theta_i z} \right) \hat{y}$$

$$= -j2E_{i0} e^{-j\beta_1 \sin \theta_i x} \sin(\beta_1 \cos \theta_i z) \hat{y}$$

onda em propagação segundo  $x$ ,  
com amplitude dependente de  $z$

máximos:  $z_{MAX} = \frac{(2n+1)\pi}{2\beta_1 \cos \theta_i}$  e  $|\vec{E}_1|_{MAX} = 2|E_{i0}|$

mínimos:  $z_{min} = \frac{n\pi}{\beta_1 \cos \theta_i}$  e  $|\vec{E}_1|_{min} = 0$



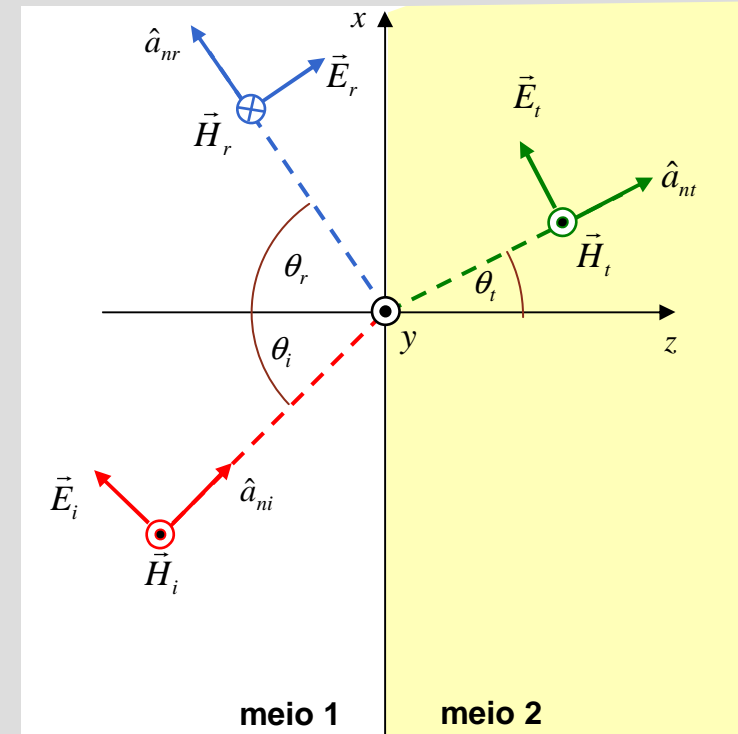
## Campo eléctrico no meio 1 – polarização paralela

meio 1 sem perdas  $\rightarrow \gamma_1 = j\beta_1$

$$\begin{aligned}\vec{E}_1 &= \vec{E}_i + \vec{E}_r = E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} (\cos \theta_i \hat{x} - \sin \theta_i \hat{z}) + E_{r0} e^{-j\beta_1 \hat{a}_{nr} \cdot \vec{r}} (\cos \theta_i \hat{x} + \sin \theta_i \hat{z}) \\ &= \tau_{\parallel} \frac{\cos \theta_t}{\cos \theta_i} E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} (\cos \theta_i \hat{x} - \sin \theta_i \hat{z}) \\ &\quad - \Gamma_{\parallel} E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} (\cos \theta_i \hat{x} - \sin \theta_i \hat{z}) + \Gamma_{\parallel} E_{i0} e^{-j\beta_1 \hat{a}_{nr} \cdot \vec{r}} (\cos \theta_i \hat{x} + \sin \theta_i \hat{z})\end{aligned}$$

$$1 + \Gamma_{\parallel} = \tau_{\parallel} \left( \frac{\cos \theta_t}{\cos \theta_i} \right)$$

$$\begin{aligned}\vec{E}_1 &= \tau_{\parallel} \frac{\cos \theta_t}{\cos \theta_i} E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} (\cos \theta_i \hat{x} - \sin \theta_i \hat{z}) \\ &\quad + \Gamma_{\parallel} E_{i0} e^{-j\beta_1 \sin \theta_i x} \left( e^{j\beta_1 \cos \theta_i z} - e^{j\beta_1 \cos \theta_i z} \right) \cos \theta_i \hat{x} \\ &\quad + \Gamma_{\parallel} E_{i0} e^{-j\beta_1 \sin \theta_i x} \left( e^{j\beta_1 \cos \theta_i z} + e^{j\beta_1 \cos \theta_i z} \right) \sin \theta_i \hat{z} \\ &= \tau_{\parallel} \frac{\cos \theta_t}{\cos \theta_i} E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} (\cos \theta_i \hat{x} - \sin \theta_i \hat{z}) \\ &\quad + j2\Gamma_{\parallel} E_{i0} e^{-j\beta_1 \sin \theta_i x} \sin(\beta_1 \cos \theta_i z) \cos \theta_i \hat{x} \\ &\quad + 2\Gamma_{\parallel} E_{i0} e^{-j\beta_1 \sin \theta_i x} \cos(\beta_1 \cos \theta_i z) \sin \theta_i \hat{z}\end{aligned}$$



# Campo eléctrico no meio 1 – polarização paralela

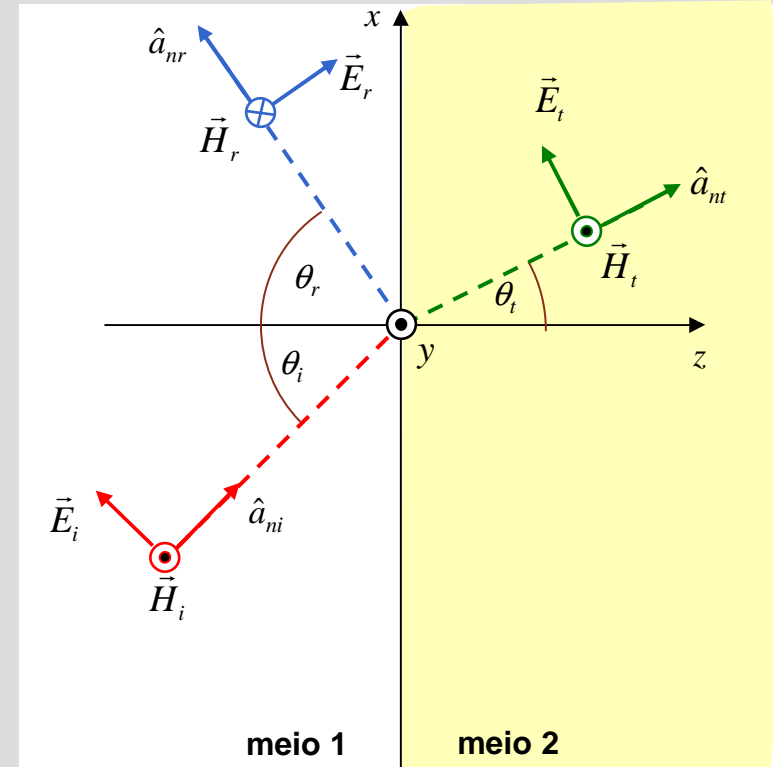
onda em propagação  
segundo  $\hat{a}_{ni}$

$$\vec{E}_1 = \tau_{\parallel} \frac{\cos \theta_t}{\cos \theta_i} E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} (\cos \theta_i \hat{x} - \sin \theta_i \hat{z})$$

$$+ j2\Gamma_{\parallel} E_{i0} e^{-j\beta_1 \sin \theta_i x} \sin(\beta_1 \cos \theta_i z) \cos \theta_i \hat{x}$$

$$+ 2\Gamma_{\parallel} E_{i0} e^{-j\beta_1 \sin \theta_i x} \cos(\beta_1 \cos \theta_i z) \sin \theta_i \hat{z}$$

ondas em propagação segundo  $x$ ,  
com amplitudes dependente de  $z$



## Máximos e mínimos no meio 1 – polarização paralela

$$\begin{aligned}\vec{E}_1 &= \vec{E}_i + \vec{E}_r = E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} (\cos \theta_i \hat{x} - \sin \theta_i \hat{z}) + E_{r0} e^{-j\beta_1 \hat{a}_{nr} \cdot \vec{r}} (\cos \theta_i \hat{x} + \sin \theta_i \hat{z}) \\ &= (E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} + E_{r0} e^{-j\beta_1 \hat{a}_{nr} \cdot \vec{r}}) \cos \theta_i \hat{x} + (-E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} + E_{r0} e^{-j\beta_1 \hat{a}_{nr} \cdot \vec{r}}) \sin \theta_i \hat{z}\end{aligned}$$

$$E_{1x} = E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} (1 + \Gamma_{\parallel} e^{j\beta_1 (\hat{a}_{ni} - \hat{a}_{nr}) \cdot \vec{r}}) \cos \theta_i$$

$$E_{1x} = E_{i0} e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} (1 + \Gamma_{\parallel} e^{j2\beta_1 \cos \theta_i z}) \cos \theta_i$$

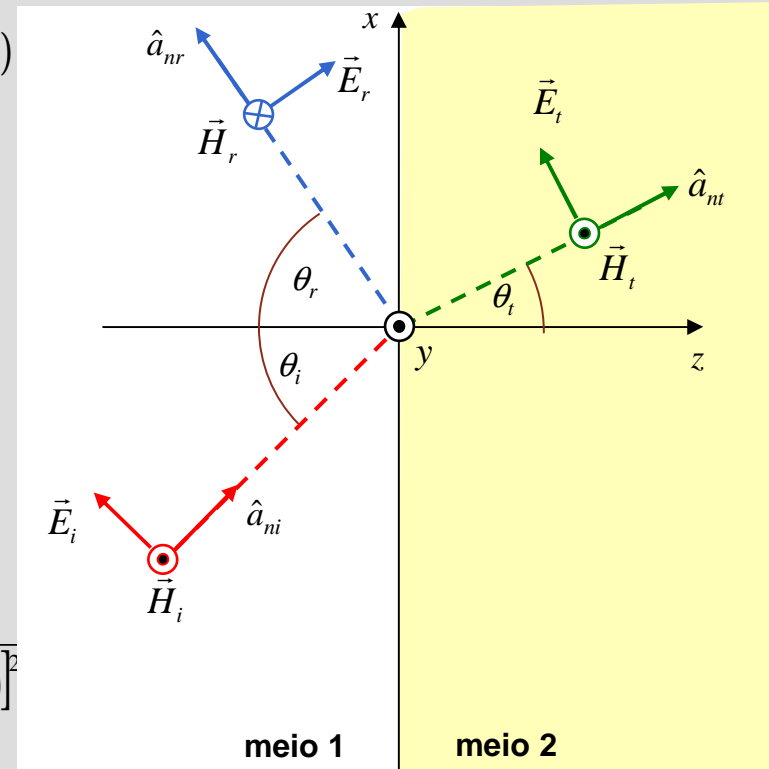
$$\Gamma_{\parallel} = |\Gamma_{\parallel}| e^{j\theta_{\Gamma}}$$

$$|E_{1x}| = |E_{i0} \cos \theta_i| \sqrt{[1 + |\Gamma_{\parallel}| \cos(\theta_{\Gamma} + 2\beta_1 \cos \theta_i z)]^2 + [|\Gamma_{\parallel}| \sin(\theta_{\Gamma} + 2\beta_1 \cos \theta_i z)]^2}$$

$$= |E_{i0} \cos \theta_i| \sqrt{1 + |\Gamma_{\parallel}|^2 + 2|\Gamma_{\parallel}| \cos(\theta_{\Gamma} + 2\beta_1 \cos \theta_i z)}$$

**máximos:**  $z_{MAX} = \frac{1}{2\beta_1 \cos \theta_i} (-\theta_{\Gamma} + 2n\pi) \rightarrow |E_{1x}|_{MAX} = |E_{i0} \cos \theta_i| (1 + |\Gamma_{\parallel}|)$

**mínimos:**  $z_{min} = \frac{1}{2\beta_1 \cos \theta_i} [-\theta_{\Gamma} + (2n+1)\pi] \rightarrow |E_{1x}|_{min} = |E_{i0} \cos \theta_i| (1 - |\Gamma_{\parallel}|)$



## Incidência num condutor ideal – polarização paralela

$$\text{meio 2 condutor ideal } (\sigma_2 = \infty) \longrightarrow \eta_2 = 0 \longrightarrow \Gamma_{\parallel} = -1$$

$$\tau_{\parallel} = 0$$

$$\vec{E}_2 = 0$$

$$\vec{E}_1 = E_{i0} \left[ \left( e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} - e^{-j\beta_1 \hat{a}_{nr} \cdot \vec{r}} \right) \cos \theta_i \hat{x} - \left( e^{-j\beta_1 \hat{a}_{ni} \cdot \vec{r}} + e^{-j\beta_1 \hat{a}_{nr} \cdot \vec{r}} \right) \sin \theta_i \hat{z} \right]$$

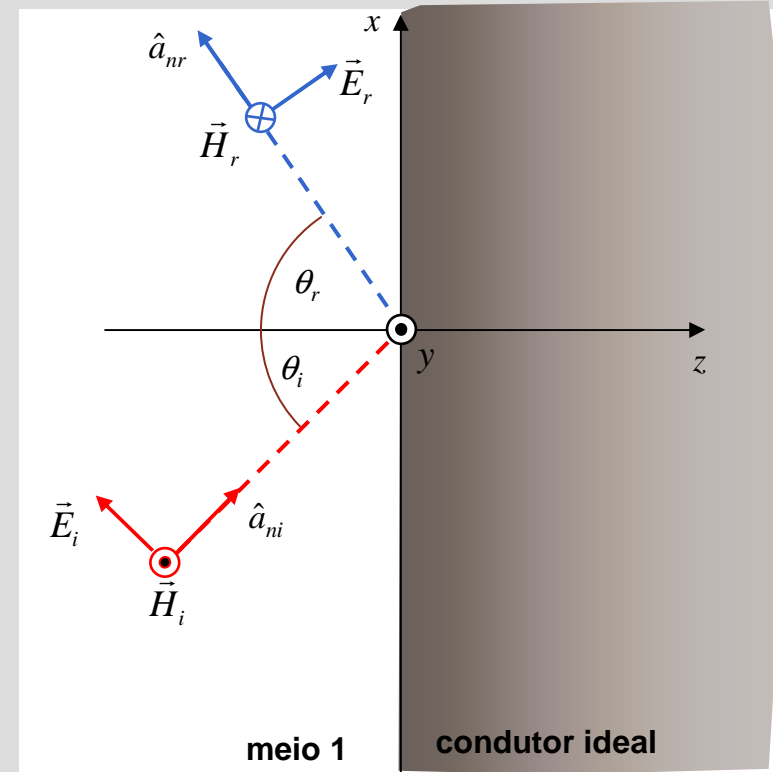
$$= -j2E_{i0} e^{-j\beta_1 \sin \theta_i x} \sin(\beta_1 \cos \theta_i z) \cos \theta_i \hat{x}$$

$$- 2E_{i0} e^{-j\beta_1 \sin \theta_i x} \cos(\beta_1 \cos \theta_i z) \sin \theta_i \hat{z}$$

ondas em propagação segundo  $x$ ,  
com amplitudes dependente de  $z$

$$\text{máximos de } E_{1x}: z_{MAX} = \frac{(2n+1)\pi}{2\beta_1 \cos \theta_i} \quad \text{e} \quad |E_{1x}|_{MAX} = 2|E_{i0} \cos \theta_i|$$

$$\text{mínimos de } E_{1x}: z_{min} = \frac{n\pi}{\beta_1 \cos \theta_i} \quad \text{e} \quad |E_{1x}|_{min} = 0$$





## Guias de onda metálicos

$$\perp : \vec{E}_1 = -j2E_{i0}e^{-j\beta_1 \sin \theta_i x} \sin(\beta_1 \cos \theta_i z) \hat{y}$$

$$\parallel : \vec{E}_1 = -j2E_{i0}e^{-j\beta_1 \sin \theta_i x} \sin(\beta_1 \cos \theta_i z) \cos \theta_i \hat{x} - 2E_{i0}e^{-j\beta_1 \sin \theta_i x} \cos(\beta_1 \cos \theta_i z) \sin \theta_i \hat{z}$$

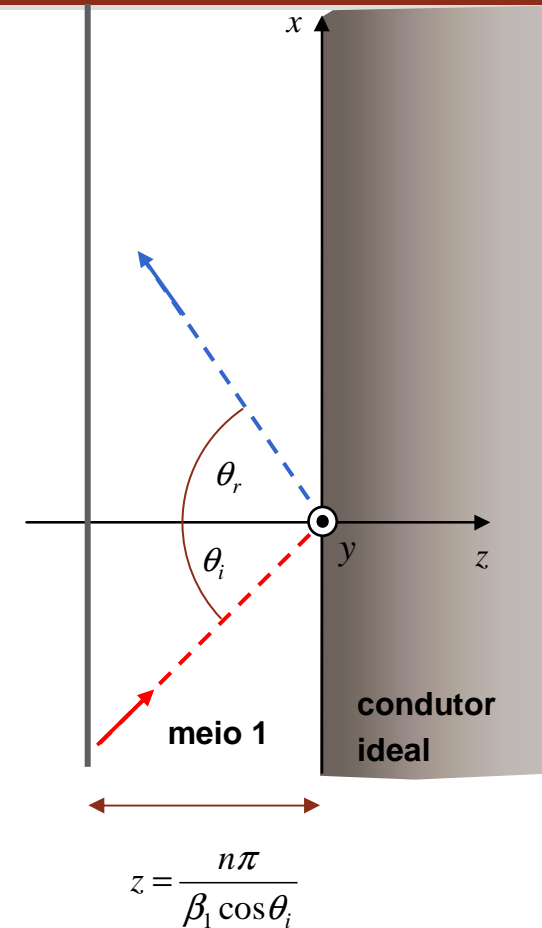


**polarização perpendicular:**  $|\vec{E}_1| = 0$  em  $z = \frac{n\pi}{\beta_1 \cos \theta_i}$

**polarização paralela:**  $|E_{1x}| = 0$  em  $z = \frac{n\pi}{\beta_1 \cos \theta_i}$



para ambas polarizações, um plano condutor paralelo ao plano xy poderia ser colocado em  $z = \frac{n\pi}{\beta_1 \cos \theta_i}$ , sem alterar o campo no meio 1



## Guias de onda metálicos

para ambas polarizações, um plano condutor paralelo ao plano xy  
poderia ser colocado em  $z = \frac{n\pi}{\beta_1 \cos\theta_i}$ , sem alterar o campo no meio 1

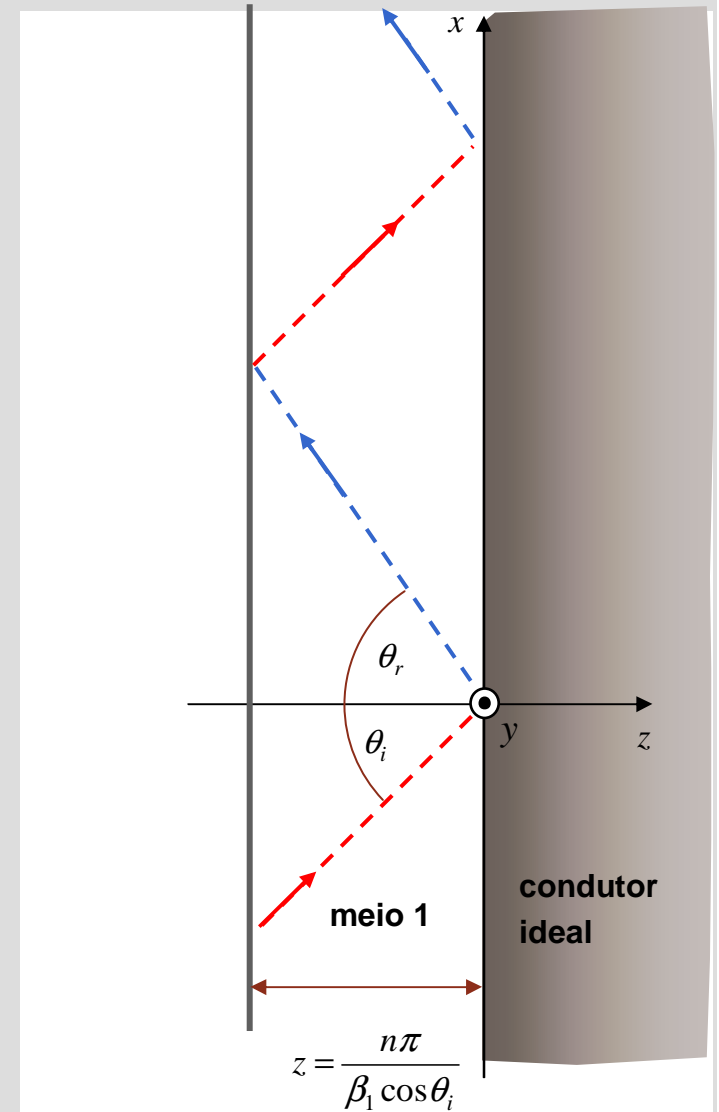


**onda electromagnética é guiada  
pelas duas superfícies condutoras**



princípio de funcionamento dos guias de onda metálicos

→ será possível guiar uma onda electromagnética  
com meios dieléctricos?



## Guias de onda metálicos

caso geral:

em cada incidência parte da onda é transmitida para o dieléctrico 2

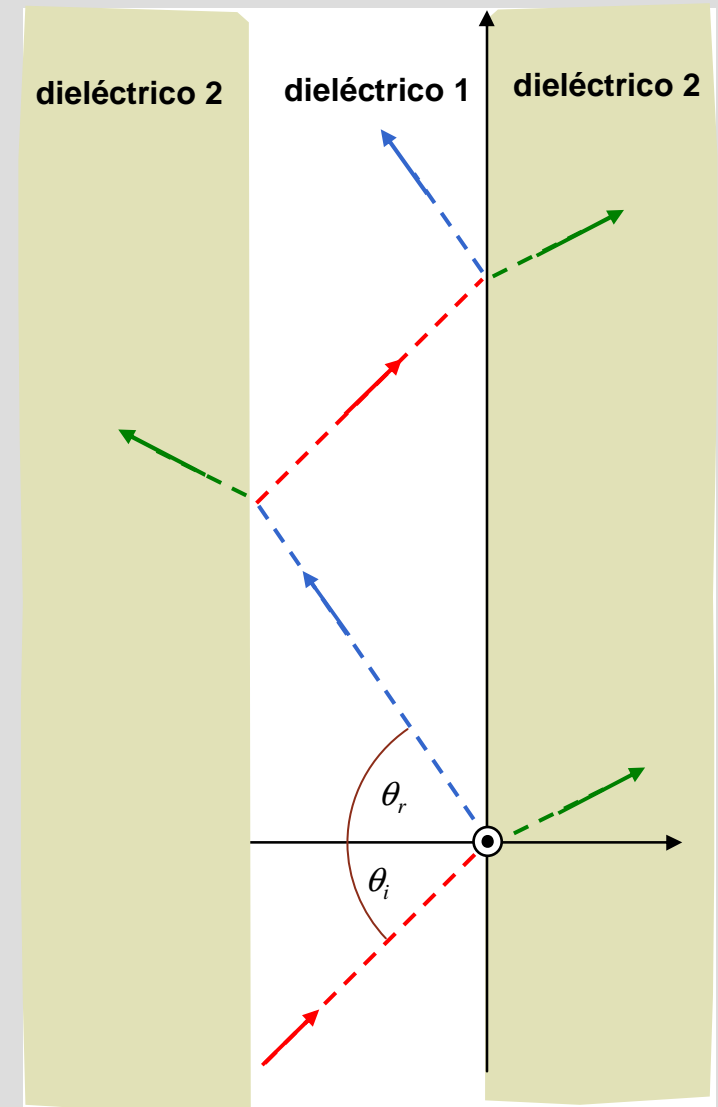


ao fim de alguma distância já a onda no dieléctrico interior se atenuou consideravelmente



no caso geral, materiais dieléctricos não permitem conduzir ondas electromagnéticas de forma eficiente

→ a solução seria garantir que não há energia transmitida para o meio 2 *será isto possível?*



# Reflexão interna total

Lei de Snell da refração  $n_1 \sin \theta_i = n_2 \sin \theta_t$

$$n_1 > n_2 \longrightarrow \theta_t > \theta_i$$

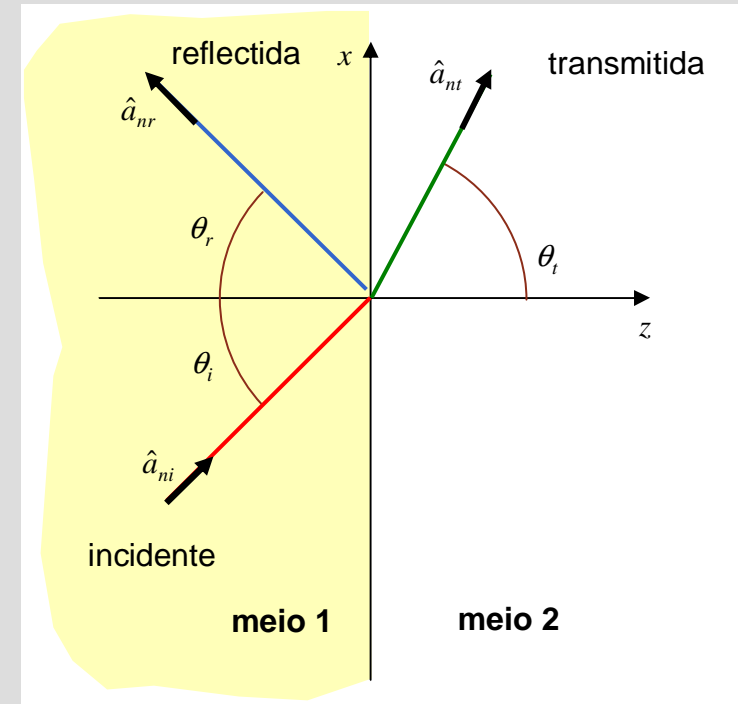
Ângulo crítico  $\theta_c = \theta_i$  talque  $\theta_t = 90^\circ \longrightarrow \theta_c = \arcsin \frac{n_2}{n_1}$

$\theta_i \geq \theta_c \longrightarrow$  não há onda transmitida para o meio 2

**Reflexão interna total**

$$\sin \theta_t = \frac{n_1}{n_2} \sin \theta_i \geq \frac{n_1}{n_2} \sin \theta_c$$

$$\sin \theta_t \geq 1 \longrightarrow \cos \theta_t = \pm \sqrt{1 - \sin^2 \theta_t} = \pm j \sqrt{\sin^2 \theta_t - 1}$$



## Reflexão interna total

**Reflexão interna total**     $\sin \theta_i \geq 1$   
 $\cos \theta_i = \pm j \sqrt{\sin^2 \theta_i - 1}$

**Meios sem perdas e não magnéticos**

$$\eta = \sqrt{\frac{\mu_0}{\epsilon}} = \frac{\eta_0}{n}$$

$$n = \sqrt{\epsilon_r}$$

**Coefficientes de reflexão**

$$\Gamma_{\perp} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$$

$$\Gamma_{\parallel} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

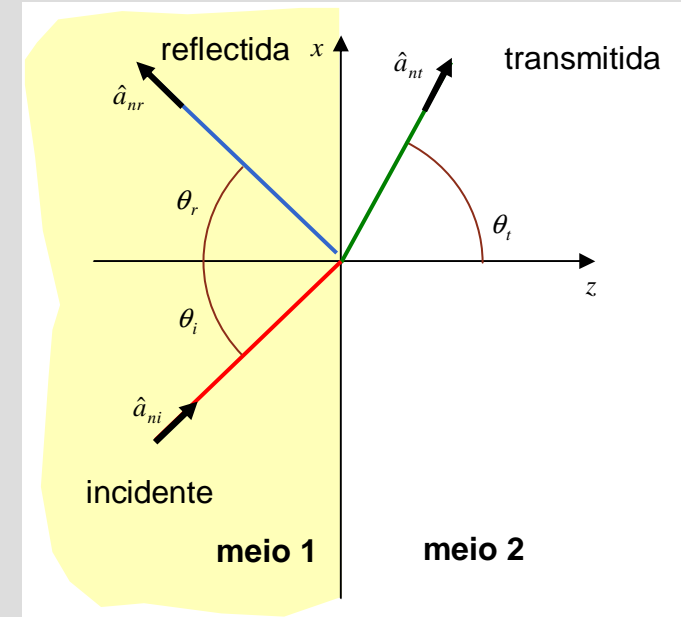
$$\Gamma_{\perp} = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t}$$

$$\Gamma_{\parallel} = \frac{n_1 \cos \theta_t - n_2 \cos \theta_i}{n_1 \cos \theta_t + n_2 \cos \theta_i}$$

$$\Gamma_{\perp} = \frac{n_1 \cos \theta_i \mp j n_2 \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2 \theta_i - 1}}{n_1 \cos \theta_i \pm j n_2 \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2 \theta_i - 1}}$$

$$\Gamma_{\parallel} = \frac{-n_2 \cos \theta_i \pm j n_1 \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2 \theta_i - 1}}{n_2 \cos \theta_i \pm j n_1 \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2 \theta_i - 1}}$$

$$|\Gamma_{\perp}| = |\Gamma_{\parallel}| = 1$$



# Reflexão interna total – campos evanescentes

dependência espacial dos campos no meio 2:  $e^{-j\beta_2 \hat{a}_{nt} \cdot \vec{r}}$

$$\hat{a}_{nt} = \sin \theta_t \hat{x} + \cos \theta_t \hat{z}$$

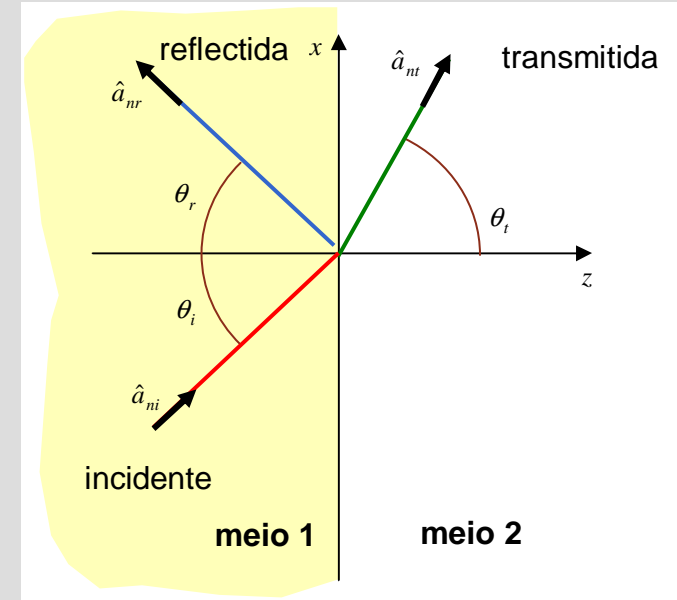
$$e^{-j\beta_2 (\sin \theta_t x + \cos \theta_t z)}$$

$$\cos \theta_t = \pm j \sqrt{\sin^2 \theta_t - 1}$$

$$e^{-\beta_2 z \sqrt{\sin^2 \theta_t - 1}} e^{-j\beta_2 x \sin \theta_t}$$

onda que se propaga segundo  $+x$   
amplitude decresce exponencialmente com  $z$

**campos evanescentes**



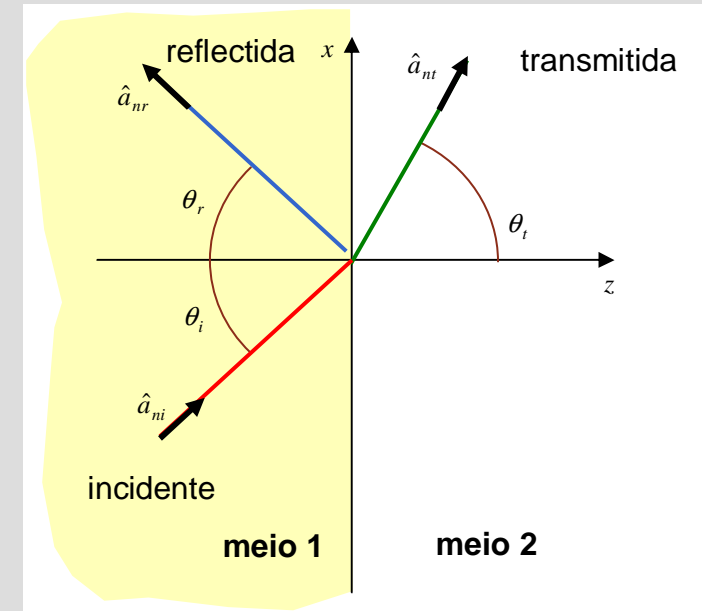
## Reflexão interna total – campos no meio 2

### Polarização perpendicular

$$\vec{E}_t = E_{t0} e^{-\beta_2 z \sqrt{\sin^2 \theta_i - 1}} e^{-j\beta_2 x \sin \theta_i} \hat{y}$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\beta_2 z \sqrt{\sin^2 \theta_i - 1}} e^{-j\beta_2 x \sin \theta_i} \left( \sin \theta_i \hat{z} + j\sqrt{\sin^2 \theta_i - 1} \hat{x} \right)$$

$$\vec{S}_{med,t} = \frac{1}{2} \operatorname{Re} \{ \vec{E}_t \times \vec{H}_t^* \} = \frac{|E_{t0}|^2}{2\eta_2} \sin \theta_i e^{-\beta_2 z \sqrt{\sin^2 \theta_i - 1}} \hat{x}$$



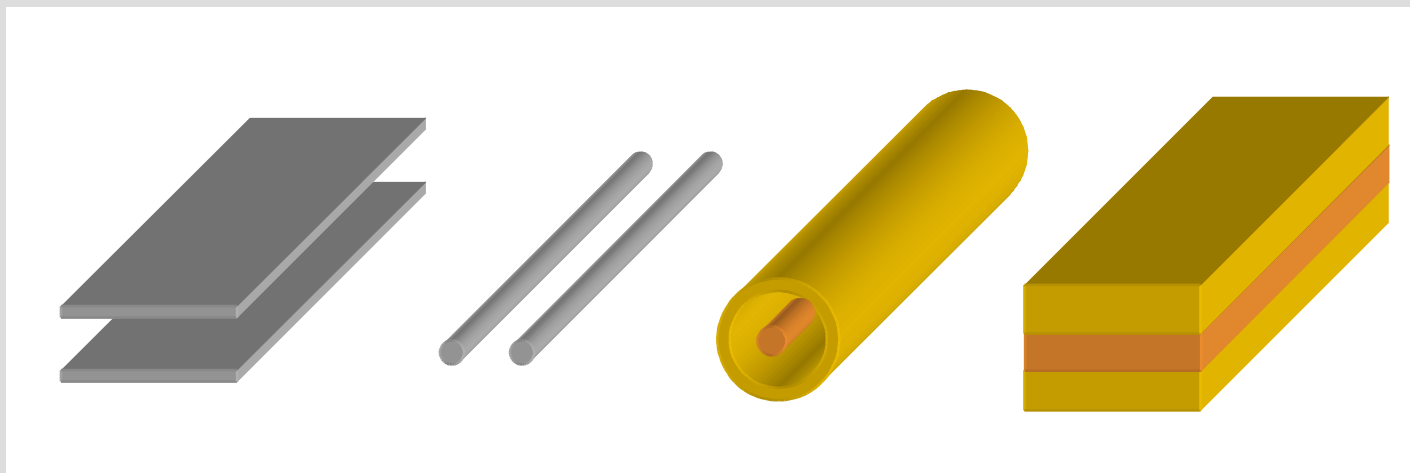
### Polarização paralela

$$\vec{E}_t = -E_{t0} e^{-\beta_2 z \sqrt{\sin^2 \theta_i - 1}} e^{-j\beta_2 x \sin \theta_i} \left( \sin \theta_i \hat{z} + j\sqrt{\sin^2 \theta_i - 1} \hat{x} \right)$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\beta_2 z \sqrt{\sin^2 \theta_i - 1}} e^{-j\beta_2 x \sin \theta_i} \hat{y}$$

$$\vec{S}_{med,t} = \frac{1}{2} \operatorname{Re} \{ \vec{E}_t \times \vec{H}_t^* \} = \frac{|E_{t0}|^2}{2\eta_2} \sin \theta_i e^{-\beta_2 z \sqrt{\sin^2 \theta_i - 1}} \hat{x}$$

# Guias de Onda e Cavidades

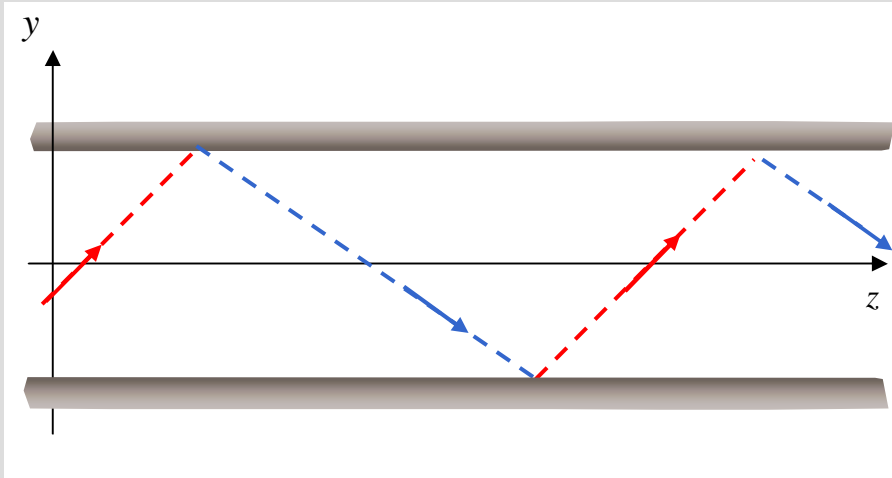


OE - MIEEC 2006/2007

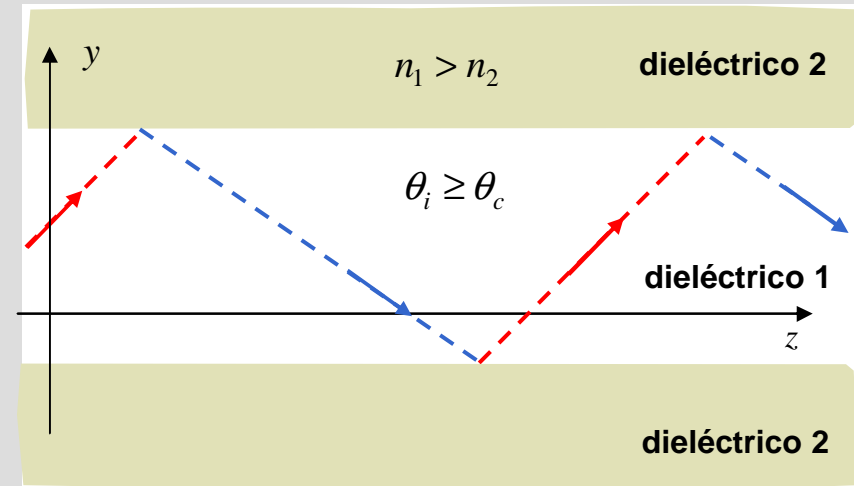


# Propagação guiada

**guia metálico**



**guia dieléctrico**



estudo dos guias de onda

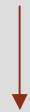


equações de Maxwell

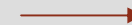


condições fronteira

campos harmónicos em meios  
LHI sem perdas e cargas

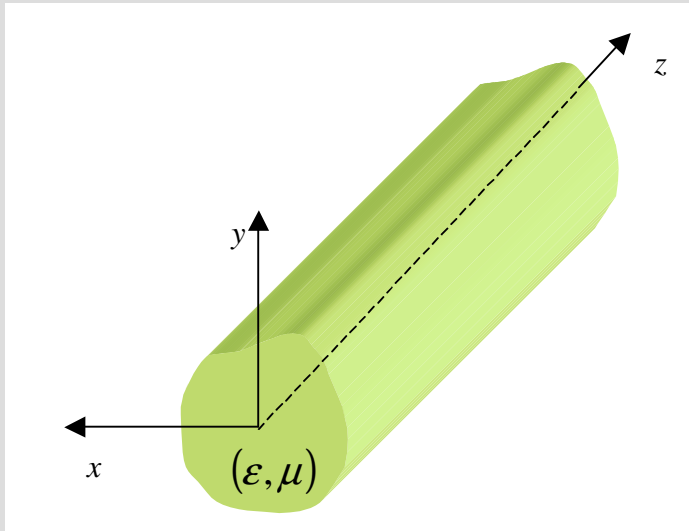


$$\begin{aligned} \nabla \times \vec{E} &= -j\omega\mu \vec{H} \\ \nabla \times \vec{H} &= j\omega\varepsilon \vec{E} \\ \nabla \cdot \vec{E} &= 0 \\ \nabla \cdot \vec{H} &= 0 \end{aligned}$$



$$\begin{aligned} \nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} &= 0 \\ \nabla^2 \vec{H} + \omega^2 \mu \varepsilon \vec{H} &= 0 \end{aligned}$$

## Guias de onda cilíndricos



guias preenchidos com meio sem perdas  
podem estar limitados por condutor ideal

comprimento infinito  $\Rightarrow$  propagação segundo  $+z$   
secção transversal não varia com  $z$

$$\vec{E}(x, y, z) = \vec{E}^0(x, y)e^{-\gamma z}$$

$$\vec{H}(x, y, z) = \vec{H}^0(x, y)e^{-\gamma z}$$

onda não uniforme

## Guias de onda cilíndricos – determinação dos campos

$$\nabla^2 \vec{E} + \omega^2 \mu \epsilon \vec{E} = 0$$

$$\nabla^2 \vec{H} + \omega^2 \mu \epsilon \vec{H} = 0$$

$$\vec{E}(x, y, z) = \vec{E}^0(x, y) e^{-\gamma z}$$

$$\vec{H}(x, y, z) = \vec{H}^0(x, y) e^{-\gamma z}$$

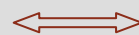
$$h^2 = \gamma^2 + \omega^2 \mu \epsilon$$

$$\nabla_{xy}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla_{xy}^2 \vec{E}^0 + h^2 \vec{E}^0 = 0$$

$$\nabla_{xy}^2 \vec{H}^0 + h^2 \vec{H}^0 = 0$$

2 eqs. vectoriais

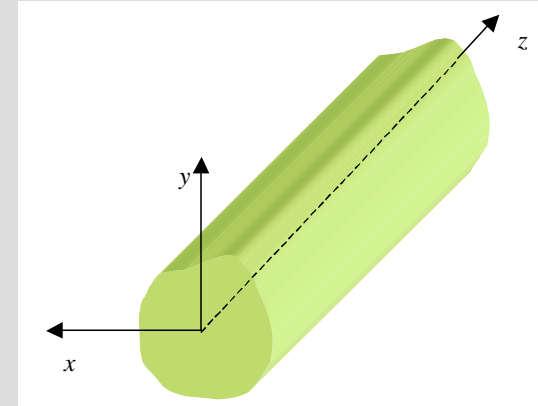


$$\nabla_{xy}^2 E_x^0 + h^2 E_x^0 = 0 \quad \nabla_{xy}^2 H_x^0 + h^2 H_x^0 = 0$$

$$\nabla_{xy}^2 E_y^0 + h^2 E_y^0 = 0 \quad \nabla_{xy}^2 H_y^0 + h^2 H_y^0 = 0$$

$$\nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0 \quad \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

6 eqs. escalares **não independentes**



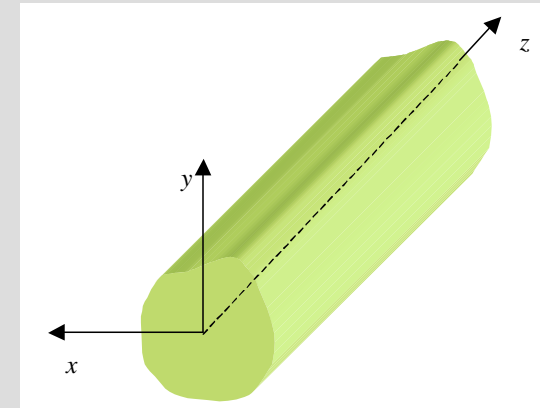
## Guias de onda cilíndricos – componentes transversais

$$\nabla \times \vec{E} = -j\omega\mu \vec{H}$$

$$\nabla \times \vec{H} = j\omega\varepsilon \vec{E}$$

$$\vec{E} = \vec{E}^0(x, y)e^{-\gamma z}$$

$$\vec{H} = \vec{H}^0(x, y)e^{-\gamma z}$$



$$\frac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0$$

$$\frac{\partial H_z^0}{\partial y} + \gamma H_y^0 = j\omega\varepsilon E_x^0$$

$$-\frac{\partial E_z^0}{\partial x} - \gamma E_x^0 = -j\omega\mu H_y^0$$

$$-\frac{\partial H_z^0}{\partial x} - \gamma H_x^0 = j\omega\varepsilon E_y^0$$

$$\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0$$

$$\frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y} = j\omega\varepsilon E_z^0$$

se  $h \neq 0$

$$H_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial x} - j\omega\varepsilon \frac{\partial E_z^0}{\partial y} \right)$$

$$H_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial y} + j\omega\varepsilon \frac{\partial E_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x} \right)$$

componentes transversais à custa  
das componentes longitudinais

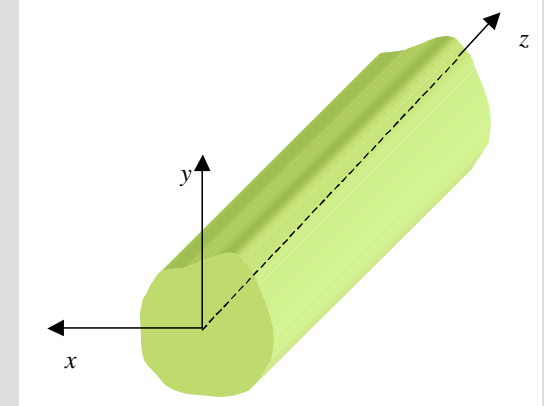
# Determinação dos campos no interior do guia

## 1. resolver

$$\nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

$$h^2 = \gamma^2 + \omega^2 \mu \epsilon$$



## 2. determinar

$$H_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial x} - j\omega \epsilon \frac{\partial E_z^0}{\partial y} \right)$$

$$H_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial y} + j\omega \epsilon \frac{\partial E_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial x} + j\omega \mu \frac{\partial H_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial y} - j\omega \mu \frac{\partial H_z^0}{\partial x} \right)$$

se  $h \neq 0$

aplicação de condições  
fronteira

## 3. obter

$$\vec{E}(x, y, z) = \vec{E}^0(x, y)e^{-\gamma z}$$

$$\vec{H}(x, y, z) = \vec{H}^0(x, y)e^{-\gamma z}$$

### Nota

$$H_z^0 = 0 \text{ e } E_z^0 = 0 \quad \leftarrow \text{ondas TEM}$$

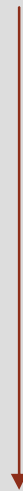
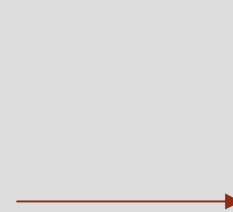
$$E_z^0 = 0 \text{ e } H_z^0 \neq 0 \quad \leftarrow \text{ondas TE}$$

$$H_z^0 \neq 0 \text{ e } E_z^0 = 0 \quad \leftarrow \text{ondas TM}$$

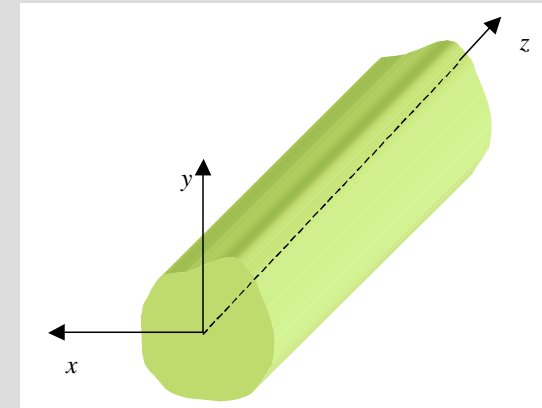
## Frequência de corte

$$h^2 = \gamma^2 + \omega^2 \mu \epsilon \quad \longrightarrow \quad \gamma = \sqrt{h^2 - \omega^2 \mu \epsilon} = \omega \sqrt{\mu \epsilon} \sqrt{\frac{h^2}{\omega^2 \mu \epsilon} - 1}$$

frequência de corte  $\rightarrow$  
$$f_c = \frac{h}{2\pi \sqrt{\mu \epsilon}}$$



$$\gamma = \omega \sqrt{\mu \epsilon} \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$



## Modos em propagação e modos evanescentes

$$\gamma = \omega \sqrt{\mu \epsilon} \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

$$f < f_c$$

$$\longrightarrow \gamma = \alpha$$

$$\begin{aligned} \vec{E}(x, y, z) &= \vec{E}^0(x, y) e^{-\alpha z} \\ \vec{H}(x, y, z) &= \vec{H}^0(x, y) e^{-\alpha z} \end{aligned}$$

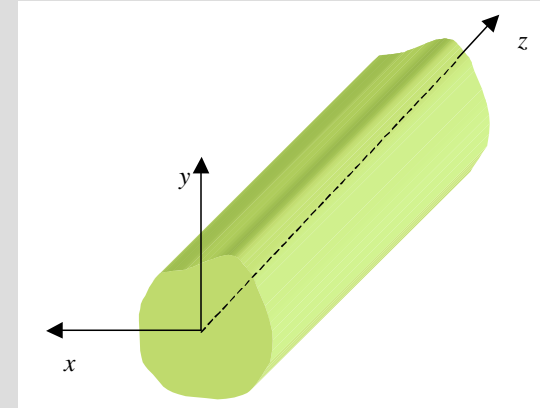
modo evanescente

$$f > f_c$$

$$\longrightarrow \gamma = j\beta$$

$$\begin{aligned} \vec{E}(x, y, z) &= \vec{E}^0(x, y) e^{-j\beta z} \\ \vec{H}(x, y, z) &= \vec{H}^0(x, y) e^{-j\beta z} \end{aligned}$$

modo em propagação



## Características dos modos em propagação

constante de fase →

$$\beta = \beta_m \sqrt{1 - \left(\frac{f_c}{f}\right)^2}, \quad \beta_m = \omega \sqrt{\mu \epsilon}$$

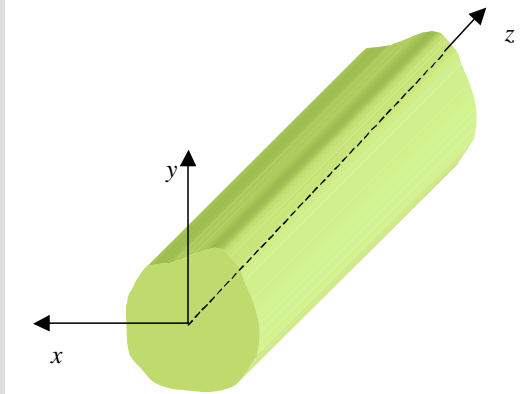
↑  
constante de fase num meio infinito de parâmetros  $(\epsilon, \mu)$

$$\lambda = \frac{2\pi}{\beta}$$

comprimento de onda →

$$\lambda = \frac{\lambda_m}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}, \quad \lambda_m = \frac{2\pi}{\beta_m}$$

se  $f_c \neq 0 \rightarrow \lambda > \lambda_m$



$$\gamma = \omega \sqrt{\mu \epsilon} \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

modo em propagação:  $\gamma = j\beta$   
( $f > f_c$ )



## Características dos modos em propagação

velocidade de fase  $\rightarrow v_f = \frac{v_m}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$  ,  $v_m = \frac{1}{\sqrt{\mu\epsilon}}$

↑  
velocidade de fase num meio infinito de parâmetros  $(\epsilon, \mu)$

se  $f_c \neq 0 \rightarrow v_f > v_m$

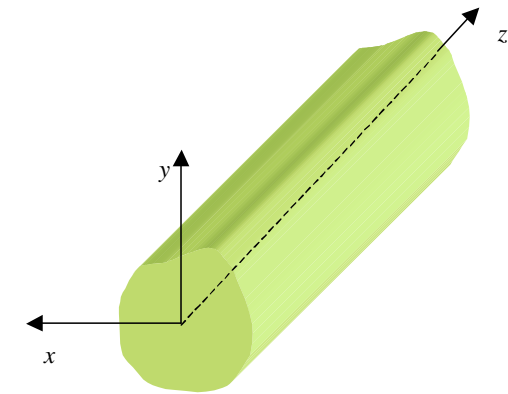
↓ se guia preenchido com ar,  $v_m = c$

$v_f > c$

velocidade de grupo  $\rightarrow v_g = v_m \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$

se  $f_c \neq 0 \rightarrow v_g < v_m$

$$v_f v_g = v_m^2$$



$$\gamma = \omega\sqrt{\mu\epsilon} \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

modo em propagação:  $\gamma = j\beta$   
( $f > f_c$ )

$$\beta = \beta_m \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

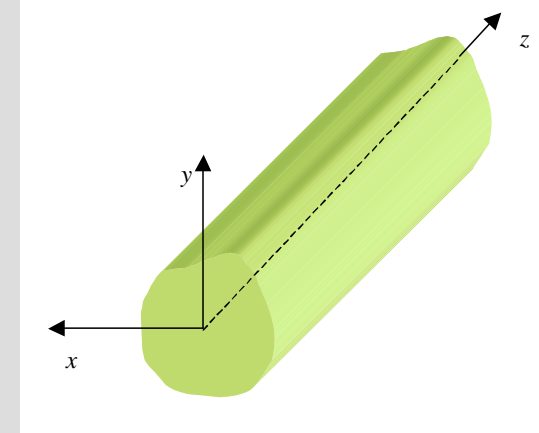
$$v_f = \frac{\omega}{\beta} \quad v_g = \frac{1}{d\beta/d\omega}$$

## Impedância de onda

ondas TEM propagando-se segundo  $+z$  num meio ilimitado com  $\eta = \sqrt{\mu/\epsilon}$

$$\vec{H} = \frac{1}{\eta} (\hat{z} \times \vec{E})$$

$$\vec{E} = -\eta (\hat{z} \times \vec{H})$$



ondas propagando-se segundo  $+z$  num guia

ondas TEM ou TM:  $\vec{H} = \frac{1}{Z} (\hat{z} \times \vec{E})$

ondas TEM ou TE:  $\vec{E} = -Z (\hat{z} \times \vec{H})$

$$H_x \hat{x} + H_y \hat{y} + H_z \hat{z} = \frac{1}{Z} (-E_y \hat{x} + E_x \hat{y}) \longrightarrow H_z = 0 \longrightarrow \text{ondas TM ou TEM}$$

$$E_x \hat{x} + E_y \hat{y} + E_z \hat{z} = -Z (-H_y \hat{x} + H_x \hat{y}) \longrightarrow E_z = 0 \longrightarrow \text{ondas TE ou TEM}$$

impedância de onda  $\rightarrow$

$$Z = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

# Potência média propagada

potência média →

$$\mathcal{P}_{med} = \int_A \vec{S}_{med} \cdot d\vec{A}$$

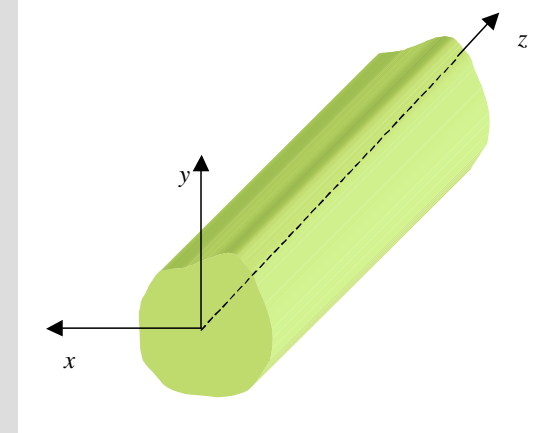
$$\vec{S}_{med} = \frac{1}{2} \mathcal{R}e\{\vec{E} \times \vec{H}^*\}$$

$$d\vec{A} = dA \hat{z}$$

$$\mathcal{P}_{med} = \frac{1}{2} \int_A \mathcal{R}e\{E_x H_y^* - E_y H_x^*\} dA$$

$$Z = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

$$\mathcal{P}_{med} = \frac{1}{2} \int_A \mathcal{R}e\left\{\frac{1}{Z}\right\} \left(|E_x|^2 + |E_y|^2\right) dA = \frac{1}{2} \int_A \mathcal{R}e\{Z\} \left(|H_x|^2 + |H_y|^2\right) dA$$



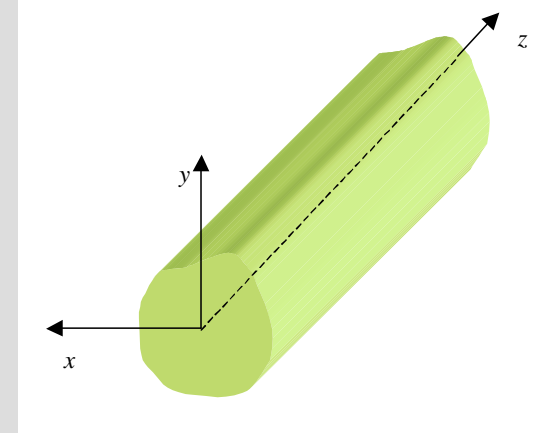
## Energia média armazenada e velocidade de transporte de energia

energia média armazenada →  
por unidade de comprimento

$$W'_{med} = \int_A (w_{e,med} + w_{m,med}) dA$$

$$w_{e,med} = \frac{\epsilon}{4} \vec{E} \cdot \vec{E}^* = \frac{\epsilon}{4} (|E_x|^2 + |E_y|^2 + |E_z|^2)$$

$$w_{m,med} = \frac{\mu}{4} \vec{H} \cdot \vec{H}^* = \frac{\mu}{4} (|H_x|^2 + |H_y|^2 + |H_z|^2)$$



velocidade de transporte de energia →

$$v_{en} = \frac{\mathcal{P}_{med}}{W'_{med}}$$

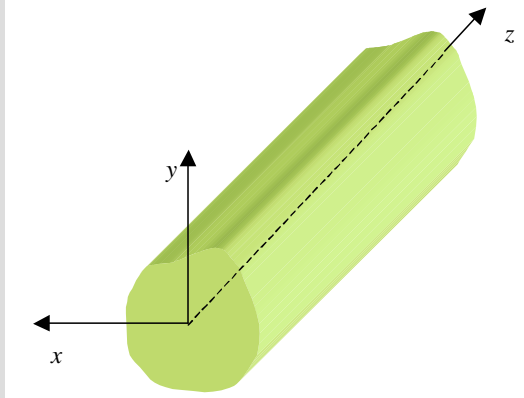
ondas TEM  $\rightarrow E_z = H_z = 0$

$$H_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y} \right)$$

$$H_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x} \right)$$



$$\gamma = \omega\sqrt{\mu\epsilon} \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

$$\beta = \beta_m \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad \beta_m = \omega\sqrt{\mu\epsilon}$$

$$h^2 = 0 \quad \rightarrow \quad f_c = \frac{h}{2\pi\sqrt{\mu\epsilon}} = 0 \quad \rightarrow$$

$$\gamma = j\beta$$

$$\beta = \beta_m$$

$$\lambda = \lambda_m$$

$$v_f = v_g = v_m$$

$$\gamma = j\omega\sqrt{\mu\epsilon}$$

usar equações  
de Maxwell

equações de Maxwell:  $\nabla \times \vec{E} = -j\omega\mu\vec{H}$

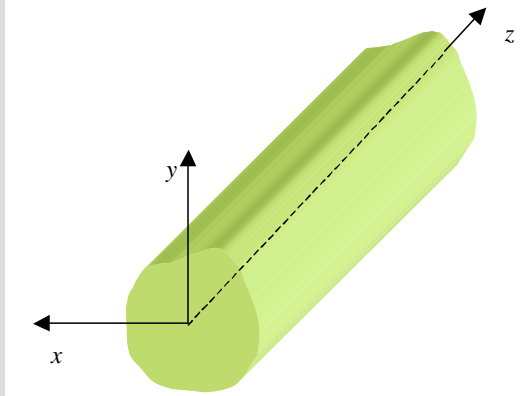


$$\begin{aligned} \gamma E_y^0 &= -j\omega\mu H_x^0 \\ -\gamma E_x^0 &= -j\omega\mu H_y^0 \\ \frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} &= 0 \end{aligned}$$

$\nabla \times \vec{H} = j\omega\epsilon\vec{E}$



$$\begin{aligned} \gamma H_y^0 &= j\omega\epsilon E_x^0 \\ -\gamma H_x^0 &= j\omega\epsilon E_y^0 \\ \frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y} &= 0 \end{aligned}$$



ondas TEM:  $E_z = H_z = 0$

$$Z = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

$$Z_{TEM} = \frac{j\omega\mu}{\gamma} = \frac{\gamma}{j\omega\epsilon} = \sqrt{\frac{\mu}{\epsilon}} = \eta$$

$$\gamma = j\omega\sqrt{\mu\epsilon}$$

ondas TM  $\rightarrow H_z^0 = 0$  e  $E_z^0 \neq 0$   $\rightarrow$

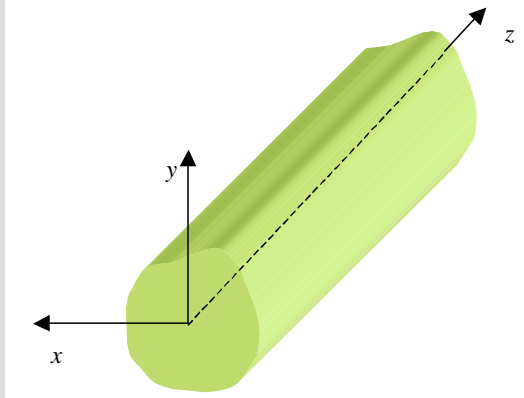
$$\nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{\partial E_z^0}{\partial y}$$

$$H_y^0 = -\frac{j\omega\epsilon}{h^2} \frac{\partial E_z^0}{\partial x}$$

$$E_x^0 = -\frac{\gamma}{h^2} \frac{\partial E_z^0}{\partial x}$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{\partial E_z^0}{\partial y}$$



$$\gamma = \omega\sqrt{\mu\epsilon} \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

$$\leftarrow Z = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

$$Z_{TM} = \frac{\gamma}{j\omega\epsilon} = \frac{\omega\sqrt{\mu\epsilon} \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}}{j\omega\epsilon} = -j\eta \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

$$Z_{TM} = -j\eta \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

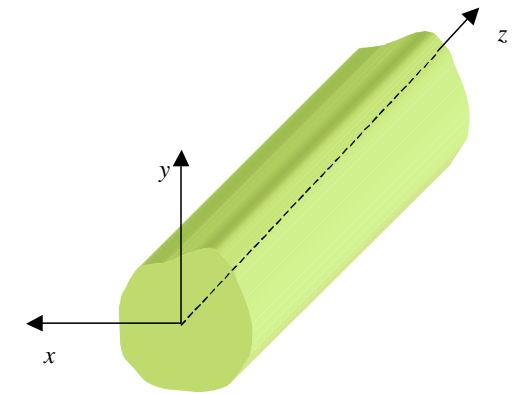
modos evanescentes  $\rightarrow f < f_c \rightarrow Z_{TM}$  é imaginário

$$\downarrow$$

$$P_{med} = 0$$

modos em propagação  $\rightarrow f > f_c \rightarrow Z_{TM} = \eta \sqrt{1 - (f_c/f)^2}$

$\uparrow$   
(real e inferior a  $\eta$ )



$$\gamma = \omega \sqrt{\mu \epsilon} \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

ondas TM  $\rightarrow H_z^0 = 0$  e  $E_z^0 \neq 0$

$$P_{med} = \frac{1}{2} \int_A \Re \left\{ \frac{1}{Z} \right\} (|E_x|^2 + |E_y|^2) dA$$



ondas TE  $\rightarrow E_z^0 = 0$  e  $H_z^0 \neq 0 \rightarrow \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$

$$H_x^0 = -\frac{\gamma}{h^2} \frac{\partial H_z^0}{\partial x}$$

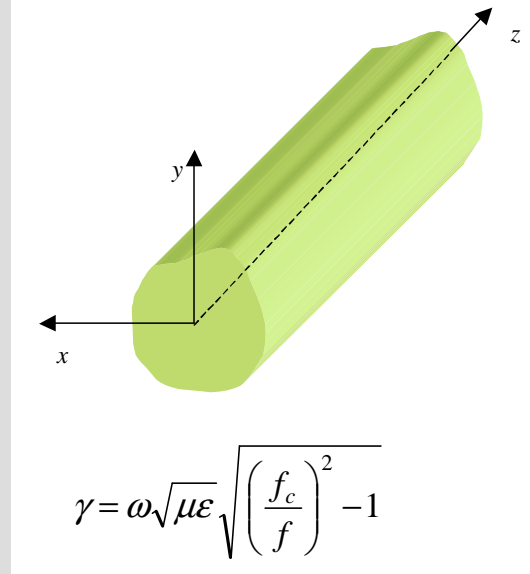
$$H_y^0 = -\frac{\gamma}{h^2} \frac{\partial H_z^0}{\partial y}$$

$$E_x^0 = -\frac{j\omega\mu}{h^2} \frac{\partial H_z^0}{\partial y}$$

$$E_y^0 = \frac{j\omega\mu}{h^2} \frac{\partial H_z^0}{\partial x}$$

$$Z = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

$$Z_{TE} = \frac{j\omega\mu}{\gamma} = \frac{j\eta}{\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}}$$



$$Z_{TE} = \frac{j\eta}{\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}}$$

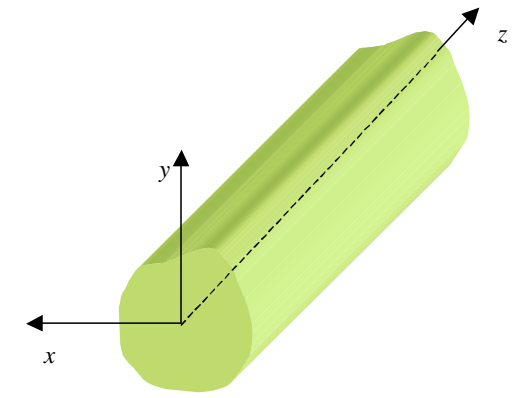
modos evanescentes  $\rightarrow f < f_c \rightarrow Z_{TE}$  é imaginário

$$\downarrow$$

$$P_{med} = 0$$

modos em propagação  $\rightarrow f > f_c \rightarrow Z_{TE} = \eta / \sqrt{1 - (f_c/f)^2}$

(real e superior a  $\eta$ )

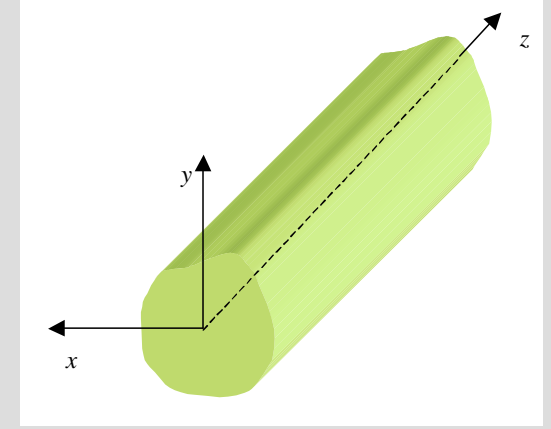
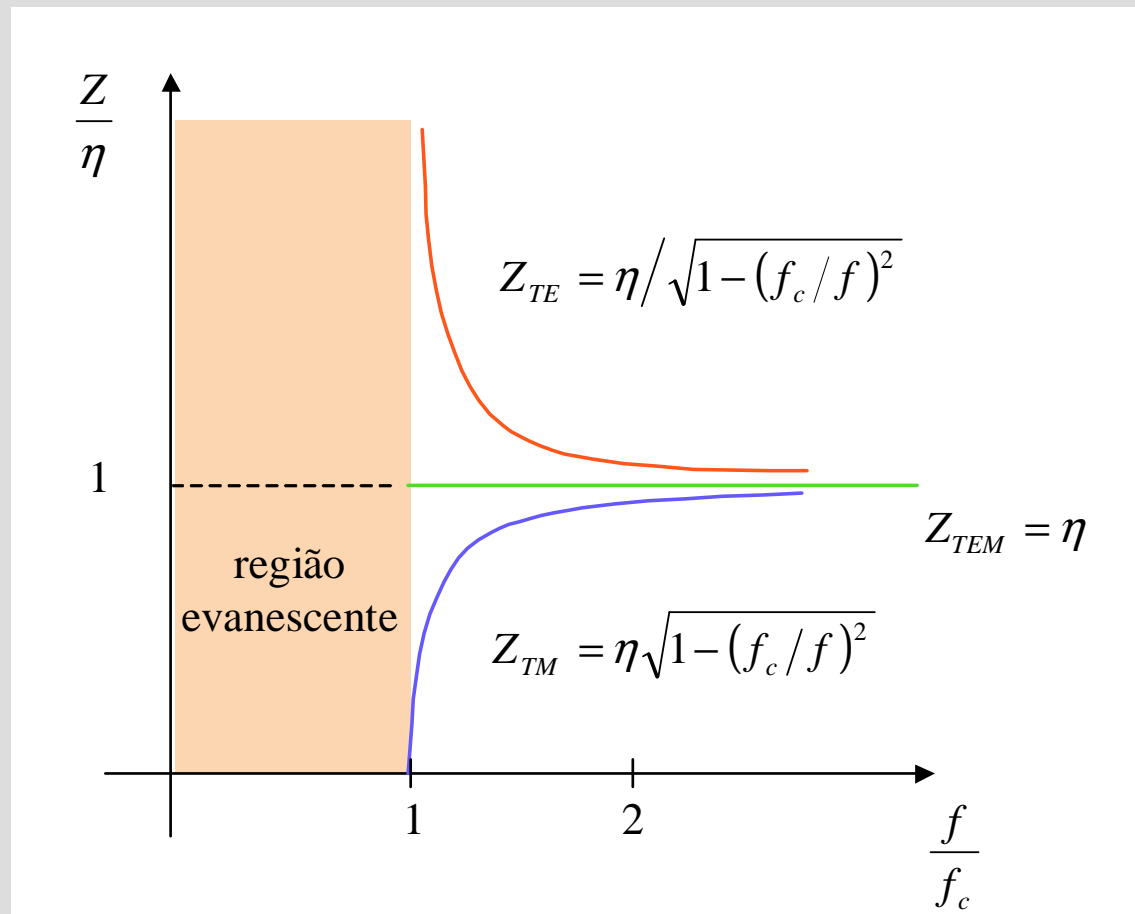


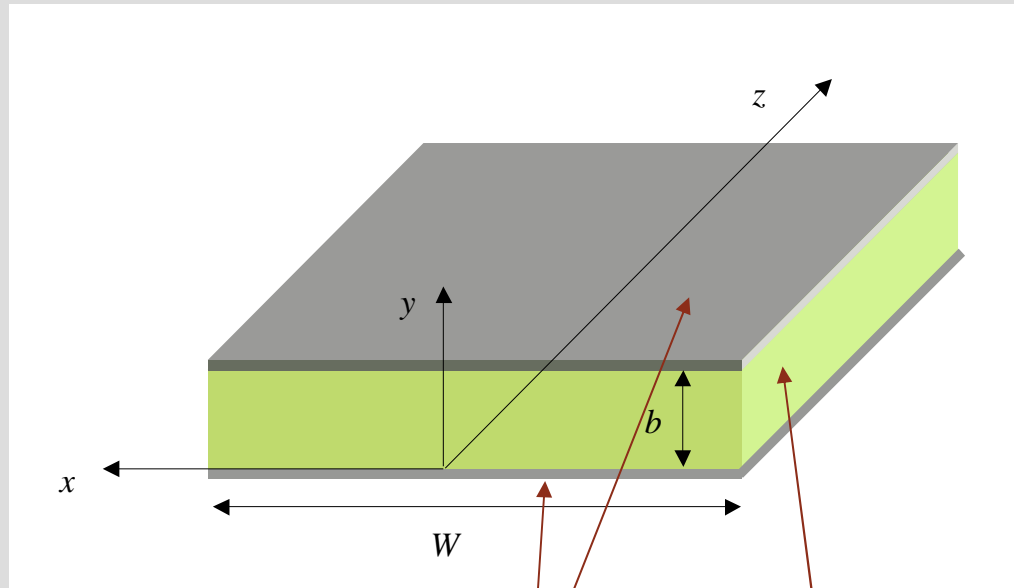
$$\gamma = \omega\sqrt{\mu\epsilon} \sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

ondas TE  $\rightarrow E_z^0 = 0$  e  $H_z^0 \neq 0$

$$P_{med} = \frac{1}{2} \int_A \Re \left\{ \frac{1}{Z} \right\} (|E_x|^2 + |E_y|^2) dA$$

# Impedância de onda vs frequência





placas condutoras ideais ( $\sigma = \infty$ )

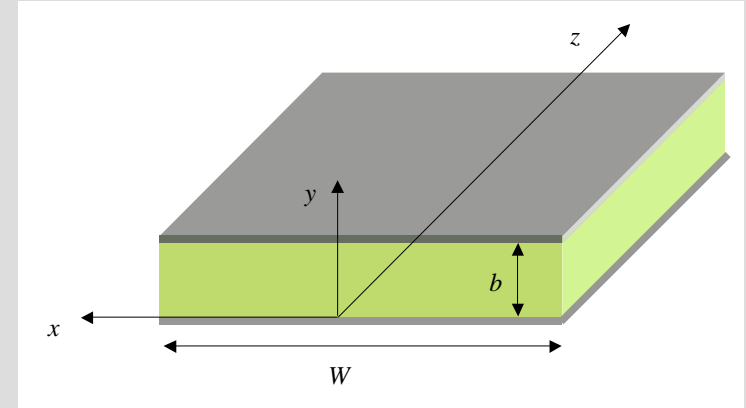
guia preenchido com material sem perdas ( $\epsilon, \mu$ )

comprimento infinito  $\rightarrow$  propagação segundo  $+z$

$W \gg b \implies$  variação dos campos com  $x$  é desprezável  $\implies \frac{\partial}{\partial x} = 0$

guias metálicos → limitados por condutores ideais

condições fronteira



$$E_{\text{tan}} \text{ contínuo e } B_{\text{norm}} \text{ contínuo}$$

$$E_{\text{cond}} = B_{\text{cond}} = 0$$

$$B = \mu H$$

$$\longrightarrow E_{\text{tan}} = H_{\text{norm}} = 0 \text{ junto aos condutores}$$

$$\begin{aligned} E_x = E_z = 0 \\ H_y = 0 \end{aligned} \quad \text{em } y=0 \text{ e } y=b$$

# Guias de placas paralelas – determinação dos campos

## 1. resolver

$$\nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

$$h^2 = \gamma^2 + \omega^2 \mu \epsilon$$

$$\frac{\partial}{\partial x} = 0$$

$$\frac{d^2 E_z^0}{dy^2} + h^2 E_z^0 = 0$$

$$\frac{d^2 H_z^0}{dy^2} + h^2 H_z^0 = 0$$

## 2. determinar (se $h \neq 0$ )

$$H_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial x} - j\omega \epsilon \frac{\partial E_z^0}{\partial y} \right)$$

$$H_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial y} + j\omega \epsilon \frac{\partial E_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial x} + j\omega \mu \frac{\partial H_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial y} - j\omega \mu \frac{\partial H_z^0}{\partial x} \right)$$

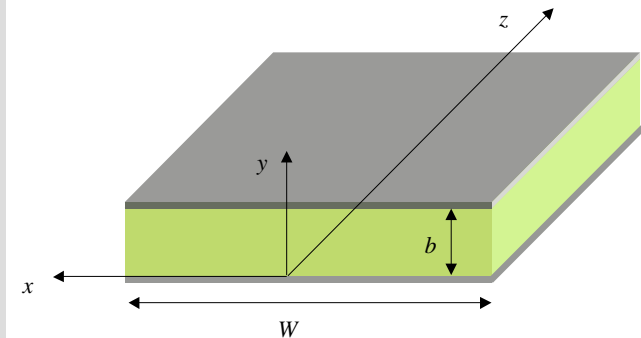
$$\frac{\partial}{\partial x} = 0$$

$$H_x^0 = \frac{j\omega \epsilon}{h^2} \frac{\partial E_z^0}{\partial y}$$

$$H_y^0 = -\frac{\gamma}{h^2} \frac{\partial H_z^0}{\partial y}$$

$$E_x^0 = -\frac{j\omega \mu}{h^2} \frac{\partial H_z^0}{\partial y}$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{\partial E_z^0}{\partial y}$$



$$\vec{E}(x, y, z) = \vec{E}^0(x, y) e^{-\gamma z}$$

$$\vec{H}(x, y, z) = \vec{H}^0(x, y) e^{-\gamma z}$$

$$H_z^0 = 0 \text{ e } E_z^0 = 0 \quad \leftarrow \text{ondas TEM}$$

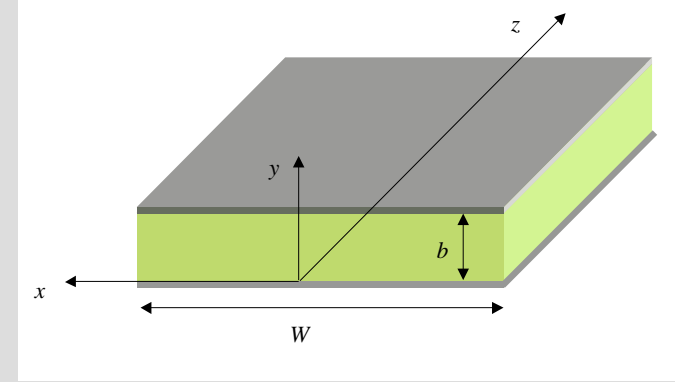
$$H_z^0 = 0 \text{ e } E_z^0 \neq 0 \quad \leftarrow \text{ondas TM}$$

$$E_z^0 = 0 \text{ e } H_z^0 \neq 0 \quad \leftarrow \text{ondas TE}$$

Nota: ondas TEM  $\rightarrow h=0$

# Ondas TEM

ondas TEM  $\rightarrow H_z^0 = 0$  e  $E_z^0 = 0$  e  $h = 0 \rightarrow$  método anterior não funciona



equações de Maxwell:

$$\nabla \times \vec{E} = -j\omega\mu \vec{H}$$

$$\nabla \times \vec{H} = j\omega\epsilon \vec{E}$$

$$\gamma E_y^0 = -j\omega\mu H_x^0$$

$$\gamma H_y^0 = j\omega\epsilon E_x^0$$

$$-\gamma E_x^0 = -j\omega\mu H_y^0$$

$$-\gamma H_x^0 = j\omega\epsilon E_y^0$$

$\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = 0$	$\frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y} = 0$
---	---

$$\partial/\partial x = 0$$

$\frac{dE_x^0}{dy} = \frac{dH_x^0}{dy} = 0$
---

$\rightarrow E_x^0$  e  $H_x^0$  são constantes

$E_x^0$  e  $H_x^0$  são constantes

$$E_y^0 = -Z H_x^0$$

$$E_y^0 = \text{constante}$$

$$H_x^0 = -\frac{E_y^0}{Z}$$

$$Z_{TEM} = \eta$$

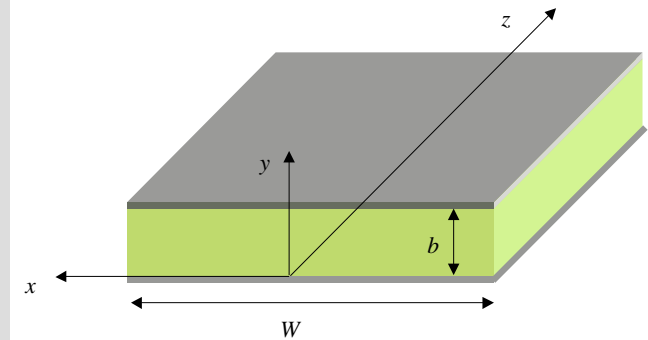
$$\vec{E}^0 = E_0 \hat{y}$$

$$\vec{H}^0 = -\frac{E_0}{\eta} \hat{x}$$

$$E_x^0(0) = E_x^0(b) = 0 \quad \rightarrow \quad E_x^0 = 0$$

$$E_x^0 = Z H_y^0$$

$$H_y^0 = 0$$



**condições fronteira**

$$E_x = E_z = 0 \quad \text{em } y=0 \text{ e } y=b$$

$$H_y = 0$$

**impedância de onda**

$$Z = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$



# Ondas TEM – densidade de carga nas placas

interior do guia:

$$\begin{aligned} \vec{E}^0 &= E_0 \hat{y} & \vec{E} &= E_0 e^{-j\beta z} \hat{y} \\ \vec{H}^0 &= -\frac{E_0}{\eta} \hat{x} & \vec{H} &= -\frac{E_0}{\eta} e^{-j\beta z} \hat{x} \end{aligned} \quad \xrightarrow{\gamma = j\beta}$$

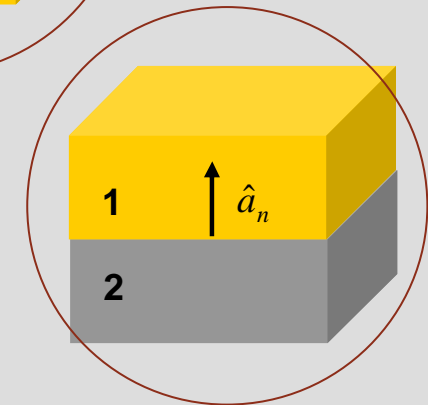
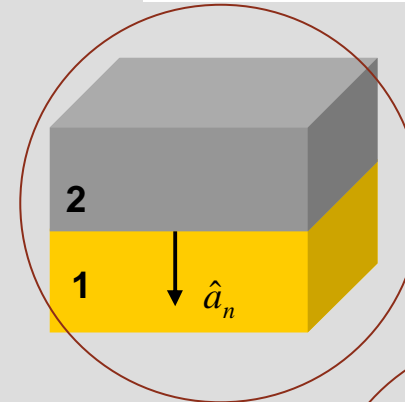
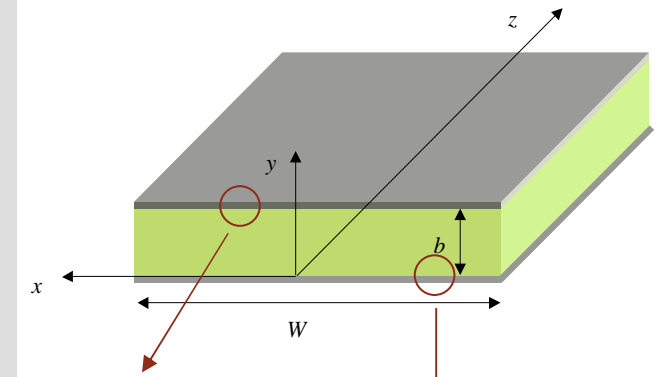
densidade de carga nas placas:  $\rho_s = \hat{a}_n \cdot (\vec{D}_1 - \vec{D}_2)$

placa superior:

$$\begin{aligned} \hat{a}_n &= -\hat{y} \\ \vec{D}_2 &= 0 \\ \vec{D}_1 &= \epsilon E_0 e^{-j\beta z} \hat{y} \end{aligned} \quad \longrightarrow \quad \rho_s(y=b) = -\epsilon E_0 e^{-j\beta z}$$

placa inferior:

$$\begin{aligned} \hat{a}_n &= \hat{y} \\ \vec{D}_2 &= 0 \\ \vec{D}_1 &= \epsilon E_0 e^{-j\beta z} \hat{y} \end{aligned} \quad \longrightarrow \quad \rho_s(y=0) = \epsilon E_0 e^{-j\beta z}$$



# Ondas TEM – densidade de corrente nas placas

interior do guia:

$$\vec{E} = E_0 e^{-j\beta z} \hat{y}$$

$$\vec{H} = -\frac{E_0}{\eta} e^{-j\beta z} \hat{x}$$

densidade de corrente nas placas:  $\vec{J}_s = \hat{a}_n \times (\vec{H}_1 - \vec{H}_2)$

placa superior:

$$\hat{a}_n = -\hat{y}$$

$$\vec{H}_2 = 0$$

$$\vec{H}_1 = -\frac{E_0}{\eta} e^{-j\beta z} \hat{x}$$

$$\vec{J}_s(y=b) = -\frac{E_0}{\eta} e^{-j\beta z} \hat{z}$$

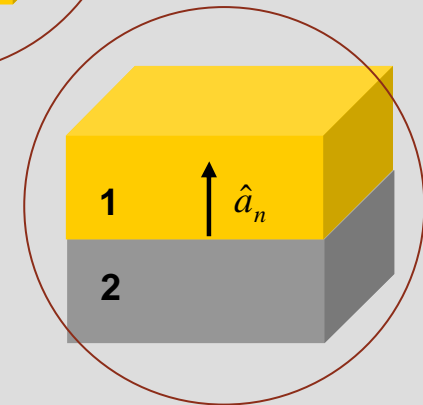
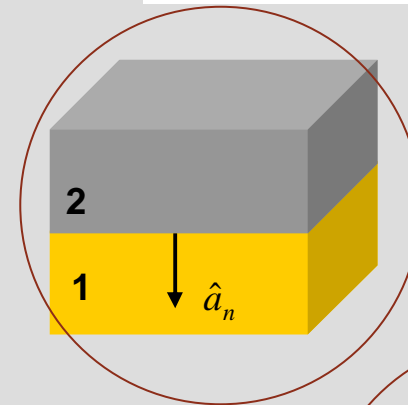
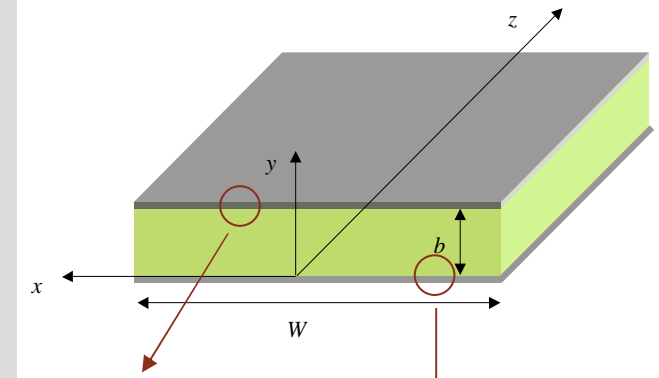
placa inferior:

$$\hat{a}_n = \hat{y}$$

$$\vec{H}_2 = 0$$

$$\vec{H}_1 = -\frac{E_0}{\eta} e^{-j\beta z} \hat{x}$$

$$\vec{J}_s(y=0) = \frac{E_0}{\eta} e^{-j\beta z} \hat{z}$$

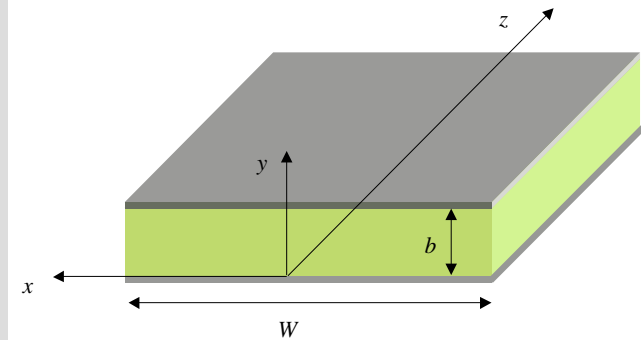


# Ondas TEM – tensão e corrente nas placas

corrente → 
$$I = \int_A \vec{J} \cdot d\vec{s}$$

corrente na placa superior: 
$$I(z) = \int_w \vec{J}_s \cdot dx \hat{z} = -W \frac{E_0}{\eta} e^{-j\beta z}$$

$$\vec{J}_s(y=b) = -\frac{E_0}{\eta} e^{-j\beta z} \hat{z}$$



interior do guia:

$$\vec{E} = E_0 e^{-j\beta z} \hat{y}$$

$$\vec{H} = -\frac{E_0}{\eta} e^{-j\beta z} \hat{x}$$

tensão → 
$$V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

tensão entre as placas: 
$$V(z) = -\int_0^b E_y dy = -bE_0 e^{-j\beta z}$$

$$E_y = E_0 e^{-j\beta z}$$

# Ondas TEM – equações das linha de transmissão

$$V(z) = -bE_0 e^{-j\beta z}$$

$$I(z) = -W \frac{E_0}{\eta} e^{-j\beta z}$$

$$\frac{dV}{dz} = j\beta bE_0 e^{-j\beta z}$$

$$\frac{dI}{dz} = j\beta W \frac{E_0}{\eta} e^{-j\beta z}$$

$$\beta = \omega\sqrt{\mu\varepsilon}$$

$$\eta = \sqrt{\mu/\varepsilon}$$

$$\frac{dV}{dz} = -j\omega \frac{\mu b}{W} I$$

$$\frac{dI}{dz} = -j\omega \frac{\varepsilon W}{b} V$$

$$L = \frac{\mu b}{W} \quad (\text{H/m})$$

$$C = \frac{\varepsilon W}{b} \quad (\text{C/m})$$

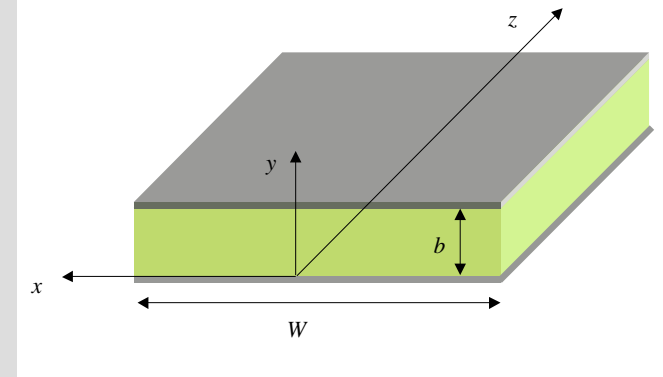
$$\frac{dV}{dz} = -j\omega L I$$

$$\frac{dI}{dz} = -j\omega C V$$

$$\frac{d^2 V}{dz^2} + \omega^2 L C V = 0$$

$$\frac{d^2 I}{dz^2} + \omega^2 L C I = 0$$

eqs. para  $V$  e  $I$  numa linha de transmissão sem perdas



# Ondas TM – componente longitudinal

ondas TM →  $H_z^0 = 0$  e  $E_z^0 \neq 0$  →  $\frac{d^2 E_z^0}{dy^2} + h^2 E_z^0 = 0$

solução geral:  $E_z^0(y) = A \sin(hy) + B \cos(hy)$

$E_z^0(0) = 0$

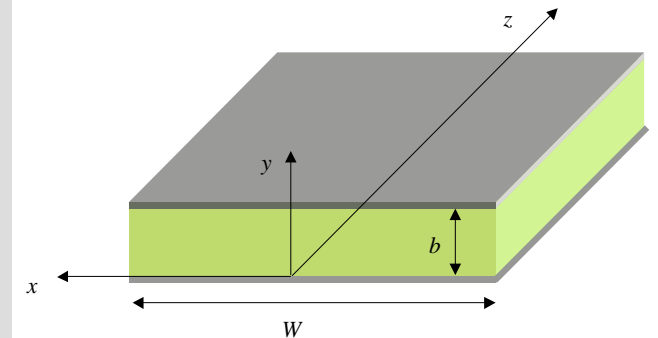
$E_z^0(y) = A \sin(hy)$  ←  $B = 0$

$E_z^0(b) = 0$

$A \sin(hb) = 0$

$h = \frac{n\pi}{b}, n = 1, 2, 3, \dots$

$h = \frac{n\pi}{b}, n = 1, 2, \dots$   
 $E_z^0 = A_n \sin\left(\frac{n\pi y}{b}\right)$



**condições fronteira**

$E_x = E_z = 0$   
 $H_y = 0$  em  $y = 0$  e  $y = b$

# Ondas TM – componentes transversais

$$h = \frac{n\pi}{b}, \quad n = 1, 2, 3, \dots$$

$$E_z^0 = A_n \sin\left(\frac{n\pi y}{b}\right)$$

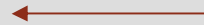


$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{\partial E_z^0}{\partial y}$$

$$H_y^0 = -\frac{\gamma}{h^2} \frac{\partial H_z^0}{\partial y} = 0$$

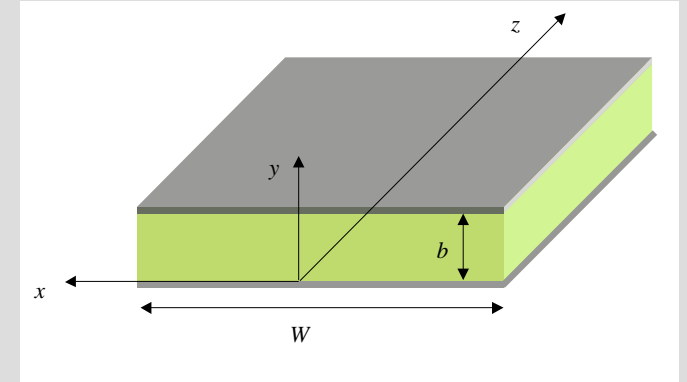
$$E_x^0 = -\frac{j\omega\mu}{h^2} \frac{\partial H_z^0}{\partial y} = 0$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{\partial E_z^0}{\partial y}$$



$$H_x^0 = \frac{j\omega\epsilon b}{n\pi} A_n \cos\left(\frac{n\pi y}{b}\right)$$

$$E_y^0 = -\frac{\gamma b}{n\pi} A_n \cos\left(\frac{n\pi y}{b}\right)$$



# Ondas TM – modo $TM_n$

modo  $TM_n$

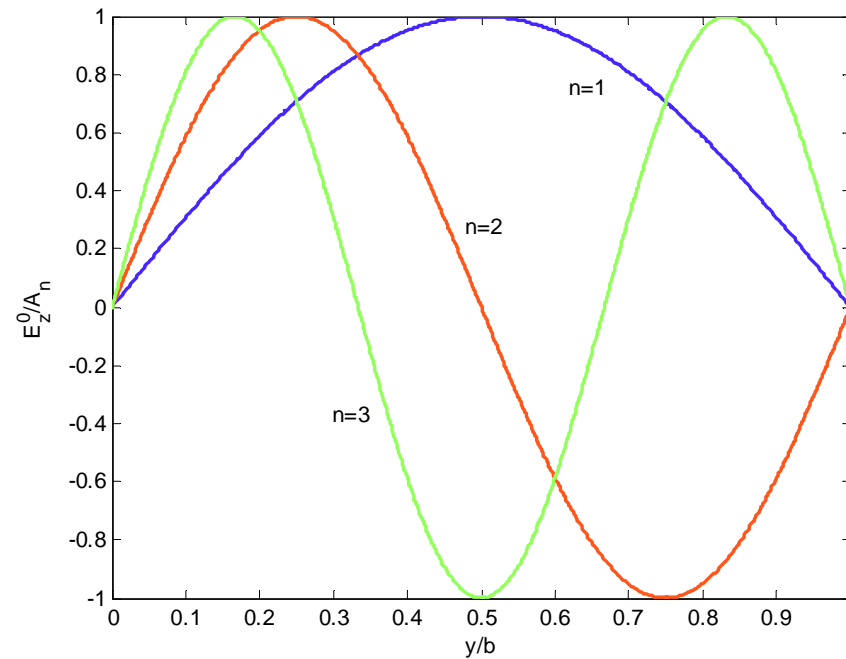
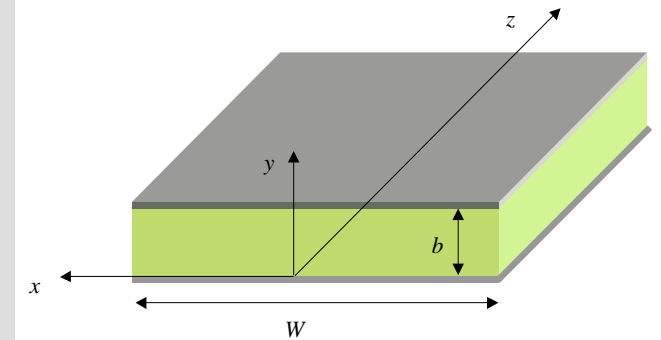
$$E_z^0 = A_n \sin\left(\frac{n\pi y}{b}\right)$$

$$H_x^0 = \frac{j\omega\epsilon b}{n\pi} A_n \cos\left(\frac{n\pi y}{b}\right)$$

$$E_y^0 = -\frac{\gamma b}{n\pi} A_n \cos\left(\frac{n\pi y}{b}\right)$$

$$h = \frac{n\pi}{b}, \quad n = 1, 2, 3, \dots$$

Nota:  
modo  $TM_n$  para  $n=0$   
é modo TEM



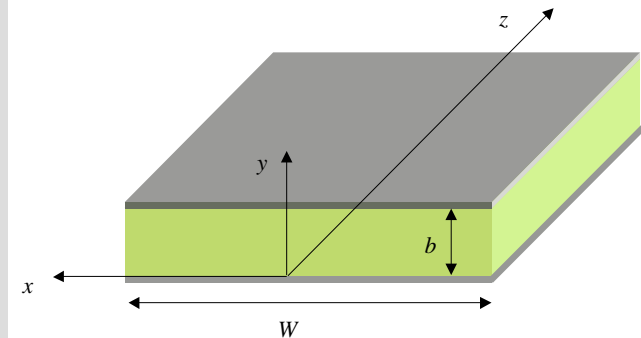
# Ondas TE – componente longitudinal

ondas TE  $\rightarrow E_z^0 = 0$  e  $H_z^0 \neq 0$   $\longrightarrow \frac{d^2 H_z^0}{dy^2} + h^2 H_z^0 = 0$

solução geral:  $H_z^0(y) = A \sin(hy) + B \cos(hy)$

nota: não existe condição fronteira para  $H_z^0$

é necessário determinar as componentes transversais para poder aplicar as **condições fronteira**



**condições fronteira**

$$\begin{aligned} E_x = E_z = 0 \\ H_y = 0 \end{aligned} \quad \text{em } y = 0 \text{ e } y = b$$



# Ondas TE – componente longitudinal

$$H_z^0(y) = A \sin(hy) + B \cos(hy)$$

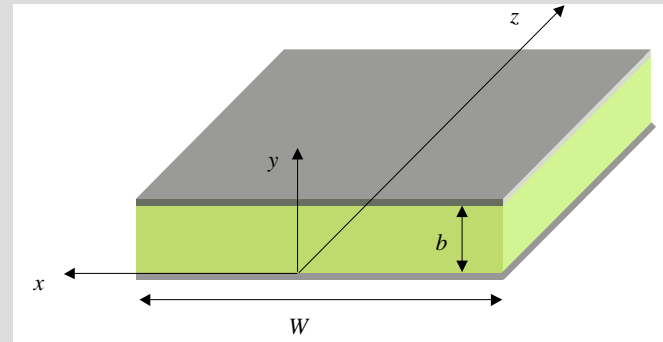


$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{\partial E_z^0}{\partial y} = 0$$

$$H_y^0 = -\frac{\gamma}{h^2} \frac{\partial H_z^0}{\partial y}$$

$$E_x^0 = -\frac{j\omega\mu}{h^2} \frac{\partial H_z^0}{\partial y}$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{\partial E_z^0}{\partial y} = 0$$



**condições fronteira**

$$E_x = E_z = 0 \quad \text{em } y=0 \text{ e } y=b$$

$$H_y = 0$$

$$H_y^0 = -\frac{\gamma}{h} [A \cos(hy) - B \sin(hy)]$$

$$E_x^0 = -\frac{j\omega\mu}{h} [A \cos(hy) - B \sin(hy)]$$

$$E_x^0(0) = H_y^0(0) = 0$$

$$A = 0$$

$$H_z^0 = B \cos(hy)$$

$$H_y^0 = \frac{\gamma}{h} B \sin(hy)$$

$$E_x^0 = \frac{j\omega\mu}{h} B \sin(hy)$$

$$E_x^0(b) = H_y^0(b) = 0$$

$$h = \frac{n\pi}{b}, \quad n = 1, 2, 3, \dots$$

$$\sin(hb) = 0$$

# Ondas TE – modo TE<sub>n</sub>

modo TE<sub>n</sub>

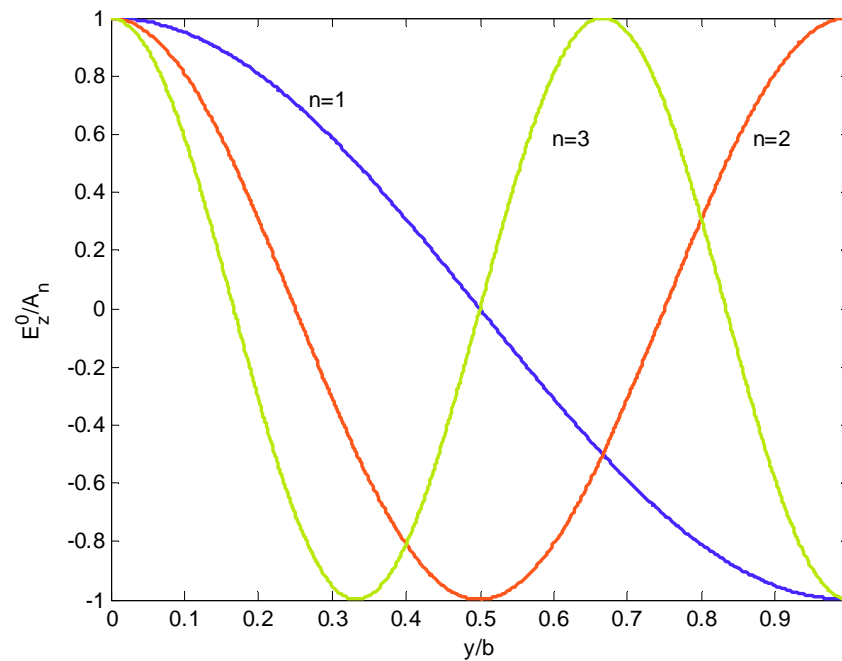
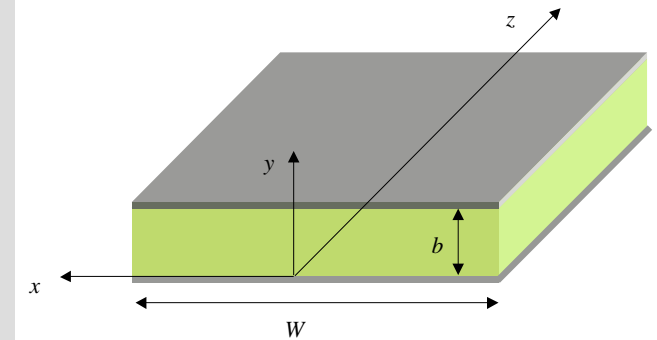
$$H_z^0 = B_n \cos\left(\frac{n\pi y}{b}\right)$$

$$H_y^0 = \frac{\gamma b}{n\pi} B_n \sin\left(\frac{n\pi y}{b}\right)$$

$$E_x^0 = j \frac{\omega\mu b}{n\pi} B_n \sin\left(\frac{n\pi y}{b}\right)$$

$$h = \frac{n\pi}{b}, \quad n = 1, 2, 3, \dots$$

Nota:  
Não existe modo TE<sub>0</sub>

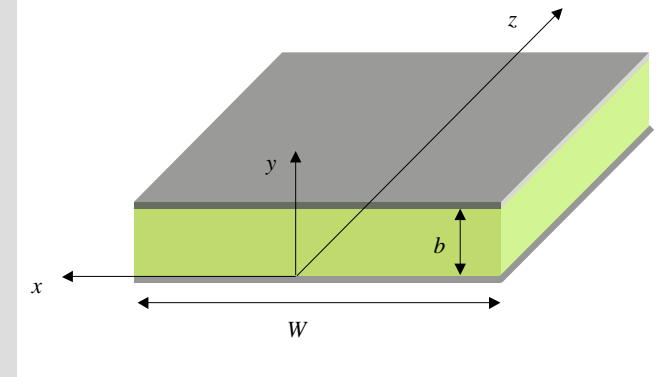


## Guias de placas paralelas – frequência de corte

$$f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}}$$

$$h_{\text{TEM}} = 0 \longrightarrow (f_c)_{\text{TEM}} = 0$$

$$h_{\text{TM,TE}} = \frac{n\pi}{b}, \quad n = 1, 2, 3, \dots \longrightarrow (f_c)_{\text{TM,TE}} = \frac{n}{2b\sqrt{\mu\varepsilon}}$$



para uma dada frequência  $f \rightarrow$  só se propagam os modos com  $f_c < f$

aumento de  $f \rightarrow$  mais modos se podem propagar

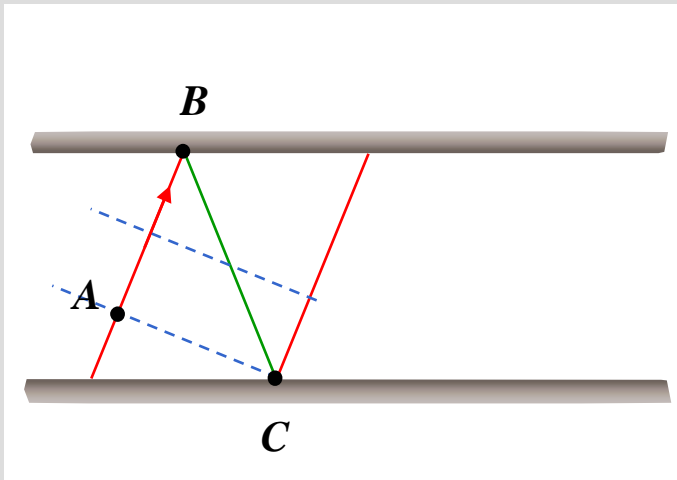
**modo dominante  $\rightarrow$  modo com menor frequência de corte**

**guias de placas paralelas  $\rightarrow$  modo dominante é o modo TEM**

como  $(f_c)_{\text{TEM}} = 0$ , modo TEM está sempre presente

## Propagação guiada – interferência de ondas

guia metálico



--- frentes de onda

$C \rightarrow$  mesmo após a reflexão

$A$  e  $C$  estão na mesma frente de onda  $\rightarrow \angle C - \angle A = \text{múltiplo de } 2\pi$

propagação ao longo de distância  $l \rightarrow$  onda adquire fase  $kl = \omega\sqrt{\mu\varepsilon}l$

em cada reflexão  $\rightarrow$  onda adquire fase  $\angle\Gamma = \pi$

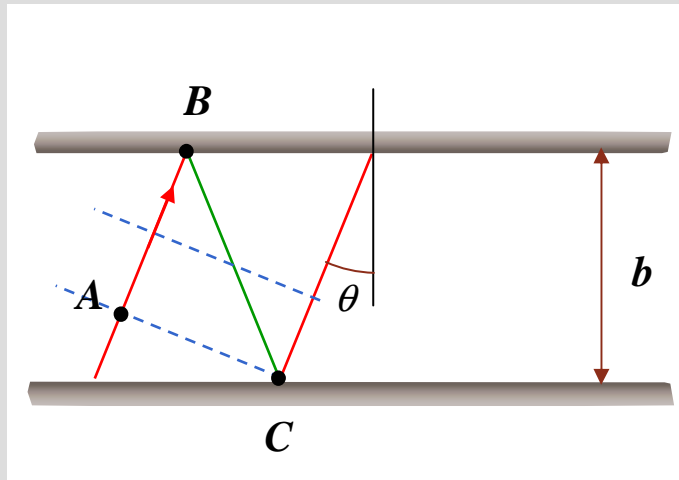
$$\Gamma = -1 \rightarrow \uparrow$$

$$k(l_{AB} + l_{BC}) = 2n\pi$$

$$\angle C = \angle A + kl_{AB} + \angle\Gamma + kl_{BC} + \angle\Gamma = \angle A + 2\pi + k(l_{AB} + l_{BC})$$

# Propagação guiada – ângulos permitidos

guia metálico



$$b = l_{BC} \cos(\theta) \longrightarrow l_{BC} = \frac{b}{\cos(\theta)}$$

$$l_{AB} = l_{BC} \cos(2\theta) = \frac{b}{\cos(\theta)} \cos(2\theta)$$

$$l_{AB} + l_{BC} = \frac{b}{\cos(\theta)} (\cos(2\theta) + 1) = 2b \cos(\theta)$$

$$k(l_{AB} + l_{BC}) = 2n\pi$$

$$2kb \cos(\theta) = 2n\pi$$

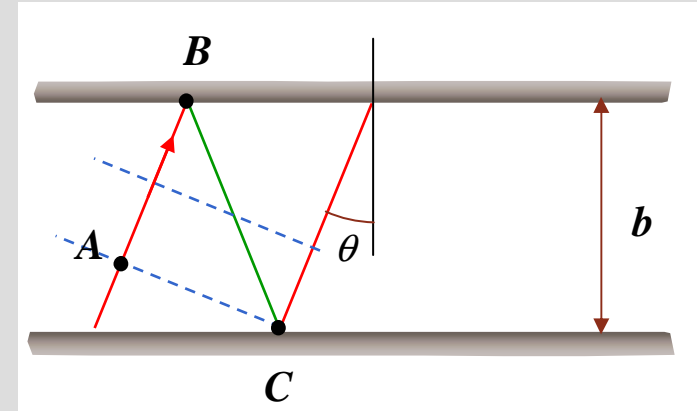
$$\cos(\theta) = \frac{n\pi}{kb} = \frac{n\pi}{b\omega\sqrt{\mu\epsilon}}$$

apenas são permitidos alguns ângulos para que haja propagação da onda ao longo do guia

## Propagação guiada – frequência de corte

$$\cos(\theta) = \frac{n\pi}{kb} = \frac{n\pi}{b\omega\sqrt{\mu\epsilon}}$$

$n$  inteiro  
 $\cos(\theta) \leq 1$



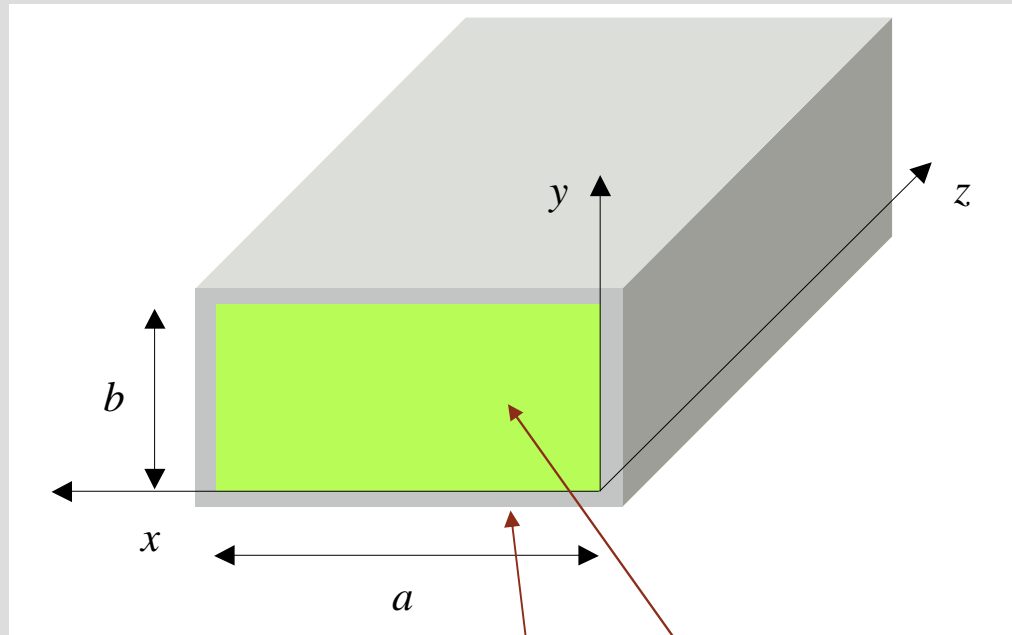
à medida que aumenta a frequência, mais ângulos são permitidos



à medida que aumenta a frequência, mais modos se podem propagar

$$\text{corte} \rightarrow \cos(\theta_{\text{corte}}) = 1 \rightarrow \frac{n\pi}{b\omega_c\sqrt{\mu\epsilon}} = 1 \rightarrow \omega_c = \frac{n\pi}{b\sqrt{\mu\epsilon}} \rightarrow f_c = \frac{n\pi}{2\pi b\sqrt{\mu\epsilon}} = \frac{h}{2\pi\sqrt{\mu\epsilon}}$$

$$h = \frac{n\pi}{b}$$



placas condutoras ideais ( $\sigma = \infty$ )

guia preenchido com material sem perdas ( $\epsilon, \mu$ )

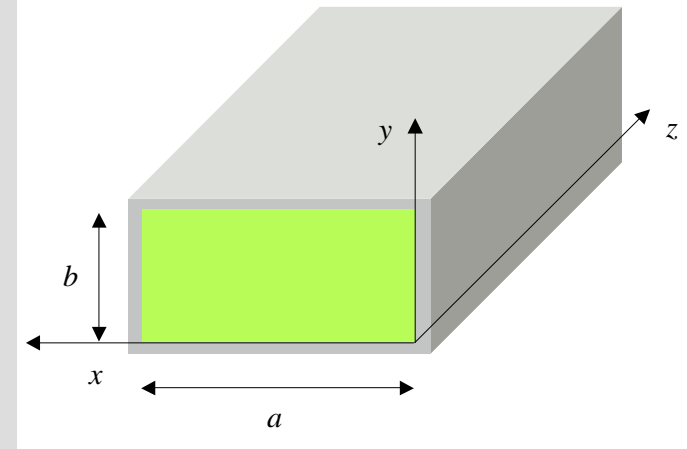
comprimento infinito  $\rightarrow$  propagação segundo  $+z$

condições fronteira

$E_{\text{tan}}$  contínuo e  $B_{\text{norm}}$  contínuo

$$E_{\text{cond}} = B_{\text{cond}} = 0$$

$$B = \mu H$$



$E_{\text{tan}} = H_{\text{norm}} = 0$  junto aos condutores

$$E_x^0 = E_z^0 = H_y^0 = 0 \quad \text{em } y=0 \text{ e } y=b$$

$$E_y^0 = E_z^0 = H_x^0 = 0 \quad \text{em } x=0 \text{ e } x=a$$



## 1. resolver

$$\nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$
$$\nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$
$$h^2 = \gamma^2 + \omega^2 \mu \epsilon$$

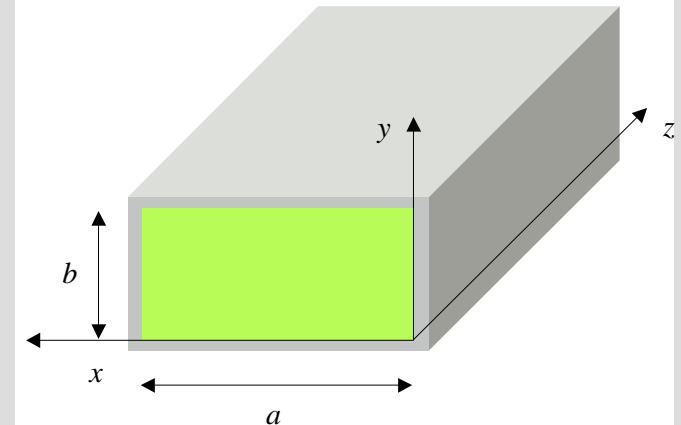
## 2. determinar (se $h \neq 0$ )

$$H_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y} \right)$$

$$H_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x} \right)$$



$$\vec{E}(x, y, z) = \vec{E}^0(x, y)e^{-\gamma z}$$
$$\vec{H}(x, y, z) = \vec{H}^0(x, y)e^{-\gamma z}$$

# Ondas TEM

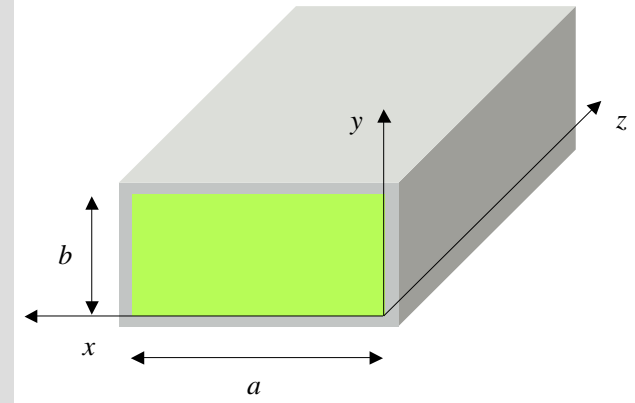
ondas TEM  $\rightarrow E_z^0 = H_z^0 = 0$

$\vec{E}$  e  $\vec{H}$  estão no plano  $xy$

$\nabla \cdot \vec{H} = 0$

linhas de  $\vec{H}$  são fechadas

$H_{\text{cond}} = 0$



linhas de  $\vec{H}$  formam percursos fechados na secção transversal do guia

$$\oint_P \vec{H} \cdot d\vec{l} = I_{\text{int}} + \epsilon \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{S}$$

$I_{\text{int}} = 0$

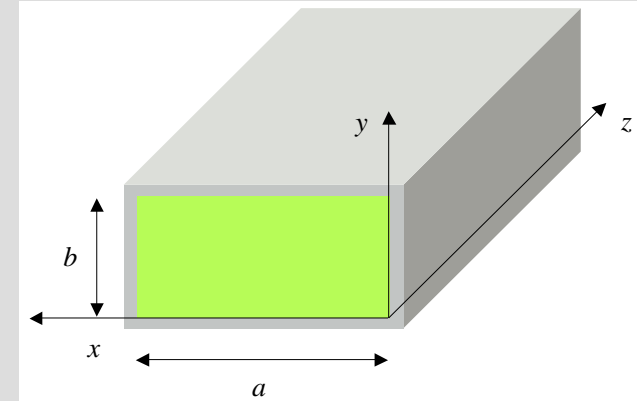
corrente no interior do guia

superfície limitada por  $P$

$$\oint_P \vec{H} \cdot d\vec{l} = I_{\text{int}} + \epsilon \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{S}$$

$I_{\text{int}} = 0$

superfície limitada por  $P$



$\vec{E}$  e  $\vec{H}$  no plano  $xy \rightarrow$  fluxo de  $\vec{E}$  através de  $S$  é nulo

$$\epsilon \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{S} = 0 \rightarrow \oint_P \vec{H} \cdot d\vec{l} = 0 \rightarrow \vec{H} = 0 \rightarrow \vec{E} = 0$$

não existem ondas TEM em guias rectangulares

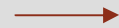
não existem ondas TEM em guias com apenas um condutor metálico

# Ondas TM e TE – determinação das componentes longitudinais

resolver

$$\nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$



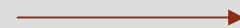
$$\nabla_{xy}^2 \psi + h^2 \psi = 0$$

$$\psi = \psi(x, y)$$

$$\nabla_{xy}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$



$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + h^2 \psi = 0$$

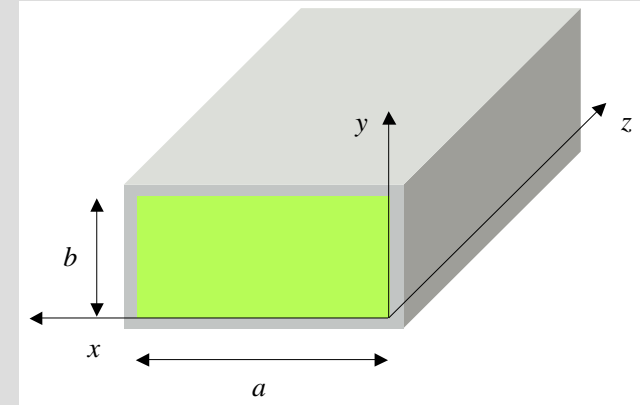


$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} + h^2 XY = 0$$

método da separação das variáveis  $\rightarrow \psi(x, y) = X(x)Y(y)$



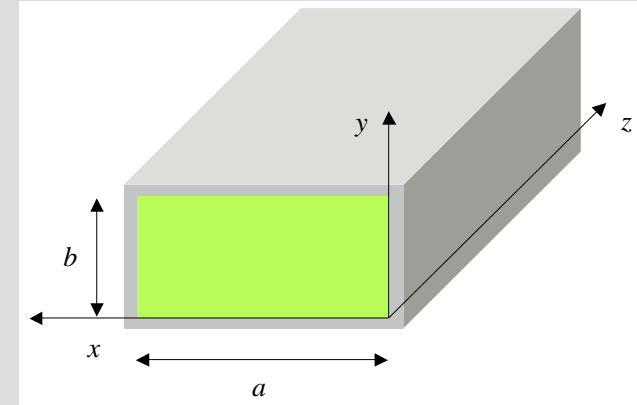
$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + h^2 = 0$$



# Método da separação das variáveis

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + h^2 = 0$$

função de  $x$       função de  $y$



→ equação anterior é satisfeita apenas quando

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \text{constante} \rightarrow \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = -k_x^2$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = \text{constante} \rightarrow \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k_y^2$$

$$\rightarrow -k_x^2 - k_y^2 + h^2 = 0 \rightarrow h^2 = k_x^2 + k_y^2$$

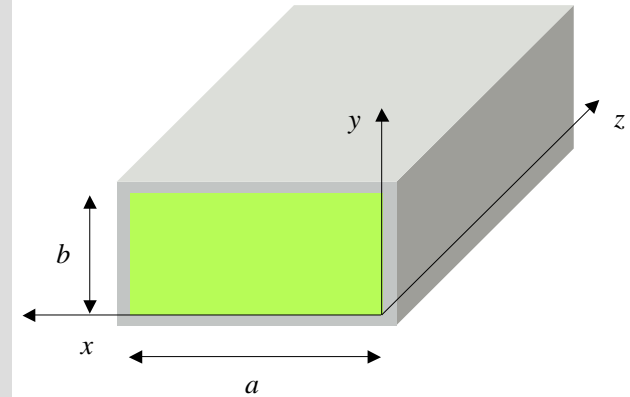
# Método da separação das variáveis

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = -k_x^2 \longrightarrow \frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0$$

$$X(x) = A \sin(k_x x) + B \cos(k_x x)$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k_y^2 \longrightarrow \frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0$$

$$Y(y) = C \sin(k_y y) + D \cos(k_y y)$$



$$\psi(x, y) = X(x)Y(y)$$

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = -k_x^2$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k_y^2$$

$$h^2 = k_x^2 + k_y^2$$

**solução geral de**  $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + h^2 \psi = 0$  **é**  $\psi(x, y) = [A \sin(k_x x) + B \cos(k_x x)][C \sin(k_y y) + D \cos(k_y y)]$

# Ondas TM – componente longitudinal

ondas TM  $\rightarrow H_z^0 = 0 \rightarrow \nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$

$$E_z^0 = [A \sin(k_x x) + B \cos(k_x x)] [C \sin(k_y y) + D \cos(k_y y)]$$

$$\underbrace{\hspace{10em}}_{X(x)} \quad \underbrace{\hspace{10em}}_{Y(y)}$$

$$E_z^0(0, y) = 0$$

$$B = 0$$

$$X(x) = A \sin(k_x x)$$

$$E_z^0(a, y) = 0$$

$$k_x = \frac{m\pi}{a}, \quad m \text{ inteiro}$$

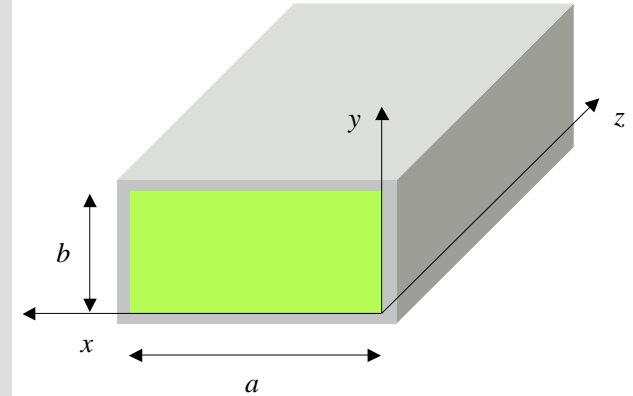
$$E_z^0(x, 0) = 0$$

$$D = 0$$

$$Y(y) = C \sin(k_y y)$$

$$E_z^0(x, b) = 0$$

$$k_y = \frac{n\pi}{b}, \quad n \text{ inteiro}$$



condições fronteira

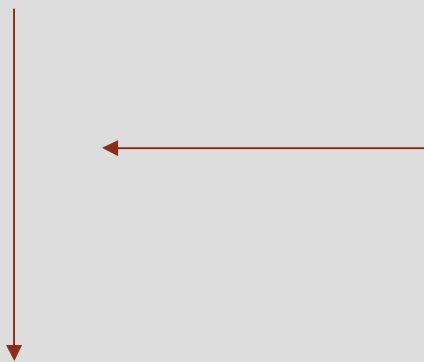
$$E_z^0 = 0 \text{ em } x=0 \text{ e } x=a$$

$$y=0 \text{ e } y=b$$

$$E_z^0 = E_{0,mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

# Ondas TM – componentes transversais

$$E_z^0 = E_{0,mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

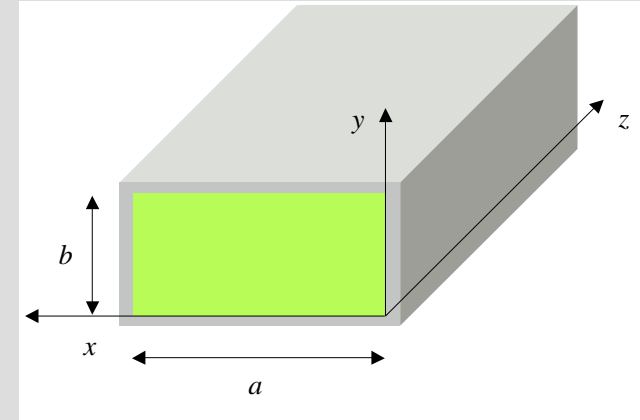


$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{\partial E_z^0}{\partial y}$$

$$H_y^0 = -\frac{j\omega\epsilon}{h^2} \frac{\partial E_z^0}{\partial x}$$

$$E_x^0 = -\frac{\gamma}{h^2} \frac{\partial E_z^0}{\partial x}$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{\partial E_z^0}{\partial y}$$



$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{n\pi}{b} E_{0,mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_y^0 = -\frac{j\omega\epsilon}{h^2} \frac{m\pi}{a} E_{0,mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_x^0 = -\frac{\gamma}{h^2} \frac{m\pi}{a} E_{0,mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{n\pi}{b} E_{0,mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$h^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

## nota

estas componentes satisfazem as condições fronteira para as componentes transversais

$$E_x^0 = H_y^0 = 0 \quad \text{em} \quad y=0 \quad \text{e} \quad y=b$$

$$E_y^0 = H_x^0 = 0 \quad \text{em} \quad x=0 \quad \text{e} \quad x=a$$



$$E_z^0 = E_{0,mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

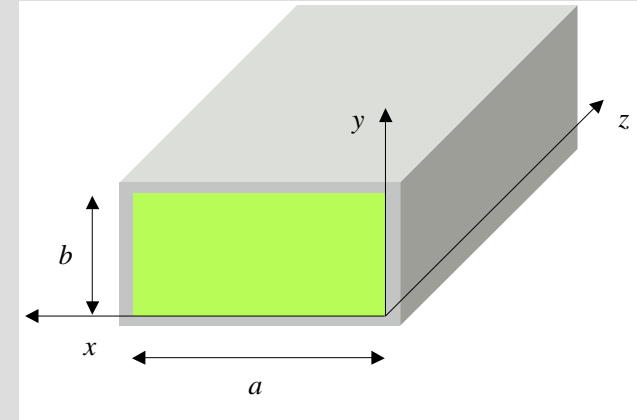
$$H_x^0 = \frac{j\omega\epsilon n\pi}{h^2 b} E_{0,mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_y^0 = -\frac{j\omega\epsilon m\pi}{h^2 a} E_{0,mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_x^0 = -\frac{\gamma m\pi}{h^2 a} E_{0,mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_y^0 = -\frac{\gamma n\pi}{h^2 b} E_{0,mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$



## notas

1.  $h \neq 0 \Rightarrow m \neq 0$  ou  $n \neq 0$
2.  $m = 0$  ou  $n = 0 \Rightarrow \vec{E} = \vec{H} = 0$



$m \geq 1 \text{ e } n \geq 1$

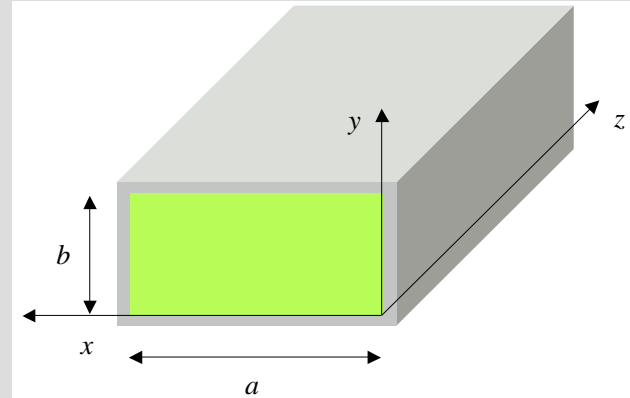
# Ondas TE – componente longitudinal

ondas TE  $\rightarrow E_z^0 = 0 \longrightarrow \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$

$$H_z^0 = [A \sin(k_x x) + B \cos(k_x x)] [C \sin(k_y y) + D \cos(k_y y)]$$

$\rightarrow$  não existem condições fronteira para aplicar a  $H_z^0$

é necessário determinar as componentes transversais para se poderem aplicar as condições fronteira



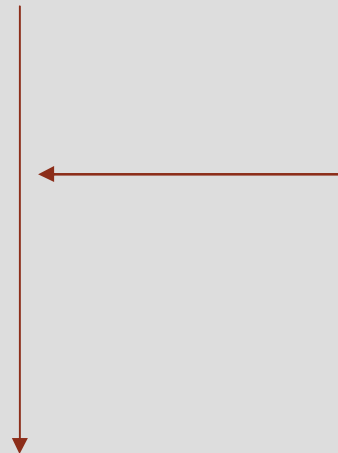
**condições fronteira**

$$E_x^0 = E_z^0 = H_y^0 = 0 \quad \text{em } y=0 \text{ e } y=b$$

$$E_y^0 = E_z^0 = H_x^0 = 0 \quad \text{em } x=0 \text{ e } x=a$$

# Ondas TE – componentes transversais

$$H_z^0 = [A \sin(k_x x) + B \cos(k_x x)] [C \sin(k_y y) + D \cos(k_y y)]$$

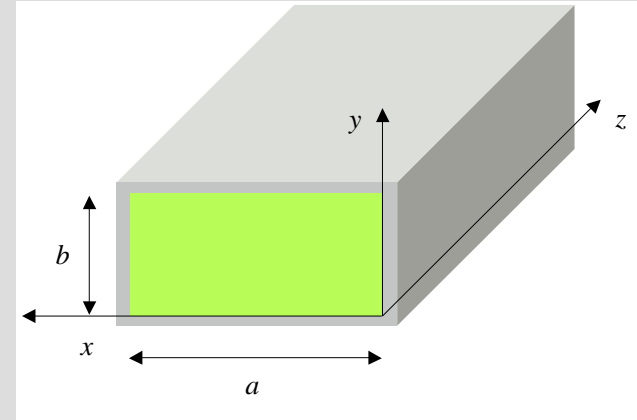


$$H_x^0 = -\frac{\gamma}{h^2} \frac{\partial H_z^0}{\partial x}$$

$$H_y^0 = -\frac{\gamma}{h^2} \frac{\partial H_z^0}{\partial y}$$

$$E_x^0 = -\frac{j\omega\mu}{h^2} \frac{\partial H_z^0}{\partial y}$$

$$E_y^0 = \frac{j\omega\mu}{h^2} \frac{\partial H_z^0}{\partial x}$$



$$H_x^0 = -\frac{\gamma k_x}{h^2} [A \cos(k_x x) - B \sin(k_x x)] [C \sin(k_y y) + D \cos(k_y y)]$$

$$H_y^0 = -\frac{\gamma k_y}{h^2} [A \sin(k_x x) + B \cos(k_x x)] [C \cos(k_y y) - D \sin(k_y y)]$$

$$E_x^0 = -\frac{j\omega\mu k_y}{h^2} [A \sin(k_x x) + B \cos(k_x x)] [C \cos(k_y y) - D \sin(k_y y)]$$

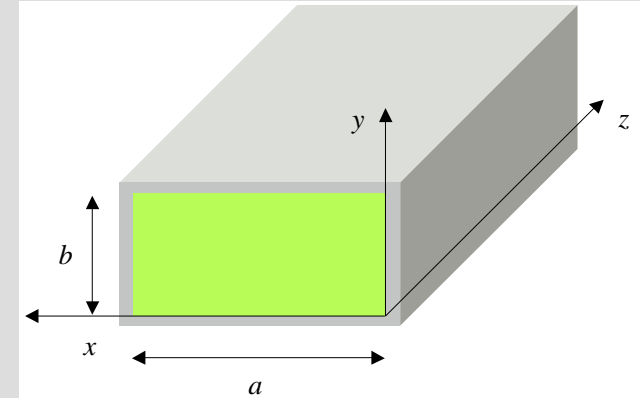
$$E_y^0 = \frac{j\omega\mu k_x}{h^2} [A \cos(k_x x) - B \sin(k_x x)] [C \sin(k_y y) + D \cos(k_y y)]$$

# Ondas TE – condições fronteira

**condições fronteira**

$$E_y^0 = H_x^0 = 0 \quad \text{em } x=0 \quad \text{e } x=a$$

$$E_x^0 = H_y^0 = 0 \quad \text{em } y=0 \quad \text{e } y=b$$



$$H_x^0 = -\frac{\gamma k_x}{h^2} [A \cos(k_x x) - B \sin(k_x x)] [C \sin(k_y y) + D \cos(k_y y)]$$

$$H_y^0 = -\frac{\gamma k_y}{h^2} [A \sin(k_x x) + B \cos(k_x x)] [C \cos(k_y y) - D \sin(k_y y)]$$

$$E_x^0 = -\frac{j\omega\mu k_y}{h^2} [A \sin(k_x x) + B \cos(k_x x)] [C \cos(k_y y) - D \sin(k_y y)]$$

$$E_y^0 = \frac{j\omega\mu k_x}{h^2} [A \cos(k_x x) - B \sin(k_x x)] [C \sin(k_y y) + D \cos(k_y y)]$$

$$H_y^0(y=0) = 0$$

$$H_y^0(y=b) = 0$$

$$\rightarrow C = 0 \quad \text{e } k_y = \frac{n\pi}{a}, \quad n \text{ inteiro}$$

$$E_y^0(x=0) = 0$$

$$E_y^0(x=a) = 0$$

$$\rightarrow A = 0 \quad \text{e } k_x = \frac{m\pi}{a}, \quad m \text{ inteiro}$$

$$H_z^0 = H_{0,mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

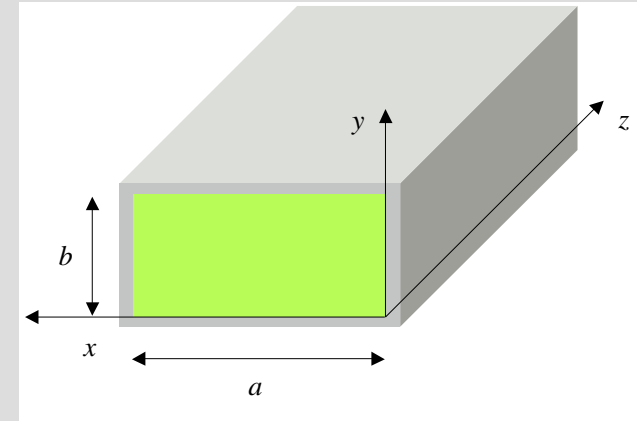
$$H_x^0 = \frac{\gamma}{h^2} \frac{m\pi}{a} H_{0,mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_y^0 = \frac{\gamma}{h^2} \frac{n\pi}{b} H_{0,mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_x^0 = \frac{j\omega\mu}{h^2} \frac{n\pi}{b} H_{0,mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_y^0 = -\frac{j\omega\mu}{h^2} \frac{m\pi}{a} H_{0,mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$



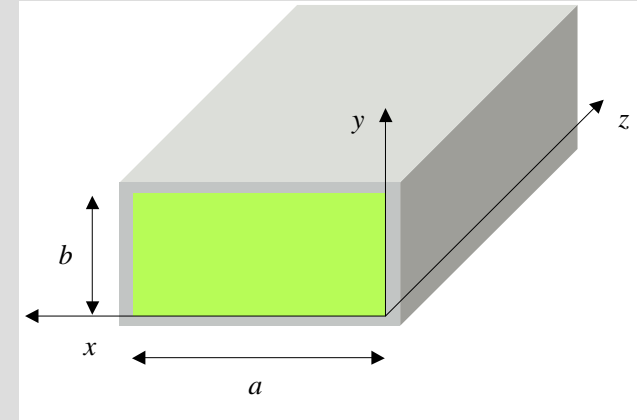
### nota

1.  $h \neq 0 \Rightarrow m \neq 0$  ou  $n \neq 0$
2. é possível ter  $m = 0$  ou  $n = 0$

# Guias rectangulares – frequência de corte

$$f_c = \frac{h}{2\pi\sqrt{\mu\epsilon}}$$

$$h_{TM,TE} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \longrightarrow f_c = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$



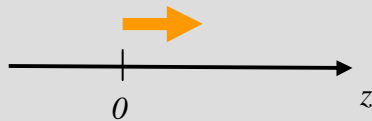
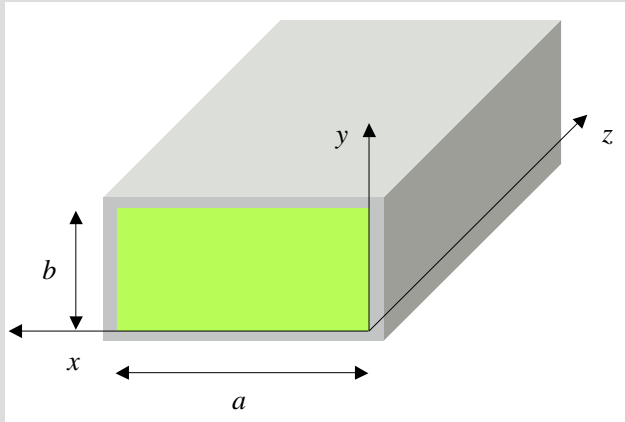
modos  $TM_{mn} \rightarrow m \geq 1$  e  $n \geq 1 \longrightarrow$  modo  $TM_{mn}$  dominante é o modo  $TM_{11}$

modos  $TE_{mn} \rightarrow m \neq 0$  ou  $n \neq 0 \longrightarrow$  se  $a > b$  modo  $TE_{mn}$  dominante é o modo  $TE_{10}$

guias rectangulares  $\rightarrow$  modo dominante é o modo  $TE_{10}$

$$(f_c)_{TE_{10}} < (f_c)_{TM_{11}}$$

**guias rectangulares**

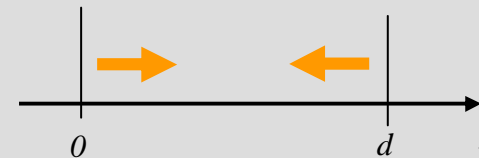
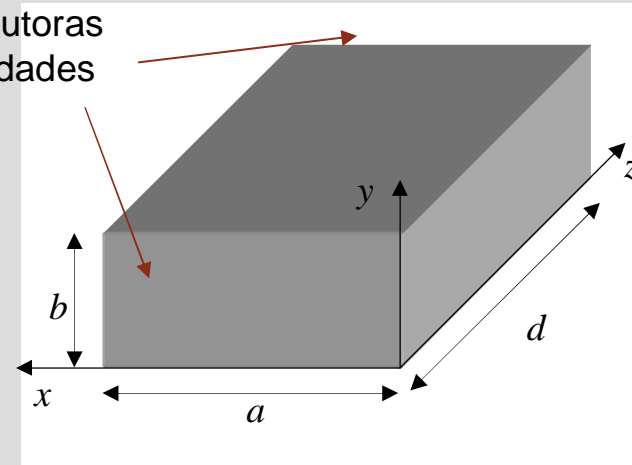


$$\vec{E}(x, y, z) = \vec{E}^0(x, y)e^{-j\beta z}$$

$$\vec{H}(x, y, z) = \vec{H}^0(x, y)e^{-j\beta z}$$

**cavidades rectangulares**

placas condutoras  
nas extremidades



$$\vec{E}(x, y, z) = \vec{E}^{0,+}(x, y)e^{-j\beta z} + \vec{E}^{0,-}(x, y)e^{j\beta z}$$

$$\vec{H}(x, y, z) = \vec{H}^{0,+}(x, y)e^{-j\beta z} + \vec{H}^{0,-}(x, y)e^{j\beta z}$$

# Cavidades rectangulares

nas expressões de

$$\begin{matrix} \vec{E}^0(x, y) \\ \vec{H}^0(x, y) \end{matrix}$$

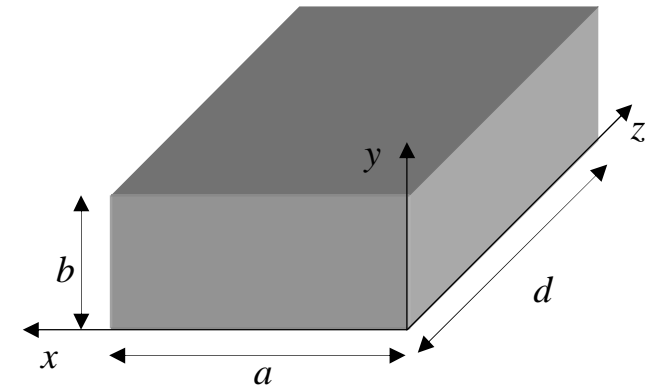
(guias rectangulares)

$$\gamma = j\beta$$

$$\gamma = -j\beta$$

$$\begin{matrix} \vec{E}^{0,+}(x, y) \\ \vec{H}^{0,+}(x, y) \end{matrix}$$

$$\begin{matrix} \vec{E}^{0,-}(x, y) \\ \vec{H}^{0,-}(x, y) \end{matrix}$$



$$\begin{aligned} \vec{E}(x, y, z) &= \vec{E}^{0,+}(x, y)e^{-j\beta z} + \vec{E}^{0,-}(x, y)e^{j\beta z} \\ \vec{H}(x, y, z) &= \vec{H}^{0,+}(x, y)e^{-j\beta z} + \vec{H}^{0,-}(x, y)e^{j\beta z} \end{aligned}$$

**condições fronteira**  $\rightarrow E_{\text{tan}} = H_{\text{normal}} = 0$  em  $z=0$  e  $z=d$

$$E_x = E_y = H_z = 0 \text{ em } z=0 \text{ e } z=d$$



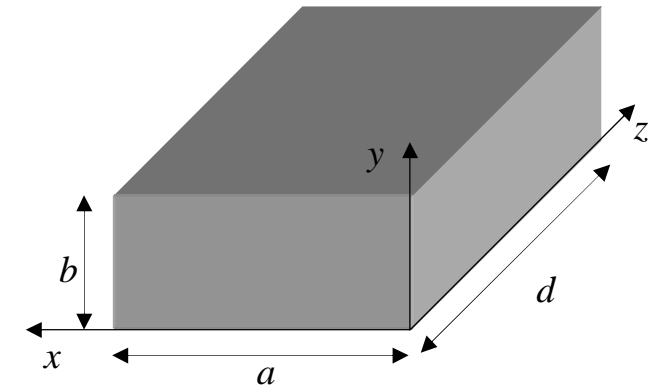
$$E_z = (E_{0,mn}^+ e^{-j\beta z} + E_{0,mn}^- e^{j\beta z}) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_x = -\frac{j\beta}{h^2} (E_{0,mn}^+ e^{-j\beta z} - E_{0,mn}^- e^{j\beta z}) \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_y = -\frac{j\beta}{h^2} (E_{0,mn}^+ e^{-j\beta z} - E_{0,mn}^- e^{j\beta z}) \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_x = \frac{j\omega\epsilon}{h^2} (E_{0,mn}^+ e^{-j\beta z} + E_{0,mn}^- e^{j\beta z}) \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_y = -\frac{j\omega\epsilon}{h^2} (E_{0,mn}^+ e^{-j\beta z} + E_{0,mn}^- e^{j\beta z}) \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$



$$\vec{E}(x, y, z) = \vec{E}^{0,+}(x, y) e^{-j\beta z} + \vec{E}^{0,-}(x, y) e^{j\beta z}$$

$$\vec{H}(x, y, z) = \vec{H}^{0,+}(x, y) e^{-j\beta z} + \vec{H}^{0,-}(x, y) e^{j\beta z}$$

$$E_x = E_y = H_z = 0 \quad \text{em } z=0 \text{ e } z=d$$

$$E_{0,mn}^+ = E_{0,mn}^- = E_0/2$$

$$\beta = \frac{p\pi}{d}, \quad p \text{ inteiro}$$

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right)$$

$$E_x = -\frac{1}{h^2} E_0 \left(\frac{m\pi}{a}\right) \left(\frac{p\pi}{d}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{d}\right)$$

$$E_y = -\frac{1}{h^2} E_0 \left(\frac{n\pi}{b}\right) \left(\frac{p\pi}{d}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{d}\right)$$

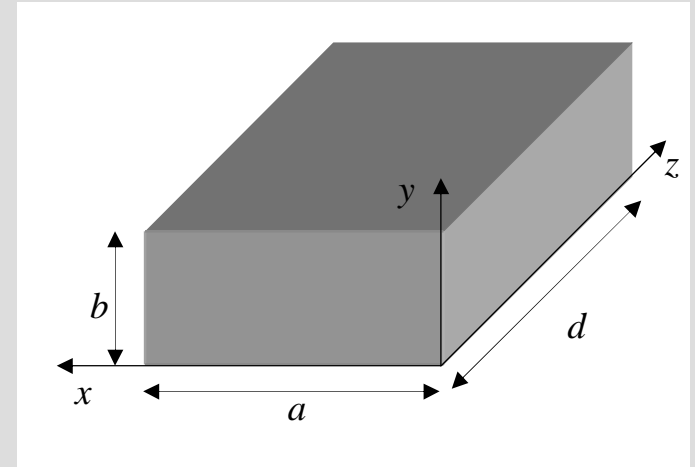
$$H_x = \frac{j\omega\epsilon}{h^2} E_0 \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right)$$

$$H_y = -\frac{j\omega\epsilon}{h^2} E_0 \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right)$$

$$h = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$m \geq 1 \text{ e } n \geq 1$$

$$p \geq 0$$



# Ondas TM – frequência de ressonância

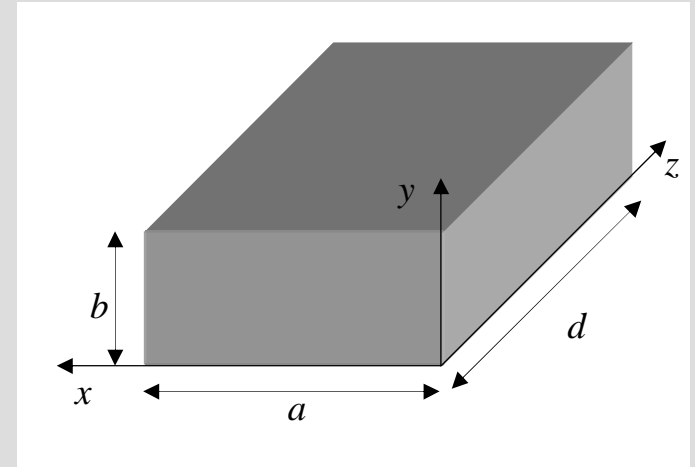
$$h^2 = \gamma^2 + \omega^2 \mu \epsilon \Leftrightarrow \omega = \sqrt{h^2 - \gamma^2} = \sqrt{h^2 + \beta^2}$$

$$h^2 = (m\pi/a)^2 + (n\pi/b)^2$$

$$\beta = p\pi/d$$

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

$$f_{mnp} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$



**modo dominante**

$$m \geq 1 \text{ e } n \geq 1 \\ p \geq 0$$

→ **modo TM<sub>110</sub>**

$$f_{110} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

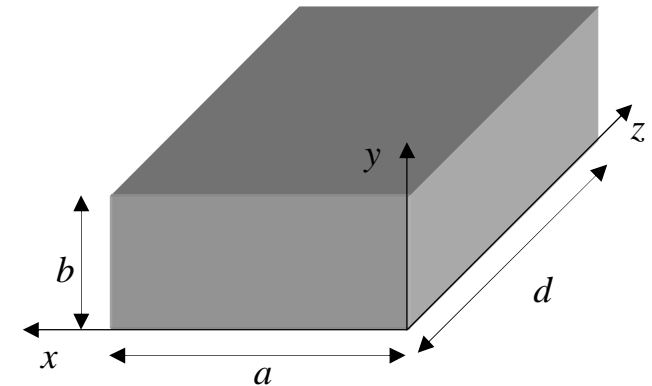
$$H_z = (H_{0,mn}^+ e^{-j\beta z} + H_{0,mn}^- e^{j\beta z}) \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$E_x = \frac{j\omega\mu}{h^2} (H_{0,mn}^+ e^{-j\beta z} + H_{0,mn}^- e^{j\beta z}) \left(\frac{n\pi}{b}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_y = -\frac{j\omega\mu}{h^2} (H_{0,mn}^+ e^{-j\beta z} + H_{0,mn}^- e^{j\beta z}) \left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_x = \frac{j\beta}{h^2} (H_{0,mn}^+ e^{-j\beta z} - H_{0,mn}^- e^{j\beta z}) \left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_y = \frac{j\beta}{h^2} (H_{0,mn}^+ e^{-j\beta z} - H_{0,mn}^- e^{j\beta z}) \left(\frac{n\pi}{b}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$



$$E_x = E_y = H_z = 0 \quad \text{em } z=0 \text{ e } z=d$$

$$H_{0,mn}^- = -H_{0,mn}^+ = H_0/2j$$

$$\beta = \frac{p\pi}{d}, \quad p \text{ inteiro}$$

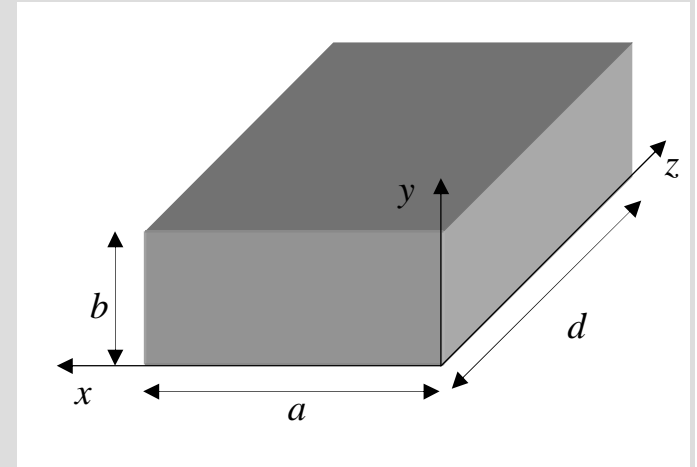
## Modo TE<sub>mnp</sub>

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{d}\right)$$
$$E_x = \frac{j\omega\mu}{h^2} H_0 \left(\frac{n\pi}{b}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{d}\right)$$
$$E_y = -\frac{j\omega\mu}{h^2} H_0 \left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{d}\right)$$
$$H_x = -\frac{1}{h^2} H_0 \left(\frac{m\pi}{a}\right) \left(\frac{p\pi}{d}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right)$$
$$H_y = -\frac{1}{h^2} H_0 \left(\frac{n\pi}{b}\right) \left(\frac{p\pi}{d}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right)$$

$$h = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

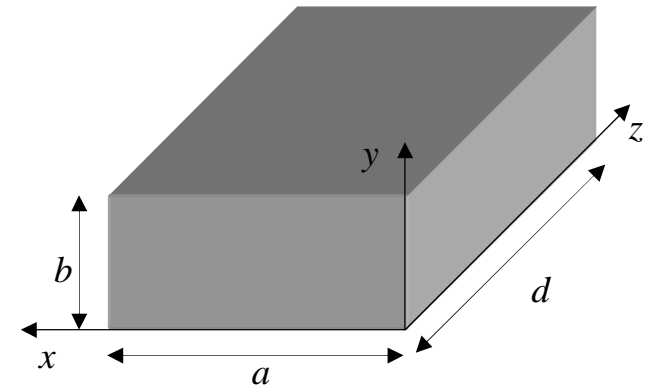
$$m \neq 0 \text{ ou } n \neq 0$$

$$p \geq 1$$



# Ondas TE – frequência de ressonância

$$f_{mnp} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$



## modo dominante

$$m + n \neq 0$$

$$p \geq 1$$

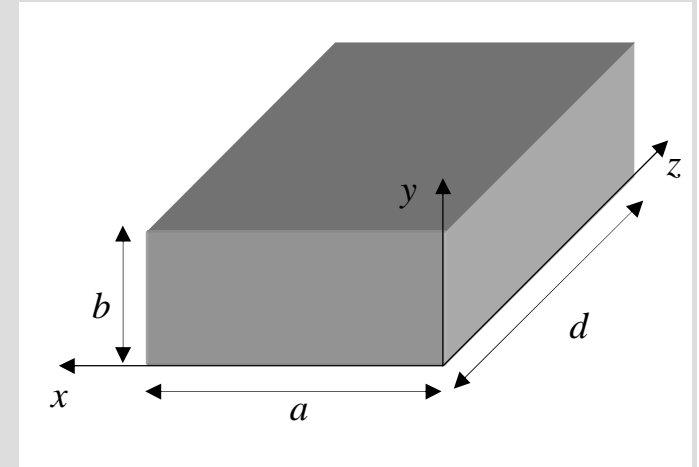
$$a > b$$

→ modo TE<sub>101</sub>

$$f_{101} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\frac{1}{a^2} + \frac{1}{d^2}}$$

# Ondas TE – frequência de ressonância

modos	$m = 0$	$n = 0$	$p = 0$	frequência de ressonância mais baixa
<b>TM</b>	não	não	sim	$f_{110} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$
<b>TE</b>	sim	não	não	$( f_{011} > f_{101} \text{ para } a > b )$
	não	sim		$f_{101} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\frac{1}{a^2} + \frac{1}{d^2}}$



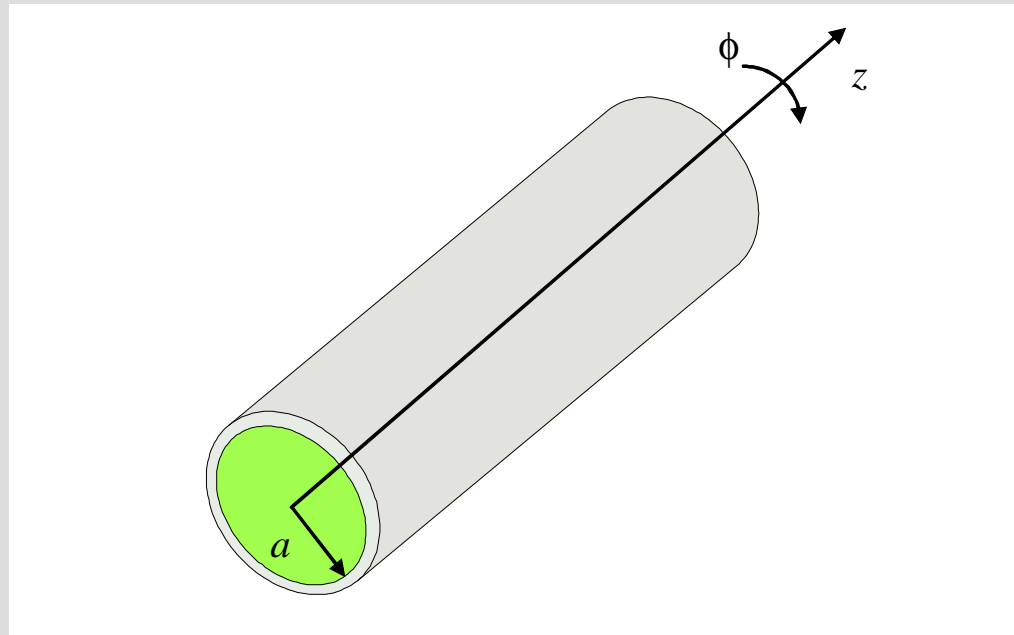
### modo dominante

modo  $TM_{110}$  se  $b > d$

modo  $TE_{101}$  se  $d > b$

se  $b = d \implies f_{101} = f_{110} \implies$

modos  $TM_{110}$  e  $TE_{101}$  são  
**modos degenerados**



superfície condutora ideal ( $\sigma = \infty$ )

guia preenchido com material sem perdas ( $\epsilon, \mu$ )

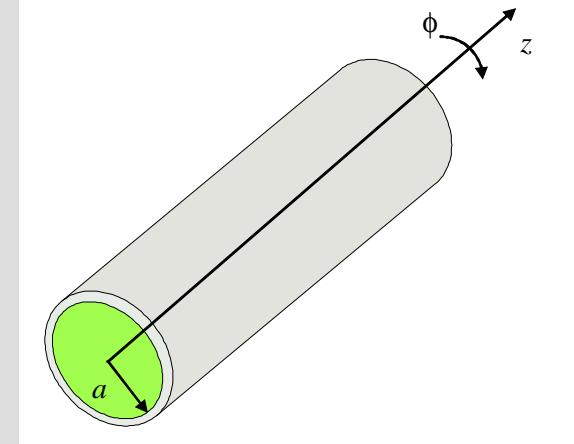
comprimento infinito  $\rightarrow$  propagação segundo  $+z$



# Guias circulares – condições fronteira

coordenadas cilíndricas →

$$\vec{E} = (E_r^0 \hat{r} + E_\phi^0 \hat{\phi} + E_z^0 \hat{z}) e^{-\gamma z}$$
$$\vec{H} = (H_r^0 \hat{r} + H_\phi^0 \hat{\phi} + H_z^0 \hat{z}) e^{-\gamma z}$$



condições fronteira

$$E_{\text{tan}} \text{ contínuo e } B_{\text{norm}} \text{ contínuo}$$
$$E_{\text{cond}} = B_{\text{cond}} = 0$$
$$B = \mu H$$

→  $E_{\text{tan}} = H_{\text{norm}} = 0$  junto ao condutor

$$E_\phi^0 = E_z^0 = H_r^0 = 0 \text{ em } r = a$$

## 1. resolver

$$\begin{aligned} \nabla_{r\phi}^2 E_z^0 + h^2 E_z^0 &= 0 \\ \nabla_{r\phi}^2 H_z^0 + h^2 H_z^0 &= 0 \end{aligned}$$

$$\nabla_{r\phi}^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

$$h^2 = \gamma^2 + \omega^2 \mu \epsilon$$

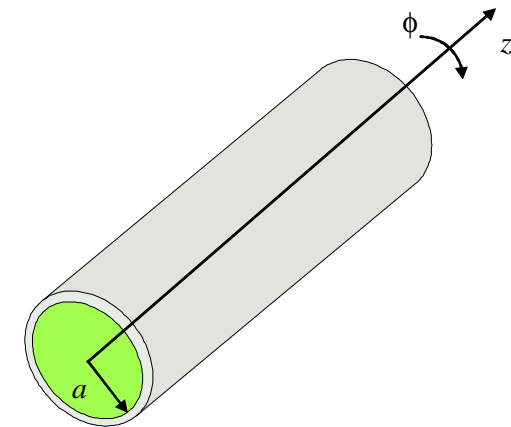
## 2. determinar (se $h \neq 0$ )

$$H_r^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial r} - \frac{j\omega\epsilon}{r} \frac{\partial E_z^0}{\partial \phi} \right)$$

$$H_\phi^0 = -\frac{1}{h^2} \left( \frac{\gamma}{r} \frac{\partial H_z^0}{\partial \phi} + j\omega\epsilon \frac{\partial E_z^0}{\partial r} \right)$$

$$E_r^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial r} + \frac{j\omega\mu}{r} \frac{\partial H_z^0}{\partial \phi} \right)$$

$$E_\phi^0 = -\frac{1}{h^2} \left( \frac{\gamma}{r} \frac{\partial E_z^0}{\partial \phi} - j\omega\mu \frac{\partial H_z^0}{\partial r} \right)$$



$$\begin{aligned} \vec{E}(r, \phi, z) &= \vec{E}^0(r, \phi) e^{-\gamma z} \\ \vec{H}(r, \phi, z) &= \vec{H}^0(r, \phi) e^{-\gamma z} \end{aligned}$$

**Nota**  
não se propagam ondas TEM  
em guias circulares

# Ondas TM e TE – determinação das componentes longitudinais

resolver

$$\nabla_{r\phi}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\nabla_{r\phi}^2 H_z^0 + h^2 H_z^0 = 0$$

$$\longrightarrow \nabla_{r\phi}^2 \psi + h^2 \psi = 0$$

$$\psi = \psi(r, \phi)$$

$$\nabla_{r\phi}^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

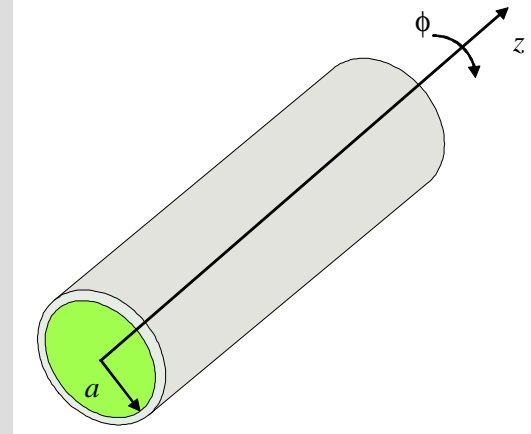
$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + h^2 \psi = 0$$

$$\Phi(\phi) \frac{d^2 R(r)}{dr^2} + \frac{\Phi(\phi)}{r} \frac{dR(r)}{dr} + \frac{R(r)}{r^2} \frac{d^2 \Phi(\phi)}{d\phi^2} + h^2 R(r) \Phi(\phi) = 0$$

método da separação  
das variáveis

$$\psi(r, \phi) = R(r) \Phi(\phi)$$

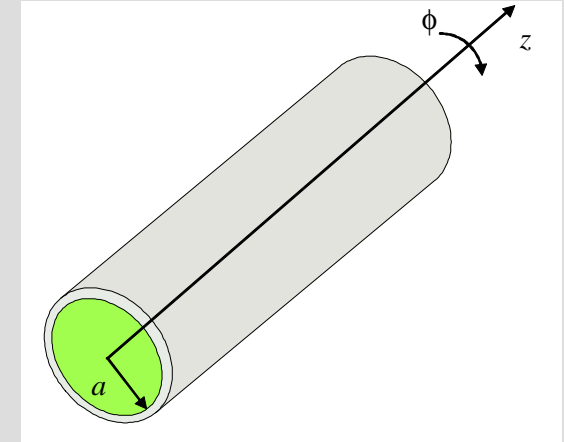
$$\frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{r}{R(r)} \frac{dR(r)}{dr} + h^2 r^2 = - \frac{1}{\Phi(\phi)} \frac{d^2 \Phi(\phi)}{d\phi^2}$$



$$\frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{r}{R(r)} \frac{dR(r)}{dr} + h^2 r^2 = - \frac{1}{\Phi(\phi)} \frac{d^2 \Phi(\phi)}{d\phi^2}$$

função de  $r$

função de  $\phi$



→ equação anterior é satisfeita apenas quando

$$\frac{1}{\Phi(\phi)} \frac{d^2 \Phi(\phi)}{d\phi^2} = \text{constante} \longrightarrow \frac{1}{\Phi(\phi)} \frac{d^2 \Phi(\phi)}{d\phi^2} = -k_\phi^2$$

$$\frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{r}{R(r)} \frac{dR(r)}{dr} + h^2 r^2 = \text{constante} \longrightarrow \frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{r}{R(r)} \frac{dR(r)}{dr} + h^2 r^2 = k_\phi^2$$

# Método da separação das variáveis

$$\frac{1}{\Phi(\phi)} \frac{d^2\Phi(\phi)}{d\phi^2} = -k_\phi^2 \longrightarrow \frac{d^2\Phi(\phi)}{d\phi^2} + k_\phi^2\Phi(\phi) = 0$$

$$\Phi(\phi) = A \sin(k_\phi \phi) + B \cos(k_\phi \phi)$$

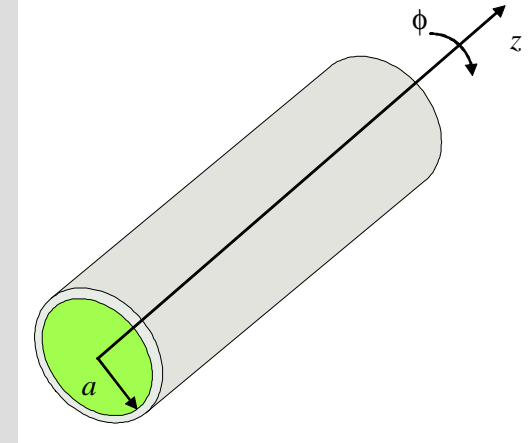
$$\Phi(\phi + 2\pi) = \Phi(\phi)$$

$$\begin{aligned} \sin(k_\phi \phi + k_\phi 2\pi) &= \sin(k_\phi \phi) \\ \cos(k_\phi \phi + k_\phi 2\pi) &= \cos(k_\phi \phi) \end{aligned}$$

$$k_\phi = n, \quad n \text{ inteiro}$$

$$\Phi(\phi) = A \sin(n\phi) + B \cos(n\phi)$$

$$\Phi(\phi) = B \cos(n\phi)$$



# Equação diferencial de Bessel

$$\frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{r}{R(r)} \frac{dR(r)}{dr} + h^2 r^2 = k_\phi^2$$

←  $k_\phi = n, n$  inteiro

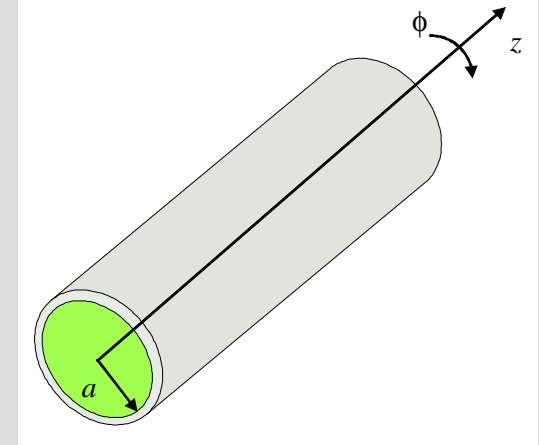
$$\frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{r}{R(r)} \frac{dR(r)}{dr} + h^2 r^2 = n^2$$

$$r^2 \frac{d^2 R(r)}{dr^2} + r \frac{dR(r)}{dr} + (h^2 r^2 - n^2) R(r) = 0$$

equação diferencial de Bessel

solução geral:  $R(r) = C \boxed{J_n(hr)} + D \boxed{N_n(hr)}$  funções de Bessel de 2ª espécie

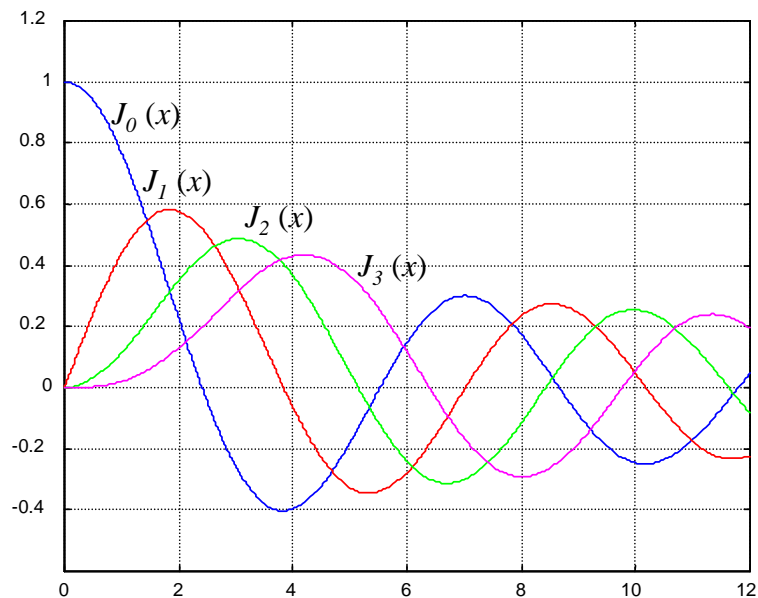
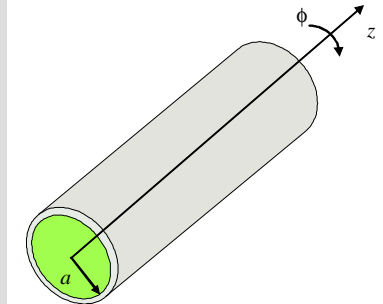
funções de Bessel de 1ª espécie



# Funções de Bessel de 1ª espécie

para  $n$  inteiro

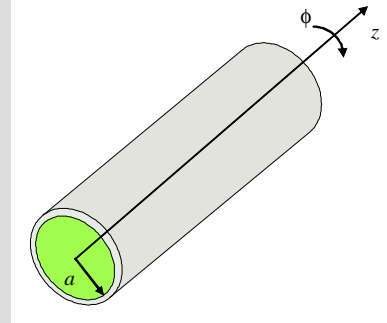
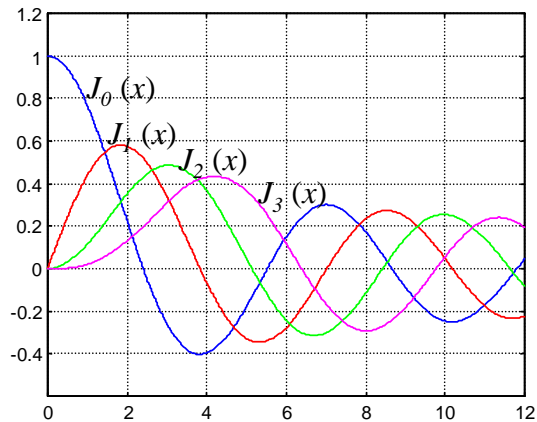
$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{n+2m}}{m!(m+n)!2^{n+2m}}$$



notas

1.  $n \neq 0 \Rightarrow J_n(0) = 0$   
 $n = 0 \Rightarrow J_n(0) = 1$
2. funções oscilatórias,  
de amplitude decrescente  
e com **zeros não periódicos**

# Zeros das funções de Bessel de 1ª espécie



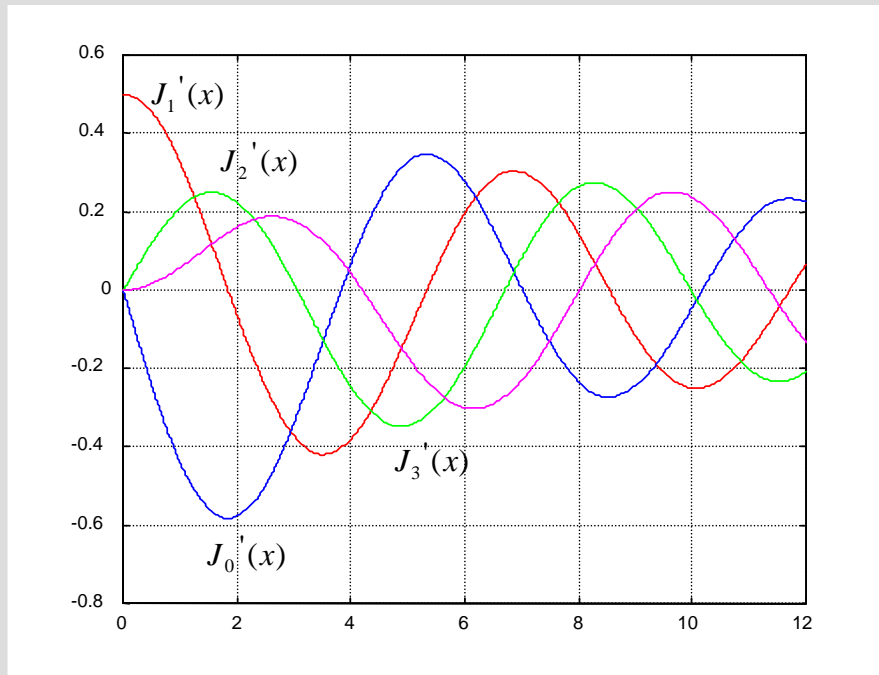
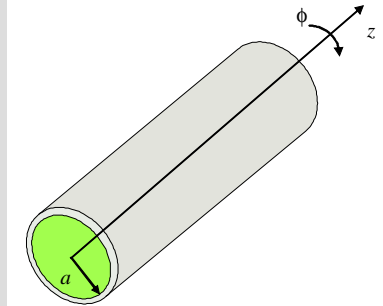
## zeros das funções de Bessel de 1ª espécie

zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$
1	2.4048	3.8317	5.1336	6.3802
2	5.5201	7.0156	8.4172	9.7610
3	8.6537	10.1735	11.6198	13.0152
4	11.7915	13.3237	14.7960	16.2235
5	14.9309	16.4706	17.9598	19.4094



# Derivadas das funções de Bessel de 1ª espécie

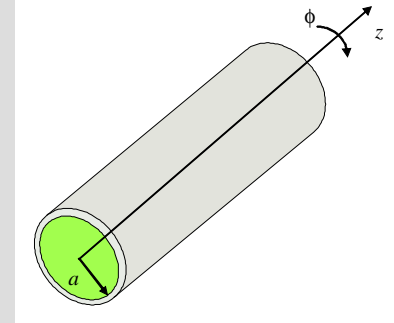
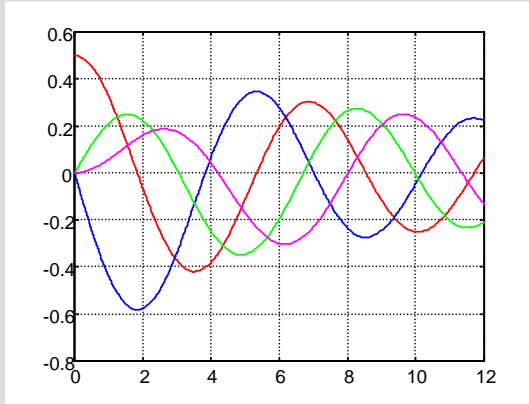
$$J'_n(x) = \frac{1}{2} [J_{n-1}(x) - J_{n+1}(x)]$$



## nota

funções oscilatórias,  
de amplitude decrescente  
e com **zeros não periódicos**

# Zeros das derivadas das funções de Bessel de 1ª espécie

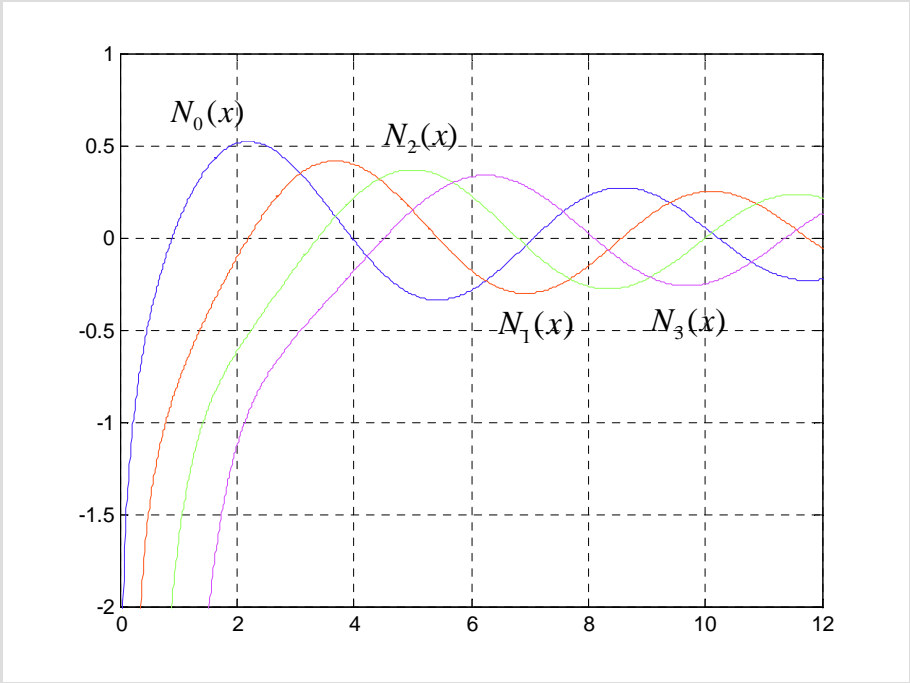
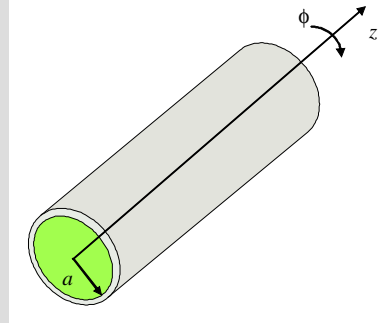


## zeros das derivadas das funções de Bessel de 1ª espécie

zero	$J'_0(x)$	$J'_1(x)$	$J'_2(x)$	$J'_3(x)$
1	3.8317	1.8412	3.0542	4.2012
2	7.0156	5.3314	6.7061	8.0152
3	10.1735	8.5363	9.9695	11.3459
4	13.3237	11.7060	13.1704	14.5858
5	16.4706	14.8636	16.3475	17.7887

# Funções de Bessel de 2ª espécie

$$N_n(x) = \lim_{p \rightarrow n} \frac{J_p(x) \cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}$$



## notas

1. funções oscilatórias, de amplitude decrescente e com zeros não periódicos

2. tomam valores infinitos quando  $x=0$

$$R(r) = C J_n(hr) + D ~~N_n(hr)~~$$

$$D = 0$$

quando região de interesse incluir a origem

# Guias circulares – solução da equação de onda

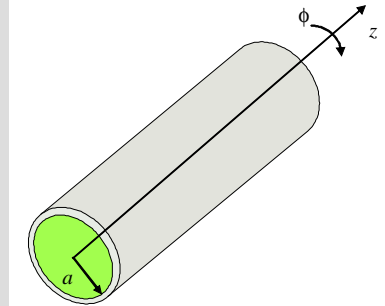
$$\nabla_{r\phi}^2 \psi + h^2 \psi = 0$$

$$\psi(r, \phi) = R(r)\Phi(\phi)$$

$$\Phi(\phi) = B \cos(n\phi)$$

$$R(r) = C J_n(hr)$$

$$\psi(r, \phi) = R(r)\Phi(\phi) = C_n J_n(hr) \cos(n\phi)$$



**nota**

**ondas TM**  $\rightarrow E_z^0 = \psi(r, \phi)$

**ondas TE**  $\rightarrow H_z^0 = \psi(r, \phi)$

# Modo $TM_{np}$ – componente longitudinal

ondas TM  $\rightarrow H_z^0 = 0$  e  $E_z^0 = C_n J_n(hr) \cos(n\phi)$

$E_z^0(r = a, \phi) = 0$

$J_n(ha) = 0$

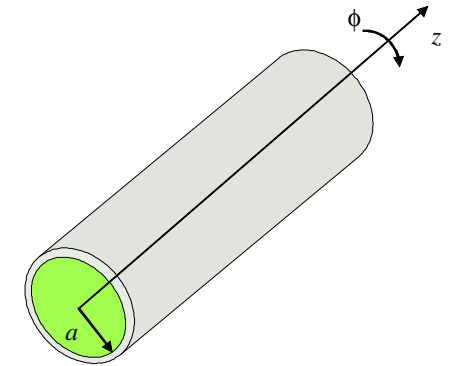
$n=0 \rightarrow h = \frac{2.4048}{a}; h = \frac{5.5201}{a}; h = \frac{8.6537}{a}; \dots$

$n=1 \rightarrow h = \frac{3.8317}{a}; h = \frac{7.0156}{a}; h = \frac{10.1735}{a}; \dots$

$\vdots$



$h = h_{TM_{np}} = \frac{p - \text{ésimo zero de } J_n}{a}$



**condições fronteira**

$E_\phi^0 = E_z^0 = H_r^0 = 0$  em  $r = a$

zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$
1	2.4048	3.8317	5.1336	6.3802
2	5.5201	7.0156	8.4172	9.7610
3	8.6537	10.1735	11.6198	13.0152
4	11.7915	13.3237	14.7960	16.2235
5	14.9309	16.4706	17.9598	19.4094

# Modos $TM_{np}$ – componentes transversais

$$E_z^0 = C_n J_n(hr) \cos(n\phi)$$

$$H_r^0 = \frac{j\omega\epsilon}{h^2 r} \frac{\partial E_z^0}{\partial \phi}$$

$$H_\phi^0 = -\frac{j\omega\epsilon}{h^2} \frac{\partial E_z^0}{\partial r}$$

$$E_r^0 = -\frac{\gamma}{h^2} \frac{\partial E_z^0}{\partial r}$$

$$E_\phi^0 = -\frac{\gamma}{h^2 r} \frac{\partial E_z^0}{\partial \phi}$$

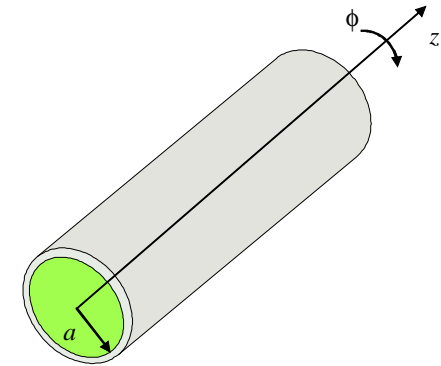
$$H_r^0 = -\frac{j\omega\epsilon n}{h^2 r} C_n J_n(hr) \sin(n\phi)$$

$$H_\phi^0 = -\frac{j\omega\epsilon}{h} C_n J'_n(hr) \cos(n\phi)$$

$$E_r^0 = -\frac{\gamma}{h} C_n J'_n(hr) \cos(n\phi)$$

$$E_\phi^0 = \frac{\gamma n}{h^2 r} C_n J_n(hr) \sin(n\phi)$$

$$h = h_{TM_{np}} = \frac{p - \text{ésimo zero de } J_n}{a}$$



**condições fronteira**

$$E_\phi^0 = E_z^0 = H_r^0 = 0 \text{ em } r = a$$

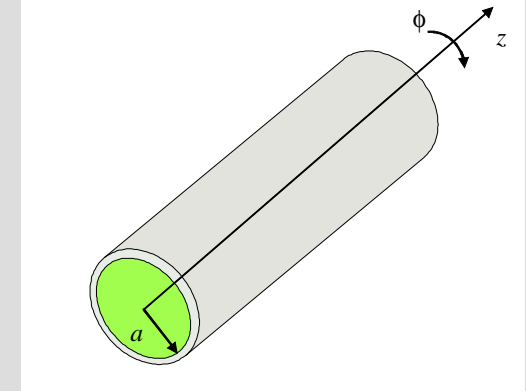
**nota**

estas componentes satisfazem as condições fronteira para as componentes transversais

# Modos $TM_{np}$ – frequência de corte

$$(f_c)_{TM_{np}} = \frac{h_{TM_{np}}}{2\pi\sqrt{\mu\epsilon}} = \frac{p - \text{ésimo zero de } J_n}{2\pi a\sqrt{\mu\epsilon}}$$

$$h_{TM_{np}} = \frac{p - \text{ésimo zero de } J_n}{a}$$



menor zero de  $J_n \rightarrow 2.4048$  ( $n=0, p=1$ )

modo TM dominante  $\rightarrow$  modo  $TM_{01}$

$$(f_c)_{TM_{01}} = \frac{2.4048}{2\pi a\sqrt{\mu\epsilon}}$$

zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$
1	2.4048	3.8317	5.1336	6.3802
2	5.5201	7.0156	8.4172	9.7610
3	8.6537	10.1735	11.6198	13.0152
4	11.7915	13.3237	14.7960	16.2235
5	14.9309	16.4706	17.9598	19.4094

# Modo TE<sub>np</sub>

ondas TE →  $E_z^0 = 0$  e

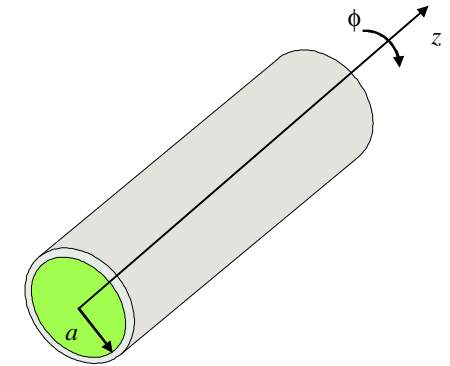
$$H_z^0 = C_n J_n(hr) \cos(n\phi)$$

$$H_r^0 = -\frac{\gamma}{h^2} \frac{\partial H_z^0}{\partial r}$$

$$H_\phi^0 = -\frac{\gamma}{h^2 r} \frac{\partial H_z^0}{\partial \phi}$$

$$E_r^0 = -\frac{j\omega\mu}{h^2 r} \frac{\partial H_z^0}{\partial \phi}$$

$$E_\phi^0 = \frac{j\omega\mu}{h^2} \frac{\partial H_z^0}{\partial r}$$



condições fronteira

$$E_\phi^0 = E_z^0 = H_r^0 = 0 \text{ em } r = a$$

$$H_r^0 = -\frac{\gamma}{h} C_n J'_n(hr) \cos(n\phi)$$

$$H_\phi^0 = \frac{\gamma n}{h^2 r} C_n J_n(hr) \sin(n\phi)$$

$$E_r^0 = \frac{j\omega\mu n}{h^2 r} C_n J_n(hr) \sin(n\phi)$$

$$E_\phi^0 = \frac{j\omega\mu}{h} C_n J'_n(hr) \cos(n\phi)$$

$$J'_n(ha) = 0$$

$$h = h_{TE_{np}} = \frac{p - \text{ésimo zero de } J'_n}{a}$$



# Modos TE<sub>np</sub> – frequência de corte

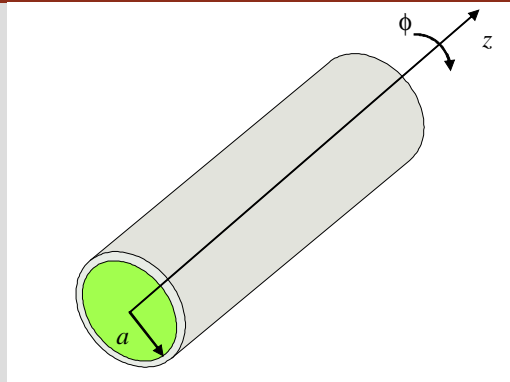
$$(f_c)_{TE_{np}} = \frac{h_{TE_{np}}}{2\pi\sqrt{\mu\epsilon}} = \frac{p - \text{ésimo zero de } J'_n}{2\pi a\sqrt{\mu\epsilon}}$$

$$h = h_{TE_{np}} = \frac{p - \text{ésimo zero de } J'_n}{a}$$

menor zero de  $J'_n \rightarrow 1.8412$  ( $n=1, p=1$ )

modo TE dominante  $\rightarrow$  modo TE<sub>11</sub>

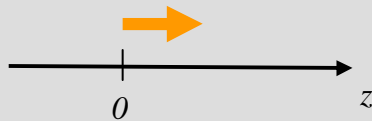
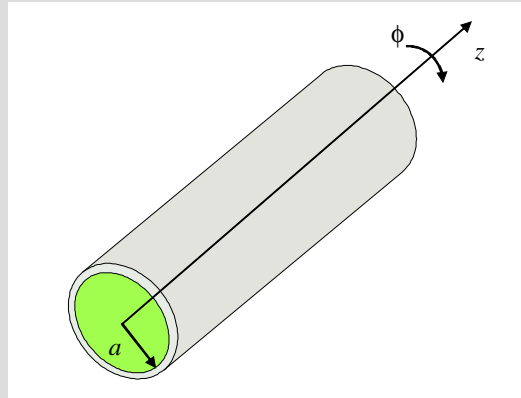
$$(f_c)_{TE_{11}} = \frac{1.8412}{2\pi a\sqrt{\mu\epsilon}}$$



zero	$J'_0(x)$	$J'_1(x)$	$J'_2(x)$	$J'_3(x)$
1	3.8317	1.8412	3.0542	4.2012
2	7.0156	5.3314	6.7061	8.0152
3	10.1735	8.5363	9.9695	11.3459
4	13.3237	11.7060	13.1704	14.5858
5	16.4706	14.8636	16.3475	17.7887

$(f_c)_{TE_{11}} < (f_c)_{TM_{01}} \rightarrow$  modo dominante nos guias circulares é o modo TE<sub>11</sub>

## guias circulares

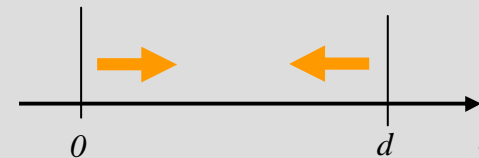
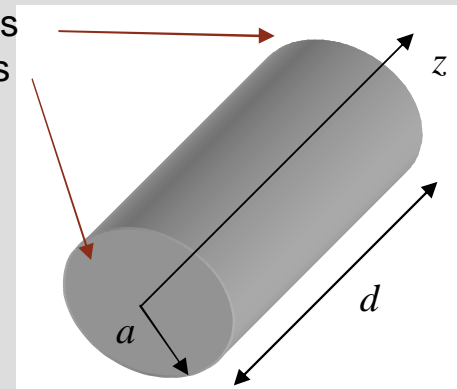


$$\vec{E}(r, \phi, z) = \vec{E}^0(r, \phi) e^{-j\beta z}$$

$$\vec{H}(r, \phi, z) = \vec{H}^0(r, \phi) e^{-j\beta z}$$

## cavidades circulares

placas condutoras  
nas extremidades



$$\vec{E}(r, \phi, z) = \vec{E}^{0,+}(r, \phi) e^{-j\beta z} + \vec{E}^{0,-}(r, \phi) e^{j\beta z}$$

$$\vec{H}(r, \phi, z) = \vec{H}^{0,+}(r, \phi) e^{-j\beta z} + \vec{H}^{0,-}(r, \phi) e^{j\beta z}$$

nas expressões de

$$\begin{matrix} \vec{E}^0(r, \phi) \\ \vec{H}^0(r, \phi) \end{matrix}$$

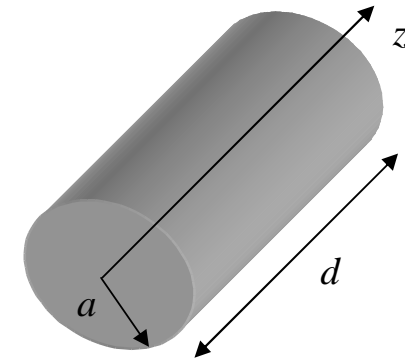
(guias circulares)

$$\gamma = j\beta$$

$$\gamma = -j\beta$$

$$\begin{matrix} \vec{E}^{0,+}(r, \phi) \\ \vec{H}^{0,+}(r, \phi) \end{matrix}$$

$$\begin{matrix} \vec{E}^{0,-}(r, \phi) \\ \vec{H}^{0,-}(r, \phi) \end{matrix}$$



$$\vec{E}(r, \phi, z) = \vec{E}^{0,+}(r, \phi)e^{-j\beta z} + \vec{E}^{0,-}(r, \phi)e^{j\beta z}$$

$$\vec{H}(r, \phi, z) = \vec{H}^{0,+}(r, \phi)e^{-j\beta z} + \vec{H}^{0,-}(r, \phi)e^{j\beta z}$$

**condições fronteira**  $\rightarrow E_{\text{tan}} = H_{\text{normal}} = 0$  em  $z=0$  e  $z=d$



$$E_r = E_\phi = H_z = 0 \text{ em } z=0 \text{ e } z=d$$

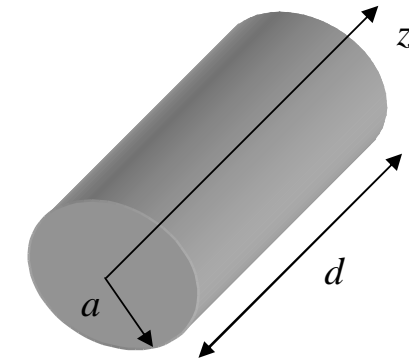
$$E_z = (E_{0,np}^+ e^{-j\beta z} + E_{0,np}^- e^{j\beta z}) J_n(hr) \cos(n\phi)$$

$$E_r = -\frac{j\beta}{h} (E_{0,np}^+ e^{-j\beta z} - E_{0,np}^- e^{j\beta z}) J'_n(hr) \cos(n\phi)$$

$$E_\phi = \frac{j\beta n}{h^2 r} (E_{0,np}^+ e^{-j\beta z} - E_{0,np}^- e^{j\beta z}) J_n(hr) \sin(n\phi)$$

$$H_r = -\frac{j\omega\epsilon n}{h^2 r} (E_{0,np}^+ e^{-j\beta z} + E_{0,np}^- e^{j\beta z}) J_n(hr) \sin(n\phi)$$

$$H_\phi = -\frac{j\omega\epsilon}{h} (E_{0,np}^+ e^{-j\beta z} + E_{0,np}^- e^{j\beta z}) J'_n(hr) \cos(n\phi)$$



$$\vec{E}(r, \phi, z) = \vec{E}^{0,+}(r, \phi) e^{-j\beta z} + \vec{E}^{0,-}(r, \phi) e^{j\beta z}$$

$$\vec{H}(r, \phi, z) = \vec{H}^{0,+}(r, \phi) e^{-j\beta z} + \vec{H}^{0,-}(r, \phi) e^{j\beta z}$$

$$E_r = E_\phi = H_z = 0 \text{ em } z=0 \text{ e } z=d$$

$$E_{0,np}^+ = E_{0,np}^- = E_0/2$$

$$\beta = \frac{q\pi}{d}, \quad q \text{ inteiro}$$

$$E_z = E_0 J_n(hr) \cos(n\phi) \cos\left(\frac{q\pi z}{d}\right)$$

$$E_r = -\frac{1}{h} E_0 \left(\frac{q\pi}{d}\right) J'_n(hr) \cos(n\phi) \sin\left(\frac{q\pi z}{d}\right)$$

$$E_\phi = \frac{n}{h^2 r} E_0 \left(\frac{q\pi}{d}\right) J_n(hr) \sin(n\phi) \sin\left(\frac{q\pi z}{d}\right)$$

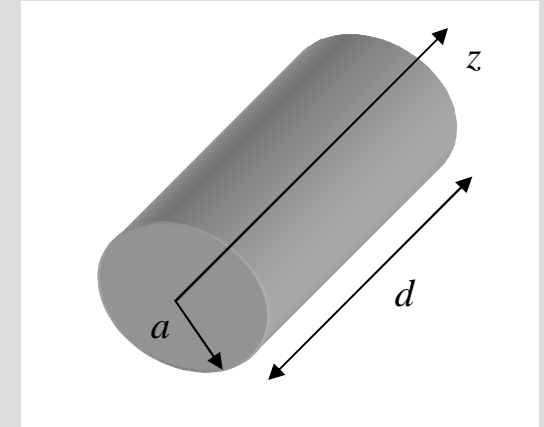
$$H_r = -\frac{j\omega\epsilon n}{h^2 r} E_0 J_n(hr) \sin(n\phi) \cos\left(\frac{q\pi z}{d}\right)$$

$$H_\phi = -\frac{j\omega\epsilon}{h} E_0 J'_n(hr) \cos(n\phi) \cos\left(\frac{q\pi z}{d}\right)$$

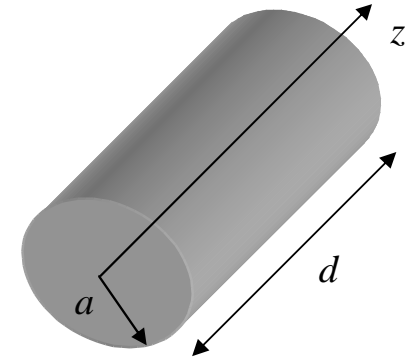
$$n \geq 0 \quad \text{e} \quad p \geq 1$$

$$h = h_{TM_{np}} = \frac{p - \text{ésimo zero de } J_n}{a}$$

$$q \geq 0$$



$$\begin{aligned}
 H_z &= (H_{0,np}^+ e^{-j\beta z} + H_{0,np}^- e^{j\beta z}) J_n(hr) \cos(n\phi) \\
 E_r &= \frac{j\omega\mu n}{h^2 r} (H_{0,np}^+ e^{-j\beta z} + H_{0,np}^- e^{j\beta z}) J_n(hr) \sin(n\phi) \\
 E_\phi &= \frac{j\omega\mu}{h} (H_{0,np}^+ e^{-j\beta z} + H_{0,np}^- e^{j\beta z}) J'_n(hr) \cos(n\phi) \\
 H_r &= -\frac{j\beta}{h} (H_{0,np}^+ e^{-j\beta z} - H_{0,np}^- e^{j\beta z}) J'_n(hr) \cos(n\phi) \\
 H_\phi &= -\frac{j\beta n}{h^2 r} (H_{0,np}^+ e^{-j\beta z} - H_{0,np}^- e^{j\beta z}) J_n(hr) \sin(n\phi)
 \end{aligned}$$



$$\begin{aligned}
 \vec{E}(r, \phi, z) &= \vec{E}^{0,+}(r, \phi) e^{-j\beta z} + \vec{E}^{0,-}(r, \phi) e^{j\beta z} \\
 \vec{H}(r, \phi, z) &= \vec{H}^{0,+}(r, \phi) e^{-j\beta z} + \vec{H}^{0,-}(r, \phi) e^{j\beta z}
 \end{aligned}$$

$$E_r = E_\phi = H_z = 0 \text{ em } z=0 \text{ e } z=d$$

$$\begin{aligned}
 H_{0,np}^+ &= H_{0,np}^- = -H_0/2j \\
 \beta &= \frac{q\pi}{d}, \quad q \text{ inteiro}
 \end{aligned}$$

$$H_z = H_0 J_n(hr) \cos(n\phi) \sin\left(\frac{q\pi z}{d}\right)$$

$$E_r = \frac{j\omega\mu n}{h^2 r} H_0 J_n(hr) \sin(n\phi) \sin\left(\frac{q\pi z}{d}\right)$$

$$E_\phi = \frac{j\omega\mu}{h} H_0 J'_n(hr) \cos(n\phi) \sin\left(\frac{q\pi z}{d}\right)$$

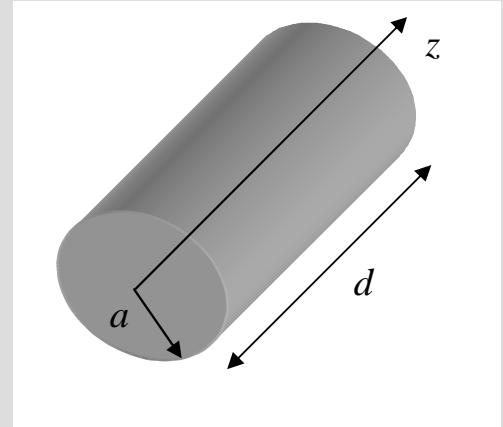
$$H_r = \frac{1}{h} \left(\frac{q\pi}{d}\right) H_0 J'_n(hr) \cos(n\phi) \cos\left(\frac{q\pi z}{d}\right)$$

$$H_\phi = -\frac{n}{h^2 r} \left(\frac{q\pi}{d}\right) H_0 J_n(hr) \sin(n\phi) \cos\left(\frac{q\pi z}{d}\right)$$

$$n \geq 0 \text{ e } p \geq 1$$

$$h = h_{TE_{np}} = \frac{p - \text{ésimo zero de } J'_n}{a}$$

$$q \geq 1$$

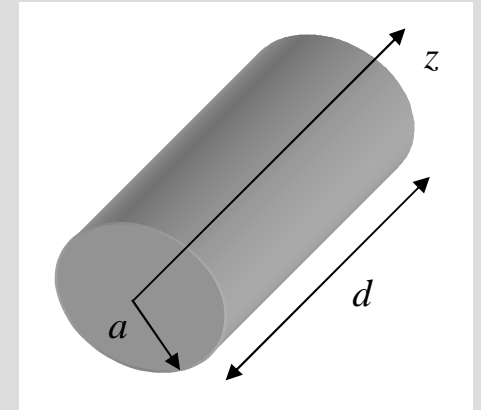


# Cavidades circulares – frequência de ressonância

$$h^2 = \gamma^2 + \omega^2 \mu \epsilon \xrightarrow{\gamma = j\beta} \omega = \sqrt{\frac{h^2 + \beta^2}{\mu \epsilon}}$$

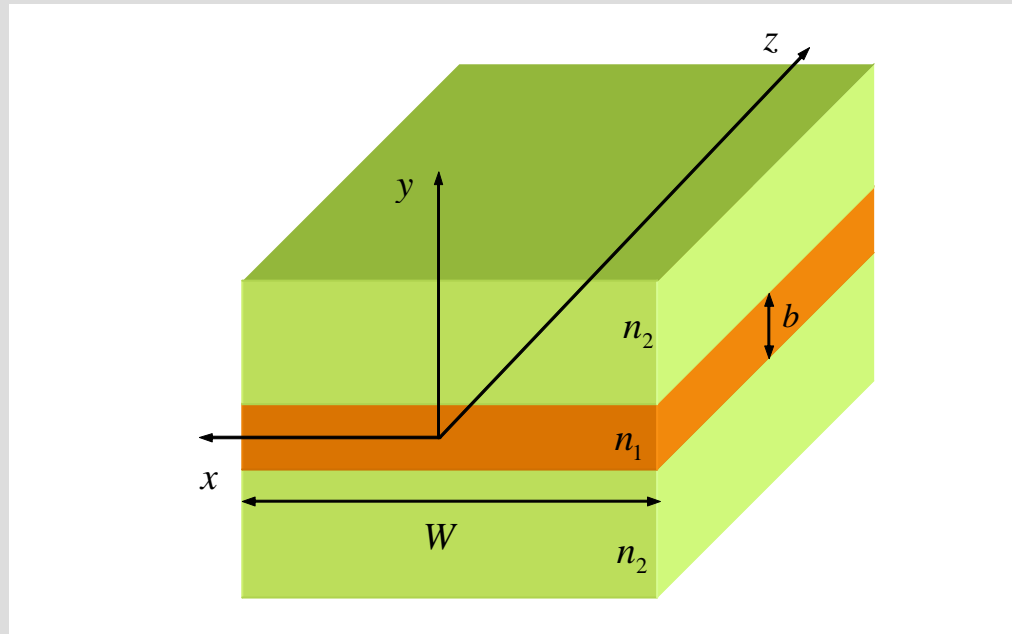
$$\beta = \frac{q\pi}{d}, \quad q \text{ inteiro}$$

$$\omega = \frac{1}{\sqrt{\mu \epsilon}} \sqrt{(h)^2 + \left(\frac{q\pi}{d}\right)^2}$$



modos	$q$	$h$	menor zero	frequência de ressonância mais baixa
TM	$q \geq 0$	$\frac{p - \text{ésimo zero de } J_n}{a}$	2.4048	$\omega_{TM_{010}} = \frac{2.4048}{a\sqrt{\mu \epsilon}}$
TE	$q \geq 1$	$\frac{p - \text{ésimo zero de } J'_n}{a}$	1.8412	$\omega_{TE_{111}} = \frac{1}{\sqrt{\mu \epsilon}} \sqrt{\left(\frac{1.8412}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2}$





materiais sem perdas ( $\sigma = 0$ )

$$n_1 > n_2$$

comprimento infinito  $\rightarrow$  propagação segundo  $+z$

$$W \gg b \quad \longrightarrow \quad \frac{\partial}{\partial x} = 0$$

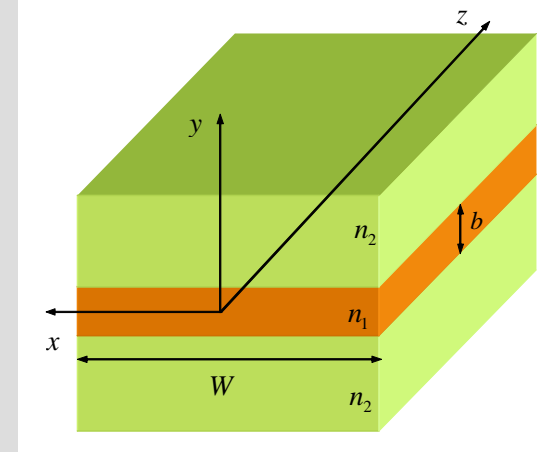
# Guias dieléctricos planares – condições fronteira

condições fronteira:

$$\begin{aligned} \hat{a}_n \times (\vec{E}_1 - \vec{E}_2) &= 0 \\ \hat{a}_n \cdot (\vec{B}_1 - \vec{B}_2) &= 0 \\ \hat{a}_n \cdot (\vec{D}_1 - \vec{D}_2) &= \rho_s \\ \hat{a}_n \times (\vec{H}_1 - \vec{H}_2) &= \vec{J}_s \end{aligned}$$

dieléctricos:

$$\begin{aligned} \rho_s &= 0 \\ \vec{J}_s &= 0 \end{aligned}$$



$$\begin{aligned} E_{\text{tan}} &\text{ contínua} \\ B_{\text{normal}} &\text{ contínua} \\ D_{\text{normal}} &\text{ contínua} \\ H_{\text{tan}} &\text{ contínua} \end{aligned}$$

$$\begin{aligned} E_z \text{ e } E_x &\text{ contínuos} \\ H_z \text{ e } H_x &\text{ contínuos} \end{aligned} \quad \text{em } y = \pm \frac{b}{2}$$

$$\frac{\partial}{\partial x} = 0$$

## 1. resolver

$$\frac{d^2 E_z^0}{dy^2} + h^2 E_z^0 = 0$$

$$\frac{d^2 H_z^0}{dy^2} + h^2 H_z^0 = 0$$

$$h^2 = \gamma^2 + \omega^2 \mu \epsilon$$

$$h^2 = \begin{cases} \gamma^2 + \left(\frac{\omega}{c} n_1\right)^2, & \text{meio1} \\ \gamma^2 + \left(\frac{\omega}{c} n_2\right)^2, & \text{meio2} \end{cases}$$

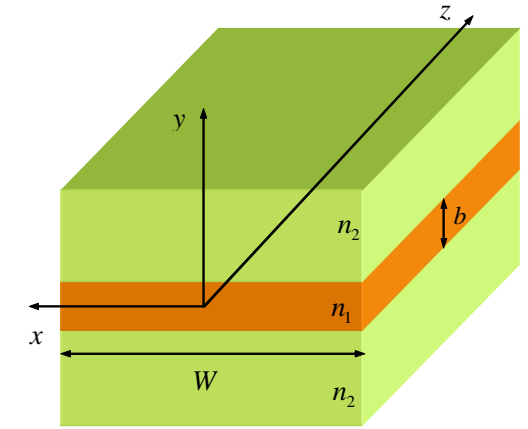
## 2. determinar (se $h \neq 0$ )

$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{dE_z^0}{dy}$$

$$H_y^0 = -\frac{\gamma}{h^2} \frac{dH_z^0}{dy}$$

$$E_x^0 = -\frac{j\omega\mu}{h^2} \frac{dH_z^0}{dy}$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{dE_z^0}{dy}$$



$$\vec{E}(x, y, z) = \vec{E}^0(x, y)e^{-\gamma z}$$

$$\vec{H}(x, y, z) = \vec{H}^0(x, y)e^{-\gamma z}$$

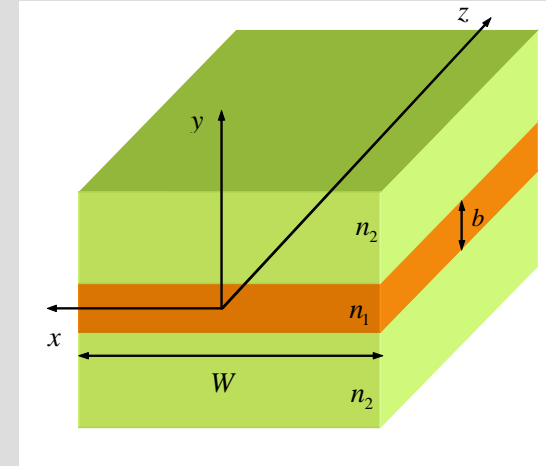
# Ondas TM e TE – determinação das componentes longitudinais

resolver

$$\frac{d^2 E_z^0}{dy^2} + h^2 E_z^0 = 0$$

$$\frac{d^2 H_z^0}{dy^2} + h^2 H_z^0 = 0$$

$$\longrightarrow \frac{d^2 \psi}{dy^2} + h^2 \psi = 0 \quad h^2 = \begin{cases} h_1^2, & \text{meio 1} \\ h_2^2, & \text{meio 2} \end{cases}$$



solução geral:

$$h^2 > 0 \iff h \text{ real}$$

$$\psi = A \sin(hy) + B \cos(hy)$$

meio 1

$$h_1 \text{ real}$$

$$h^2 < 0 \iff h = j\nu$$

$$\psi = C e^{-\nu y} + D e^{+\nu y}$$

meio 2

$$h_2 = j\nu$$

$$h_2^2 = -\nu^2$$

$$\nu = \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2}$$

$$\begin{cases} h_1^2 = \gamma^2 + \left(\frac{\omega}{c} n_1\right)^2 \\ \nu^2 = -\gamma^2 - \left(\frac{\omega}{c} n_2\right)^2 \end{cases}$$

# Valores limites da constante de fase

modos em propagação  $\rightarrow \gamma = j\beta$

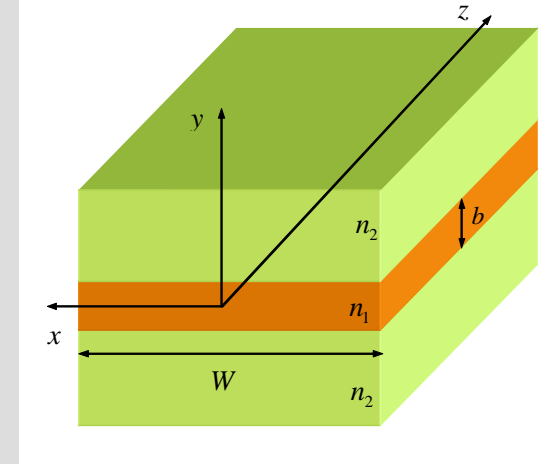
$$h_1^2 = \gamma^2 + (\omega n_1 / c_1)^2$$
$$v^2 = -\gamma^2 - (\omega n_2 / c_1)^2$$

$$\beta = \sqrt{\left(\frac{\omega}{c} n_1\right)^2 - h_1^2} = \sqrt{\left(\frac{\omega}{c} n_2\right)^2 + v^2}$$

$$\beta < \frac{\omega}{c} n_1 \quad \beta > \frac{\omega}{c} n_2$$

$$\frac{\omega}{c} n_1 > \beta > \frac{\omega}{c} n_2$$

$$n_1 > n_2$$



# Determinação das componentes longitudinais

meio 1

$$\psi = A \sin(h_1 y) + B \cos(h_1 y)$$

meio 2

$$\psi = C e^{-\nu y} + D e^{+\nu y}$$

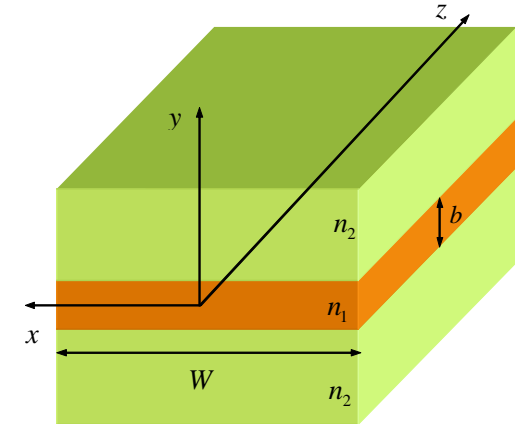
$$\psi(y) = \begin{cases} C e^{-\nu y}, & y > \frac{b}{2} \\ A \sin(h_1 y) + B \cos(h_1 y), & |y| \leq \frac{b}{2} \\ D e^{+\nu y}, & y < -\frac{b}{2} \end{cases}$$

decaimento exponencial no meio 2

$\psi$  contínua em  $y = \pm b/2$

$$C e^{-\frac{\nu b}{2}} = A \sin\left(\frac{h_1 b}{2}\right) + B \cos\left(\frac{h_1 b}{2}\right)$$

$$D e^{-\frac{\nu b}{2}} = -A \sin\left(\frac{h_1 b}{2}\right) + B \cos\left(\frac{h_1 b}{2}\right)$$



ondas TM  $\rightarrow E_z^0 = \psi$

ondas TE  $\rightarrow H_z^0 = \psi$

condições fronteira

$E_z$  e  $H_z$  contínuos em  $y = \pm b/2$

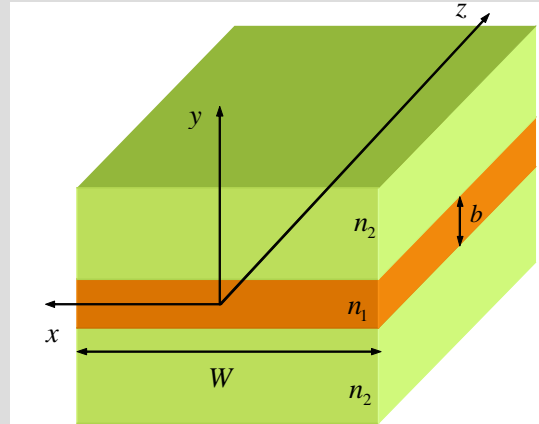
# Determinação das componentes longitudinais

$$C e^{-\frac{v b}{2}} = A \sin\left(\frac{h_1 b}{2}\right) + B \cos\left(\frac{h_1 b}{2}\right)$$

$$D e^{-\frac{v b}{2}} = -A \sin\left(\frac{h_1 b}{2}\right) + B \cos\left(\frac{h_1 b}{2}\right)$$

$$C = \left[ A \sin\left(\frac{h_1 b}{2}\right) + B \cos\left(\frac{h_1 b}{2}\right) \right] e^{\frac{v b}{2}}$$

$$D = \left[ -A \sin\left(\frac{h_1 b}{2}\right) + B \cos\left(\frac{h_1 b}{2}\right) \right] e^{\frac{v b}{2}}$$



$$\psi(y) = \begin{cases} C e^{-v y}, & y > \frac{b}{2} \\ A \sin(h_1 y) + B \cos(h_1 y), & |y| \leq \frac{b}{2} \\ D e^{v y}, & y < -\frac{b}{2} \end{cases}$$

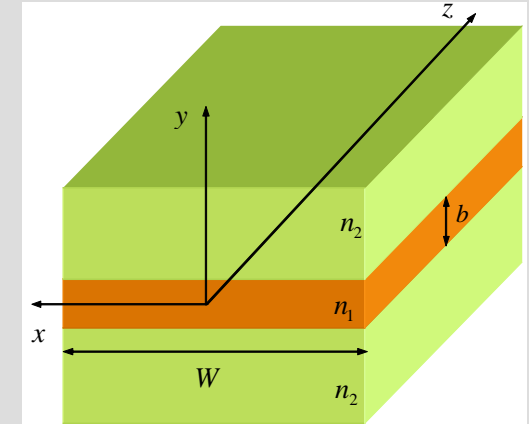
$$\psi(y) = \begin{cases} \left[ A \sin\left(\frac{h_1 b}{2}\right) + B \cos\left(\frac{h_1 b}{2}\right) \right] e^{-v\left(y - \frac{b}{2}\right)}, & y > \frac{b}{2} \\ A \sin(h_1 y) + B \cos(h_1 y), & |y| \leq \frac{b}{2} \\ \left[ -A \sin\left(\frac{h_1 b}{2}\right) + B \cos\left(\frac{h_1 b}{2}\right) \right] e^{v\left(y + \frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

# Modos pares e ímpares

$$\psi(y) = \begin{cases} \left[ A \sin\left(\frac{h_1 b}{2}\right) + B \cos\left(\frac{h_1 b}{2}\right) \right] e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ A \sin(h_1 y) + B \cos(h_1 y), & |y| \leq \frac{b}{2} \\ \left[ -A \sin\left(\frac{h_1 b}{2}\right) + B \cos\left(\frac{h_1 b}{2}\right) \right] e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$A = 0$

$B = 0$



ondas TM  $\rightarrow E_z^0 = \psi$   
ondas TE  $\rightarrow H_z^0 = \psi$

**modos pares**

$$\psi_{\text{par}} = \begin{cases} B \cos\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ B \cos(h_1 y), & |y| \leq \frac{b}{2} \\ B \cos\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

**modos ímpares**

$$\psi_{\text{ímpar}} = \begin{cases} A \sin\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ A \sin(h_1 y), & |y| \leq \frac{b}{2} \\ A \sin\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

- ondas TM pares  $\rightarrow E_z^0 = \psi_{\text{par}}$
- ondas TM ímpares  $\rightarrow E_z^0 = \psi_{\text{ímpar}}$
- ondas TE pares  $\rightarrow H_z^0 = \psi_{\text{par}}$
- ondas TE ímpares  $\rightarrow H_z^0 = \psi_{\text{ímpar}}$



# Modos TM pares

ondas TM pares  $\rightarrow H_z^0 = 0$  e  $E_z^0 = \psi_{\text{par}}$

$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{dE_z^0}{dy}$$

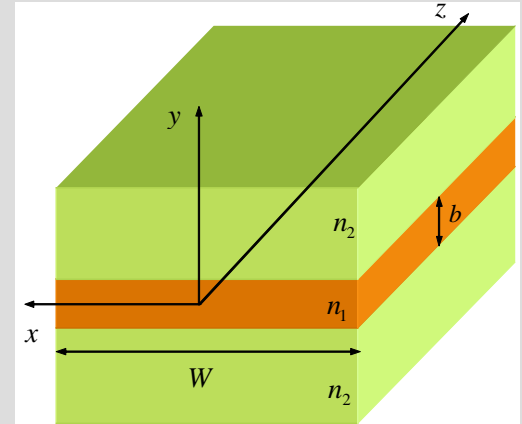
$$H_y^0 = -\frac{\gamma}{h^2} \frac{dH_z^0}{dy} = 0$$

$$E_x^0 = -\frac{j\omega\mu}{h^2} \frac{dH_z^0}{dy} = 0$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{dE_z^0}{dy}$$

$$E_y^0 = \begin{cases} -\frac{j\beta}{\nu} B \cos\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ \frac{j\beta}{h_1} B \sin(h_1 y), & |y| < \frac{b}{2} \\ \frac{j\beta}{\nu} B \cos\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$$H_x^0 = \begin{cases} \frac{j\omega\epsilon_2}{\nu} B \cos\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ -\frac{j\omega\epsilon_1}{h_1} B \sin(h_1 y), & |y| < \frac{b}{2} \\ -\frac{j\omega\epsilon_2}{\nu} B \cos\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$



$$\psi_{\text{par}} = \begin{cases} B \cos\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ B \cos(h_1 y), & |y| \leq \frac{b}{2} \\ B \cos\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$$\frac{j\omega\epsilon_2}{\nu} B \cos\left(\frac{h_1 b}{2}\right) = -\frac{j\omega\epsilon_1}{h_1} B \sin\left(\frac{h_1 b}{2}\right)$$

**condições fronteira**  
 $H_x$  contínuo em  $y = \pm b/2$

# Modos TM pares – relação característica

$$\frac{j\omega\epsilon_2}{\nu} B \cos\left(\frac{h_1 b}{2}\right) = -\frac{j\omega\epsilon_1}{h_1} B \sin\left(\frac{h_1 b}{2}\right) \longrightarrow \nu = -h_1 \frac{\epsilon_2}{\epsilon_1} \cot\left(\frac{h_1 b}{2}\right)$$

$$n = \sqrt{\mu_r \epsilon_r} = \sqrt{\epsilon_r}$$

se  $\mu_r = 1$

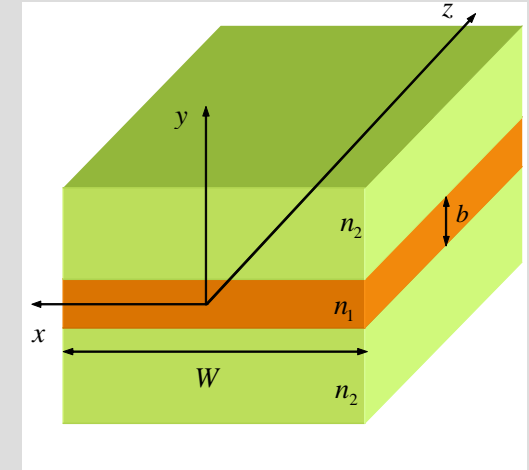
$$\nu = -h_1 \left(\frac{n_2}{n_1}\right)^2 \cot\left(\frac{h_1 b}{2}\right)$$

$$\nu = \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2}$$

**relação característica**

$$\left(\frac{n_1}{n_2}\right)^2 \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2} = -h_1 \cot\left(\frac{h_1 b}{2}\right)$$

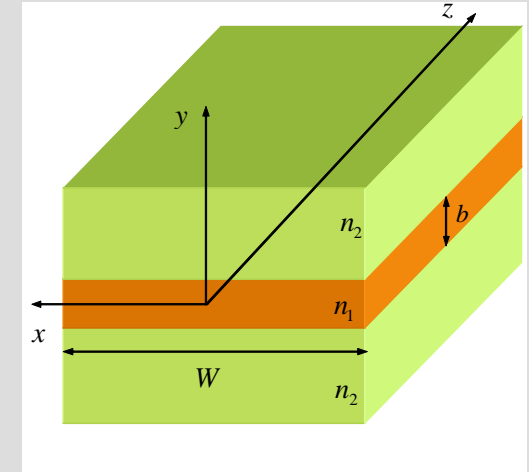
permite obter o valor característico  $h_1$  em função do guia e da frequência



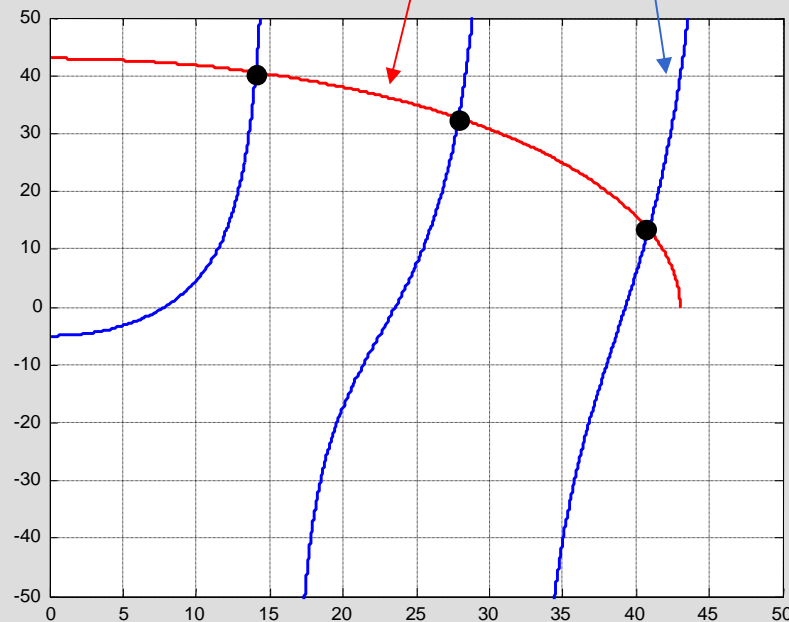
# Modos TM pares – soluções da equação característica

relação característica

$$\left(\frac{n_1}{n_2}\right)^2 \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2} = -h_1 \cot\left(\frac{h_1 b}{2}\right) \rightarrow \text{equação do tipo:}$$
$$\sqrt{A^2 - x^2} = -x \cot(Bx)$$



apenas 3 soluções!  
(no caso representado)



## notas

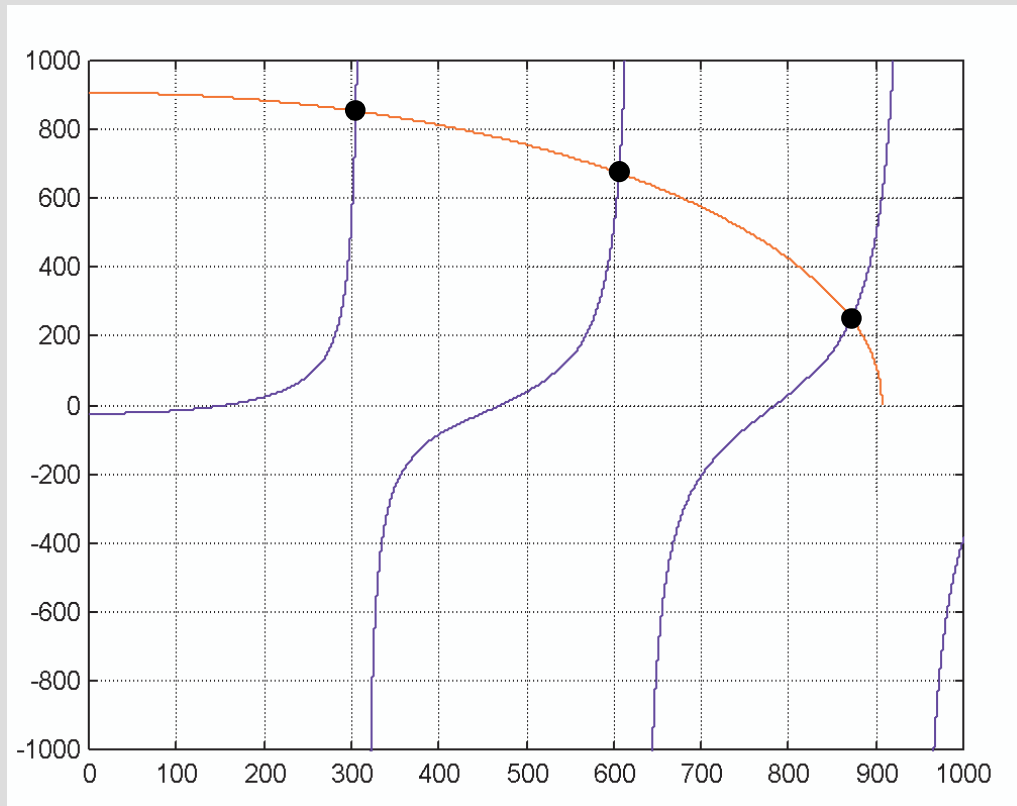
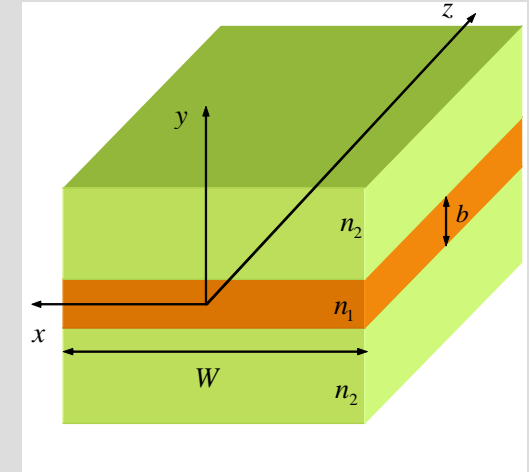
1. soluções em número finito
2. a cada solução corresponde um modo em propagação
3. número de modos em propagação aumenta com a frequência

# Modos TM pares – exemplo

$f = 25 \text{ GHz}$   
 $\epsilon_2 = \epsilon_0$   
 $\epsilon_1 = 4\epsilon_0$   
 $b = 2 \text{ cm}$

$n_2 = 1$   
 $n_1 = 2$

$$4\sqrt{\frac{(500\pi)^2}{3} - h_1^2} = -h_1 \cot(0.01h_1)$$



$$h_{1,1} = 305.3$$

$$h_{1,2} = 606.2$$

$$h_{1,3} = 871.2$$

# Modo $TM_1$ par – exemplo

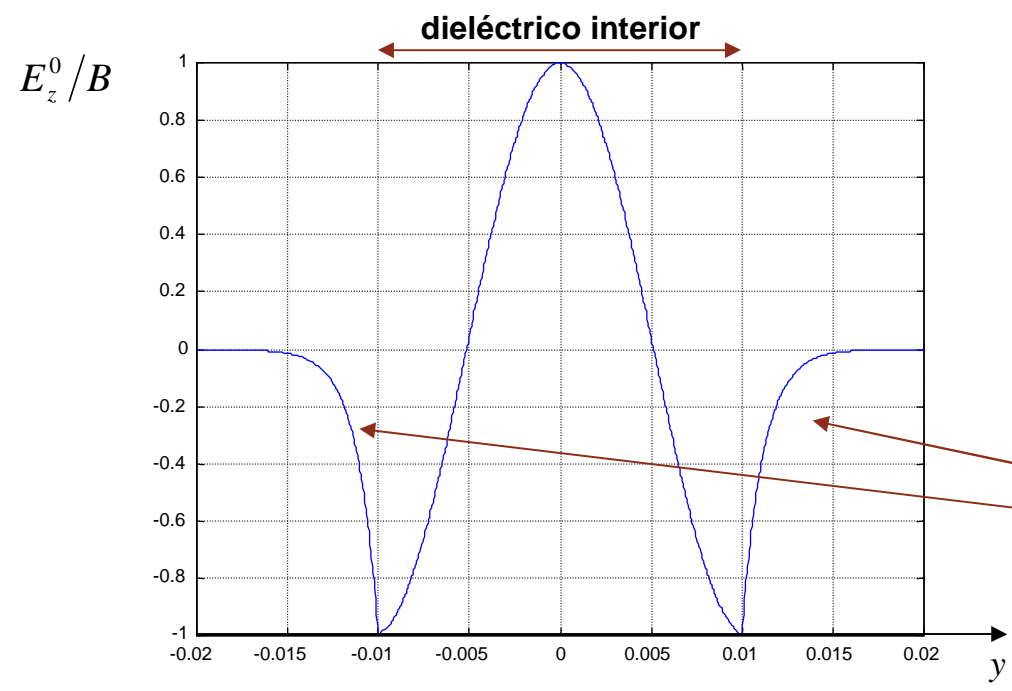
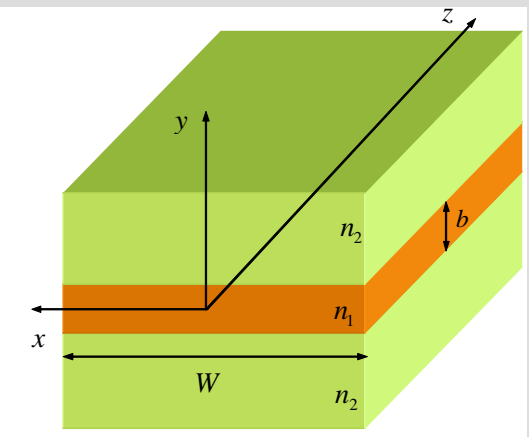
$f = 25 \text{ GHz}$   
 $\epsilon_2 = \epsilon_0$   
 $\epsilon_1 = 4\epsilon_0$   
 $b = 2 \text{ cm}$

$n_2 = 1$   
 $n_1 = 2$

$h_{1,1} = 305.3$

$\nu_1 = 854$

$$\nu = \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2}$$



$$E_z^0 = \begin{cases} B \cos\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ B \cos(h_1 y), & |y| \leq \frac{b}{2} \\ B \cos\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

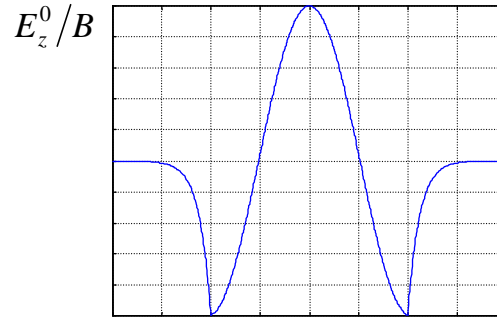
decaimento exponencial no meio 2

# Modos pares $TM_1$ , $TM_2$ e $TM_3$ – exemplo

**modo  $TM_1$**

$$h_{1,1} = 305.3$$

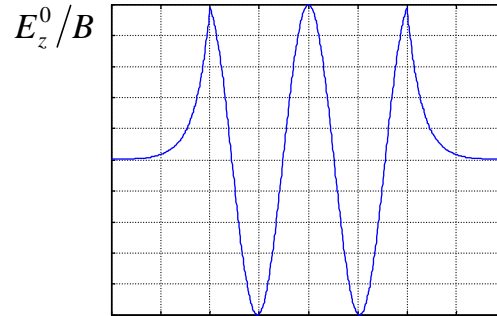
$$\nu_1 = 854$$



**modo  $TM_2$**

$$h_{1,2} = 606.2$$

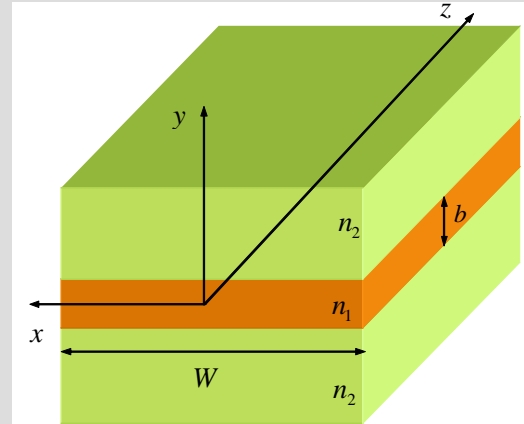
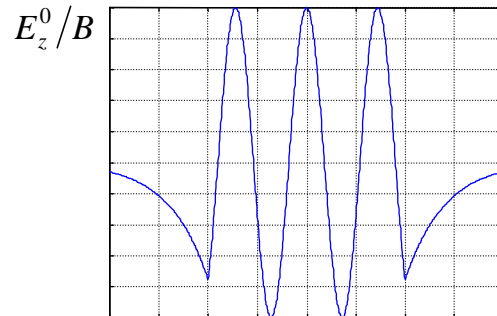
$$\nu_2 = 674.5$$



**modo  $TM_3$**

$$h_{1,3} = 871.2$$

$$\nu_3 = 252$$



**exemplo**

$$f = 25 \text{ GHz}$$

$$\epsilon_2 = \epsilon_0$$

$$\epsilon_1 = 4\epsilon_0$$

$$b = 2 \text{ cm}$$

taxa de decaimento  
diminui com o  
aumento da ordem  
do modo

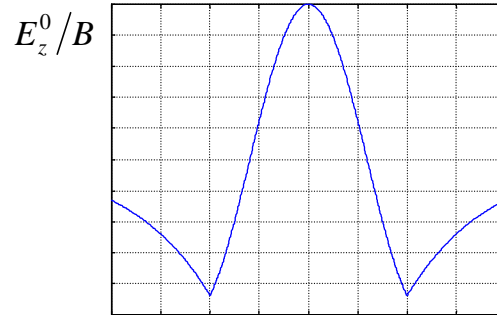
## Variação do modos $TM_1$ par com a frequência– exemplo

**modo  $TM_1$**

$f = 8\text{GHz}$

$h_{1,1} = 264$

$\nu_1 = 120.5$

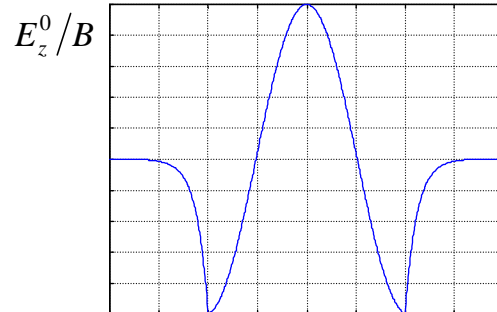


**modo  $TM_1$**

$f = 25\text{GHz}$

$h_{1,1} = 305.3$

$\nu_1 = 854$

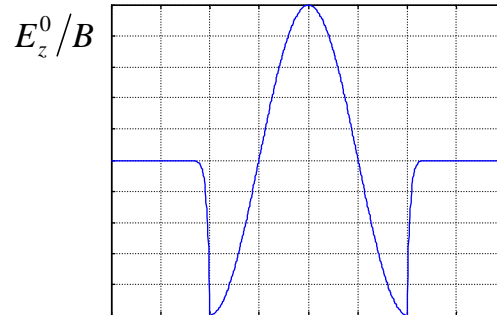


**modo  $TM_1$**

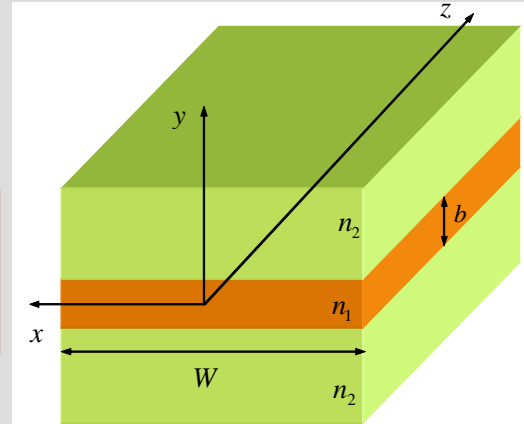
$f = 100\text{GHz}$

$h_{1,1} = 312$

$\nu_1 = 3614.2$



**taxa de decaimento  
diminui com a  
diminuição da frequência**



**exemplo**  $\epsilon_2 = \epsilon_0$   
 $\epsilon_1 = 4\epsilon_0$   
 $b = 2\text{ cm}$

$f$  pequeno

$\nu = 0 \rightarrow$  onda não confinada ao guia

**frequência de corte**

# Modos TM par – frequência de corte

condição de corte:  $\nu = 0$

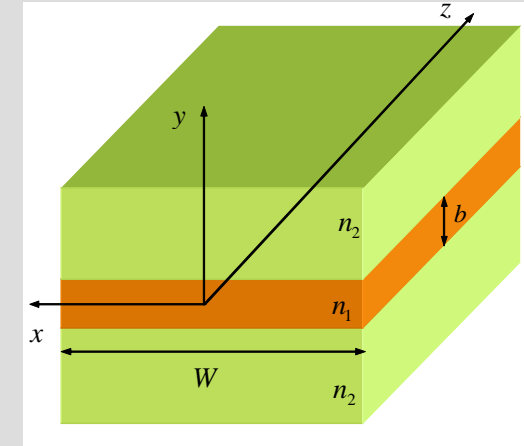
$$\nu = -h_1 \left( \frac{n_2}{n_1} \right)^2 \cot \left( \frac{h_1 b}{2} \right) = 0 \longrightarrow -\cot \left( \frac{h_1 b}{2} \right) = 0 \longrightarrow \frac{h_1 b}{2} = \left( n - \frac{1}{2} \right) \pi, \quad n = 1, 2, \dots$$

$$\nu^2 = -\gamma^2 - \left( \frac{\omega}{c} n_2 \right)^2 = 0 \longrightarrow \gamma^2 = - \left( \frac{\omega}{c} n_2 \right)^2$$

$$h_1^2 = \gamma^2 + \left( \frac{\omega}{c} n_1 \right)^2$$

$$h_1^2 = \left( \frac{\omega}{c} \right)^2 (n_1^2 - n_2^2) \longrightarrow h_1 = \frac{\omega}{c} \sqrt{n_1^2 - n_2^2}$$

$$(f_c)_{TM \text{ par}} = \frac{\left( n - \frac{1}{2} \right) c}{b \sqrt{n_1^2 - n_2^2}}, \quad n = 1, 2, \dots$$





## Modos TM par – modo dominante

$$(f_c)_{TM\ par} = \frac{\left(n - \frac{1}{2}\right) c}{b\sqrt{n_1^2 - n_2^2}}, \quad n = 1, 2, \dots$$



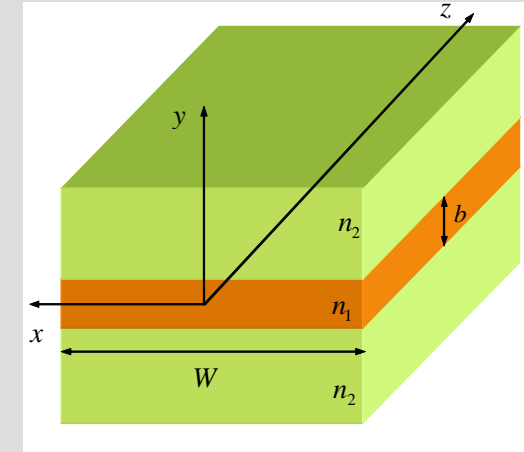
frequência de corte aumenta à medida que a largura do guia diminui

modo TM par dominante → modo  $TM_1$  par

$$(f_c)_{TM_1\ par} = \frac{c}{2b\sqrt{n_1^2 - n_2^2}}$$

modo TM par seguinte → modo  $TM_2$  par

$$(f_c)_{TM_2\ par} = \frac{3c}{2b\sqrt{n_1^2 - n_2^2}}$$



→ propagação de apenas um modo TM par se

$$(f_c)_{TM_2\ par} > f$$



$$b < \frac{3c}{2f\sqrt{n_1^2 - n_2^2}}$$

## Frequência de corte dos modos TM par – exemplo

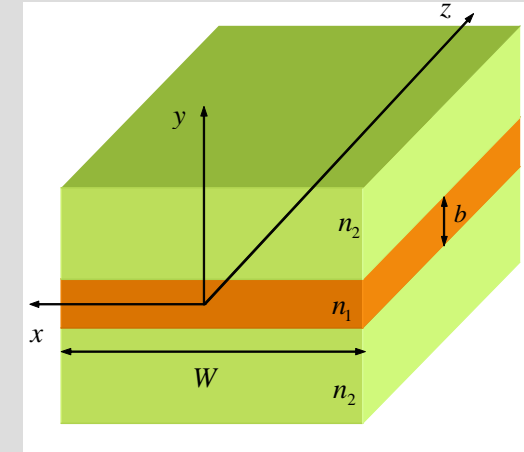
$$(f_c)_{TM_{par}} = \frac{\left(n - \frac{1}{2}\right) c}{b\sqrt{n_1^2 - n_2^2}}, \quad n = 1, 2, \dots$$

$$f = 25 \text{ GHz}$$

$$\epsilon_2 = \epsilon_0$$

$$\epsilon_1 = 4\epsilon_0$$

$$b = 2 \text{ cm}$$



$$(f_c)_{TM_{1par}} = 4.3 \text{ GHz}$$

$$(f_c)_{TM_{2par}} = 13 \text{ GHz}$$

$$(f_c)_{TM_{3par}} = 21.7 \text{ GHz}$$

$$(f_c)_{TM_{4par}} = 30.3 \text{ GHz}$$

⋮

← modos em propagação

← modos evanescentes

# Potência média propagada – modos TM par

$$P_{med} = \frac{1}{2} \int_A \Re \left\{ \frac{1}{Z} \right\} \left( |E_x|^2 + |E_y|^2 \right) dA$$

$$E_x^0 = 0$$

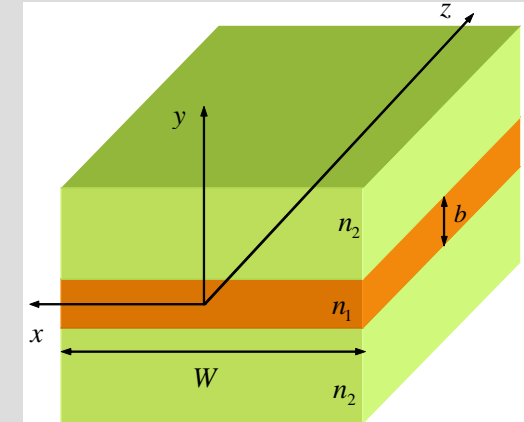
$$E_y^0 = \begin{cases} -\frac{j\beta}{\nu} B \cos\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ \frac{j\beta}{h_1} B \sin(h_1 y), & |y| < \frac{b}{2} \\ \frac{j\beta}{\nu} B \cos\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$$E_y = E_y^0 e^{-j\beta z} \longrightarrow |E_y| = |E_y^0|$$

$$Z_{TM} = \eta \sqrt{1 - (f_c/f)^2}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \begin{cases} \frac{\eta_0}{n_1}, & \text{meio1} \\ \frac{\eta_0}{n_2}, & \text{meio2} \end{cases}$$

$$(f_c)_{TM\ par} = \frac{\left(n - \frac{1}{2}\right) c}{b \sqrt{n_1^2 - n_2^2}}, \quad n = 1, 2, \dots$$



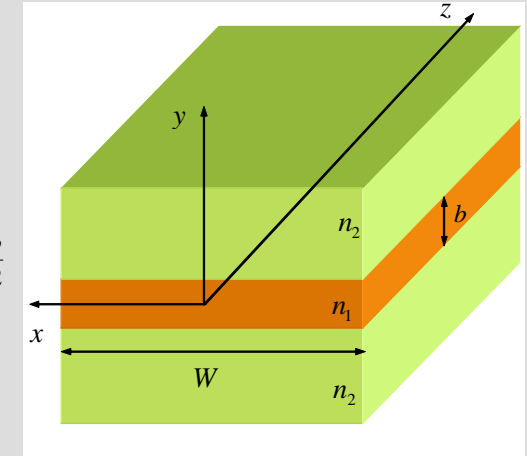
$$P_{med} = \int_A \vec{S}_{med} \cdot d\vec{A}$$

$$\vec{S}_{med} = \frac{1}{2} \Re \{ \vec{E} \times \vec{H}^* \}$$

# Potência média nos meios 1 e 2 – modos TM par

## potência no meio 1

$$\begin{aligned}
 P_{med} &= \frac{W\eta_0 B^2 \beta^2}{2n_1 h_1^2} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \int_{-\frac{b}{2}}^{\frac{b}{2}} \sin^2(h_1 y) dy = \frac{W\eta_0 B^2 \beta^2}{2n_1 h_1^2} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \left[ \frac{y}{2} - \frac{\sin(2h_1 y)}{4h_1} \right]_{-\frac{b}{2}}^{\frac{b}{2}} \\
 &= \frac{W\eta_0 B^2 \beta^2}{2n_1 h_1^2} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \left[ \frac{b}{2} - \frac{\sin(h_1 b)}{2h_1} \right]
 \end{aligned}$$



## potência no meio 2

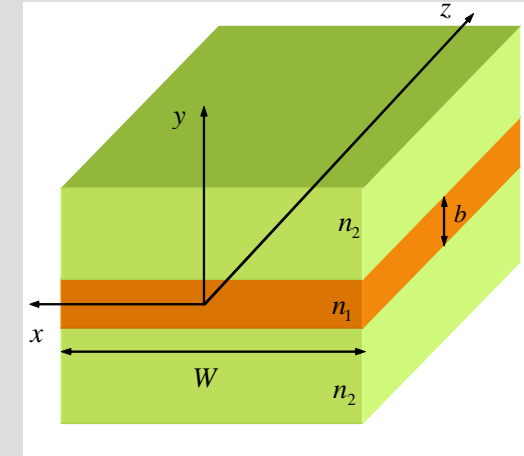
$$\begin{aligned}
 P_{med} &= \frac{W\eta_0 B^2 \beta^2}{2n_2 \nu^2} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \cos^2\left(\frac{h_1 b}{2}\right) \left\{ \int_{-\infty}^{-\frac{b}{2}} e^{2\nu\left(y+\frac{b}{2}\right)} dy + \int_{\frac{b}{2}}^{\infty} e^{-2\nu\left(y-\frac{b}{2}\right)} dy \right\} \\
 &= \frac{W\eta_0 B^2 \beta^2}{2n_2 \nu^3} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \cos^2\left(\frac{h_1 b}{2}\right)
 \end{aligned}$$

→ potência propagada no meio 2 não é nula

# Percentagem de potência média no meio 2 – modos TM par

percentagem de potência no meio 2:

$$\frac{P_{med, meio2}}{P_{med, total}} = \frac{\frac{\cos^2\left(\frac{h_1 b}{2}\right)}{n_2 v^3}}{\frac{\cos^2\left(\frac{h_1 b}{2}\right)}{n_2 v^3} + \frac{b}{2h_1^2 n_1} - \frac{\sin(h_1 b)}{2n_1 h_1^3}}$$



exemplo

$$\begin{aligned} f &= 25 \text{ GHz} \\ \epsilon_2 &= \epsilon_0 \\ \epsilon_1 &= 4\epsilon_0 \\ b &= 2 \text{ cm} \end{aligned}$$

modo  $TM_1$  par

$$\begin{aligned} h_{1,1} &= 305.3 \\ v_1 &= 854 \\ (f_c)_{TM_1 \text{ par}} &= 4.3 \text{ GHz} \end{aligned}$$

$$\frac{P_{med, meio2}}{P_{med, total}} = 0.72\%$$

modo  $TM_2$  par

$$\begin{aligned} h_{1,2} &= 606.2 \\ v_2 &= 674.5 \\ (f_c)_{TM_2 \text{ par}} &= 13 \text{ GHz} \end{aligned}$$

$$\frac{P_{med, meio2}}{P_{med, total}} = 5.22\%$$

modo  $TM_3$  par

$$\begin{aligned} h_{1,3} &= 871.2 \\ v_3 &= 252 \\ (f_c)_{TM_3 \text{ par}} &= 21.7 \text{ GHz} \end{aligned}$$

$$\frac{P_{med, meio2}}{P_{med, total}} = 56.2\%$$

ondas menos confinadas ao dieléctrico interior  
à medida que o modo se aproxima do corte

# Modos TM ímpares

ondas TM ímpares  $\rightarrow H_z^0 = 0$  e  $E_z^0 = \psi_{\text{ímpar}}$

$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{dE_z^0}{dy}$$

$$H_y^0 = -\frac{\gamma}{h^2} \frac{dH_z^0}{dy} = 0$$

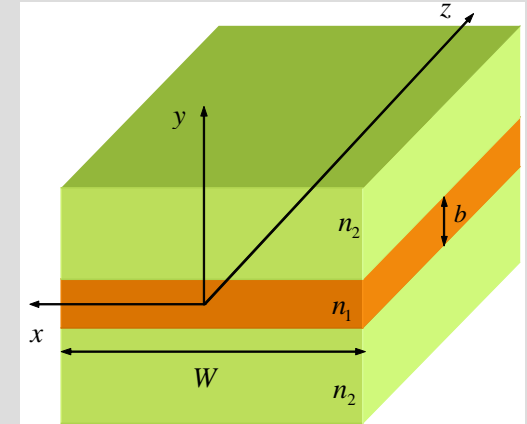
$$E_x^0 = -\frac{j\omega\mu}{h^2} \frac{dH_z^0}{dy} = 0$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{dE_z^0}{dy}$$

$$E_y^0 = \begin{cases} -\frac{j\beta}{\nu} A \sin\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ -\frac{j\beta}{h_1} A \cos(h_1 y), & |y| < \frac{b}{2} \\ -\frac{j\beta}{\nu} A \sin\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$$H_x^0 = \begin{cases} \frac{j\omega\epsilon_2}{\nu} A \sin\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ \frac{j\omega\epsilon_1}{h_1} A \cos(h_1 y), & |y| < \frac{b}{2} \\ \frac{j\omega\epsilon_2}{\nu} A \sin\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$$\frac{j\omega\epsilon_2}{\nu} A \sin\left(\frac{h_1 b}{2}\right) = \frac{j\omega\epsilon_1}{h_1} A \cos\left(\frac{h_1 b}{2}\right)$$



$$\psi_{\text{ímpar}} = \begin{cases} A \sin\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ A \sin(h_1 y), & |y| \leq \frac{b}{2} \\ A \sin\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

**condições fronteira**  
 $H_x$  contínuo em  $y = \pm b/2$

# Modos TM ímpares – relação característica

$$\frac{j\omega\epsilon_2}{\nu} A \sin\left(\frac{h_1 b}{2}\right) = \frac{j\omega\epsilon_1}{h_1} A \cos\left(\frac{h_1 b}{2}\right) \longrightarrow \nu = \frac{\epsilon_2}{\epsilon_1} h_1 \tan\left(\frac{h_1 b}{2}\right)$$

$$n = \sqrt{\mu_r \epsilon_r} = \sqrt{\epsilon_r}$$

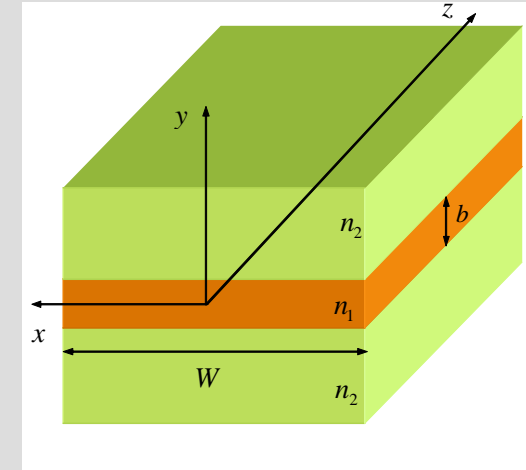
se  $\mu_r = 1$

$$\nu = h_1 \left(\frac{n_2}{n_1}\right)^2 \tan\left(\frac{h_1 b}{2}\right)$$

$$\nu = \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2}$$

relação característica

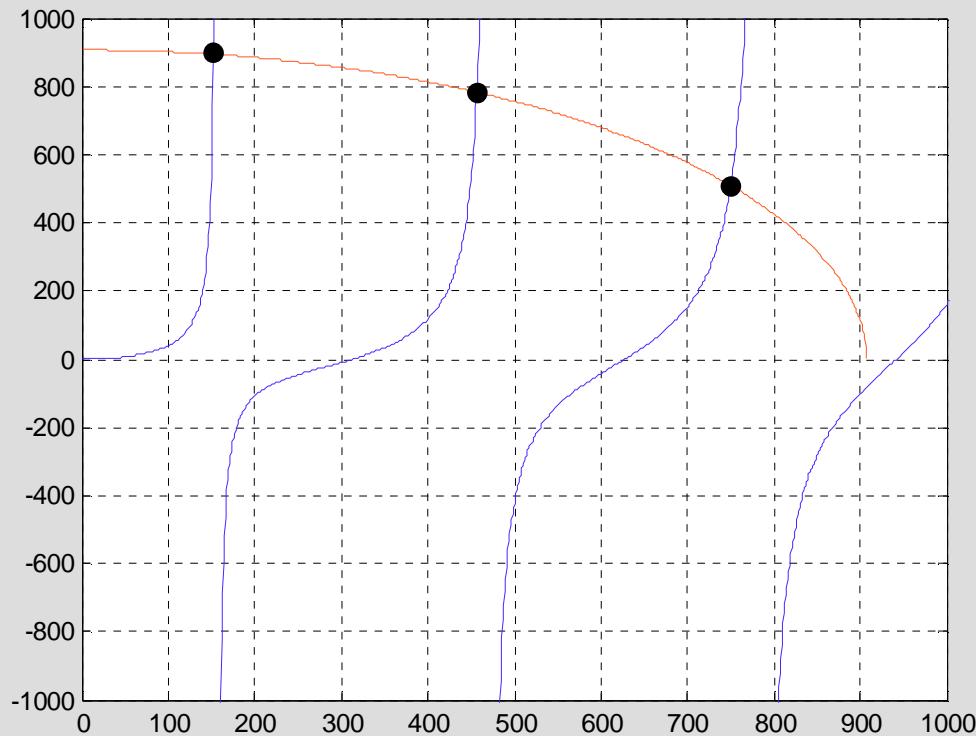
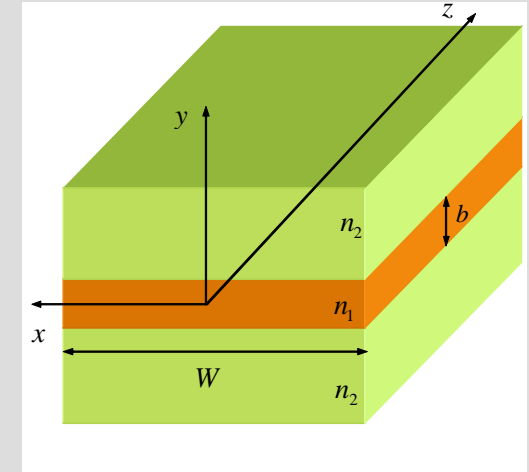
$$\left(\frac{n_1}{n_2}\right)^2 \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2} = h_1 \tan\left(\frac{h_1 b}{2}\right)$$



# Modos TM ímpares – exemplo

$f = 25 \text{ GHz}$   
 $\epsilon_2 = \epsilon_0 \rightarrow n_2 = 1$   
 $\epsilon_1 = 4\epsilon_0 \rightarrow n_1 = 2$   
 $b = 2 \text{ cm}$

$$\sqrt{\frac{25\pi^2 \times 10^4}{3} - h_1^2} = \frac{1}{4} h_1 \tan(0.01 h_1)$$



$h_{1,1} = 152.8$	$\rightarrow$	$v_1 = 893.9$
$h_{1,2} = 456.8$	$\rightarrow$	$v_2 = 783.5$
$h_{1,3} = 750.1$	$\rightarrow$	$v_3 = 509.7$

$$v = \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2}$$



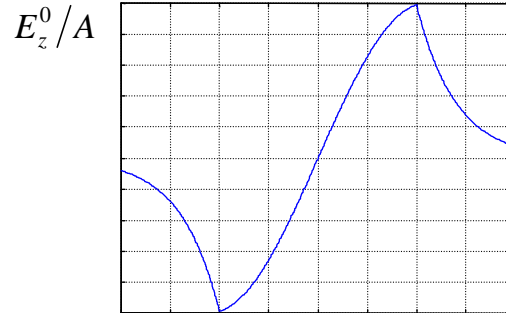
# Variação do modo $TM_1$ ímpar com a frequência– exemplo

**modo  $TM_1$**

$$f = 8\text{GHz}$$

$$h_{1,1} = 143$$

$$\nu_1 = 252.5$$

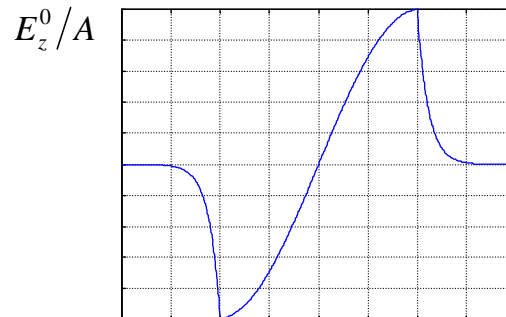


**modo  $TM_1$**

$$f = 25\text{GHz}$$

$$h_{1,1} = 152.8$$

$$\nu_1 = 893.9$$

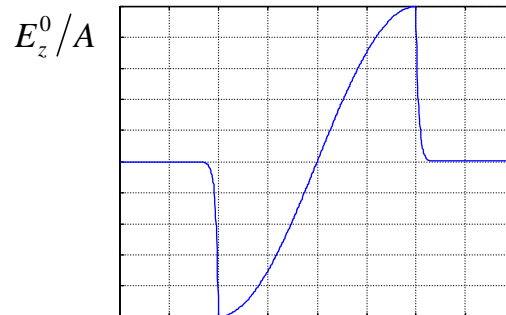


**modo  $TM_1$**

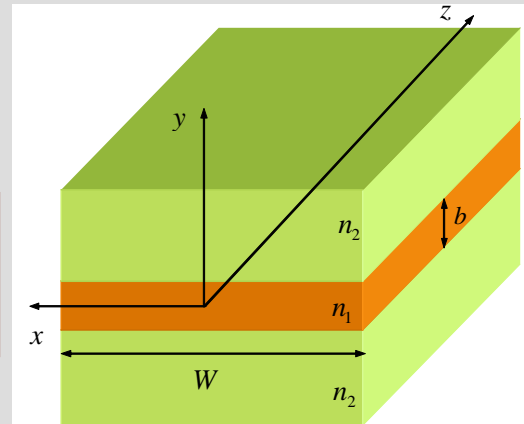
$$f = 100\text{GHz}$$

$$h_{1,1} = 156.0$$

$$\nu_1 = 3624.2$$



**taxa de decaimento  
diminui com a  
diminuição da frequência**



**exemplo**  $\epsilon_2 = \epsilon_0$   
 $\epsilon_1 = 4\epsilon_0$   
 $b = 2\text{ cm}$

# Modos TM ímpar – frequência de corte

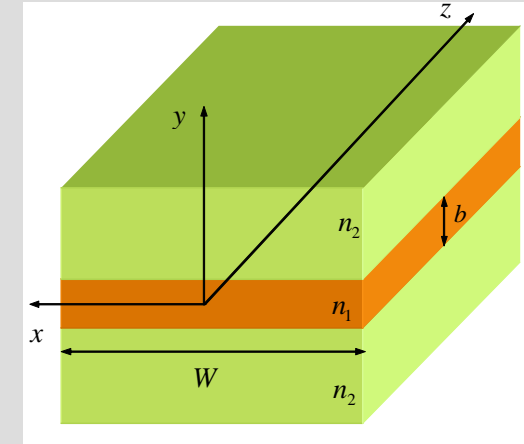
condição de corte:  $\nu = 0$

$$\nu = h_1 \left( \frac{n_2}{n_1} \right)^2 \tan\left( \frac{h_1 b}{2} \right) = 0 \longrightarrow \tan\left( \frac{h_1 b}{2} \right) = 0 \longrightarrow \frac{h_1 b}{2} = (n-1)\pi, \quad n = 1, 2, \dots$$

$$\nu^2 = -\gamma^2 - \left( \frac{\omega}{c} n_2 \right)^2 = 0 \longrightarrow \gamma^2 = -\left( \frac{\omega}{c} n_2 \right)^2$$

$$h_1^2 = \gamma^2 + \left( \frac{\omega}{c} n_1 \right)^2$$

$$h_1^2 = \left( \frac{\omega}{c} \right)^2 (n_1^2 - n_2^2) \longrightarrow h_1 = \frac{\omega}{c} \sqrt{n_1^2 - n_2^2}$$



$$(f_c)_{\text{TM ímpar}} = \frac{(n-1)c}{b\sqrt{n_1^2 - n_2^2}}, \quad n = 1, 2, \dots$$

## Modos TM ímpar – modo dominante

$$(f_c)_{\text{TM ímpar}} = \frac{(n-1)c}{b\sqrt{n_1^2 - n_2^2}}, \quad n=1, 2, \dots$$

modo TM ímpar dominante → modo  $\text{TM}_1$  ímpar

$$(f_c)_{\text{TM}_1 \text{ ímpar}} = 0$$

→ está sempre presente

modo TM ímpar seguinte → modo  $\text{TM}_2$  ímpar

$$(f_c)_{\text{TM}_2 \text{ ímpar}} = \frac{c}{b\sqrt{n_1^2 - n_2^2}}$$

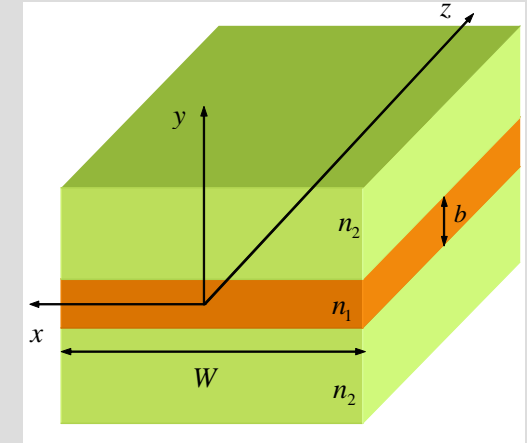
→ propagação de apenas um modo TM ímpar se  $(f_c)_{\text{TM}_2 \text{ par}} > f$

nota: modo dominante TM par → modo  $\text{TM}_1$

$$(f_c)_{\text{TM}_1 \text{ par}} = \frac{c}{2b\sqrt{n_1^2 - n_2^2}}$$

→

1º                      2º                      3º  
 $\text{TM}_1$  ímpar     $\text{TM}_1$  par     $\text{TM}_2$  ímpar    ...



# Modos TE pares

ondas TE pares  $\rightarrow E_z^0 = 0$  e  $H_z^0 = \psi_{\text{par}}$

$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{dE_z^0}{dy} = 0$$

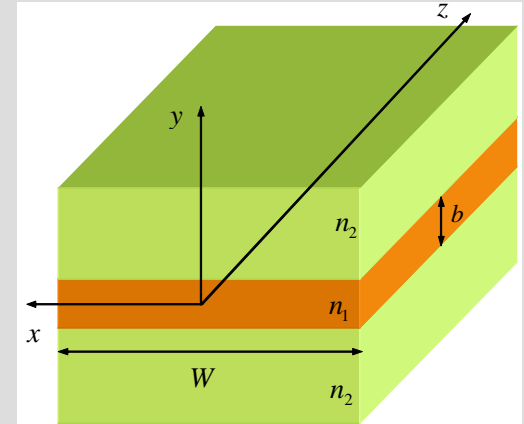
$$H_y^0 = -\frac{\gamma}{h^2} \frac{dH_z^0}{dy}$$

$$E_x^0 = -\frac{j\omega\mu}{h^2} \frac{dH_z^0}{dy}$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{dE_z^0}{dy} = 0$$

$$H_y^0 = \begin{cases} -\frac{j\beta}{\nu} B \cos\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ \frac{j\beta}{h_1} B \sin(h_1 y), & |y| < \frac{b}{2} \\ \frac{j\beta}{\nu} B \cos\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$$E_x^0 = \begin{cases} -\frac{j\omega\mu_0}{\nu} B \cos\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ \frac{j\omega\mu_0}{h_1} B \sin(h_1 y), & |y| < \frac{b}{2} \\ \frac{j\omega\mu_0}{\nu} B \cos\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$



$$\psi_{\text{par}} = \begin{cases} B \cos\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ B \cos(h_1 y), & |y| \leq \frac{b}{2} \\ B \cos\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$$-\frac{j\omega\mu_0}{\nu} B \cos\left(\frac{h_1 b}{2}\right) = \frac{j\omega\mu_0}{h_1} B \sin\left(\frac{h_1 b}{2}\right)$$

**condições fronteira**  
 $E_x$  contínuo em  $y = \pm b/2$

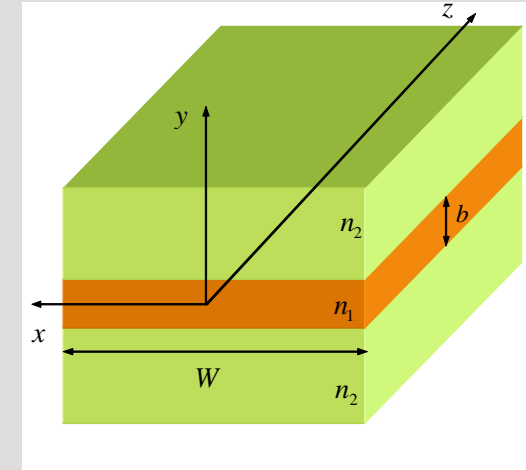
## Modos TE pares – relação característica e frequência de corte

$$-\frac{j\omega\mu_0}{\nu} B \cos\left(\frac{h_1 b}{2}\right) = \frac{j\omega\mu_0}{h_1} B \sin\left(\frac{h_1 b}{2}\right) \longrightarrow \nu = -h_1 \cot\left(\frac{h_1 b}{2}\right)$$

$$\nu = \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2}$$

$$\sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2} = -h_1 \cot\left(\frac{h_1 b}{2}\right)$$

$$(f_c)_{\text{TE par}} = \frac{\left(n - \frac{1}{2}\right)c}{b\sqrt{n_1^2 - n_2^2}}, \quad n = 1, 2, \dots$$



**nota:**

$$(f_c)_{\text{TE par}} = (f_c)_{\text{TM par}}$$

# Modos TE ímpares

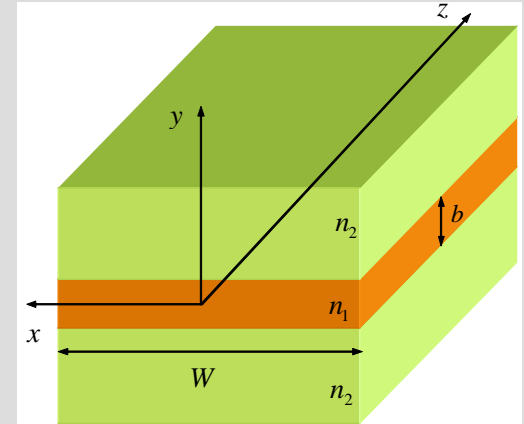
ondas TE ímpares  $\rightarrow E_z^0 = 0$  e  $H_z^0 = \psi_{\text{ímpar}}$

$$H_x^0 = \frac{j\omega\epsilon}{h^2} \frac{dE_z^0}{dy} = 0$$

$$H_y^0 = -\frac{\gamma}{h^2} \frac{dH_z^0}{dy}$$

$$E_x^0 = -\frac{j\omega\mu}{h^2} \frac{dH_z^0}{dy}$$

$$E_y^0 = -\frac{\gamma}{h^2} \frac{dE_z^0}{dy} = 0$$



$$H_y^0 = \begin{cases} -\frac{j\beta}{\nu} A \sin\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ -\frac{j\beta}{h_1} A \cos(h_1 y), & |y| < \frac{b}{2} \\ -\frac{j\beta}{\nu} A \sin\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$$E_x^0 = \begin{cases} -\frac{j\omega\mu_0}{\nu} A \sin\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ -\frac{j\omega\mu_0}{h_1} A \cos(h_1 y), & |y| < \frac{b}{2} \\ -\frac{j\omega\mu_0}{\nu} A \sin\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$$\psi_{\text{ímpar}} = \begin{cases} A \sin\left(\frac{h_1 b}{2}\right) e^{-\nu\left(y-\frac{b}{2}\right)}, & y > \frac{b}{2} \\ A \sin(h_1 y), & |y| \leq \frac{b}{2} \\ A \sin\left(\frac{h_1 b}{2}\right) e^{\nu\left(y+\frac{b}{2}\right)}, & y < -\frac{b}{2} \end{cases}$$

$$-\frac{j\omega\mu_0}{\nu} A \sin\left(\frac{h_1 b}{2}\right) = -\frac{j\omega\mu_0}{h_1} A \cos\left(\frac{h_1 b}{2}\right)$$

**condições fronteira**  
 $E_x$  contínuo em  $y = \pm b/2$

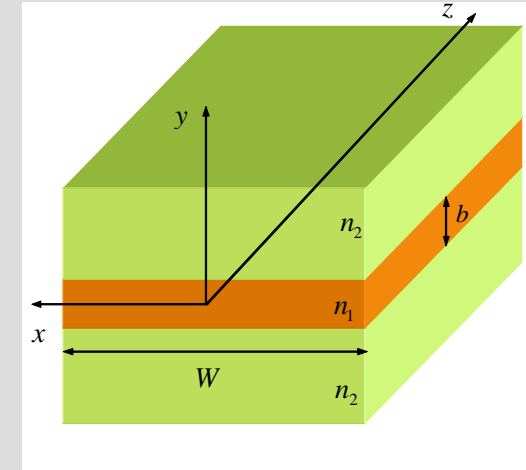
# Modos TE ímpares – relação característica e frequência de corte

$$-\frac{j\omega\mu_0}{\nu} A \sin\left(\frac{h_1 b}{2}\right) = -\frac{j\omega\mu_0}{h_1} A \cos\left(\frac{h_1 b}{2}\right) \rightarrow \nu = h_1 \tan\left(\frac{h_1 b}{2}\right)$$

$$\nu = \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2}$$

$$\sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2} = -h_1 \tan\left(\frac{h_1 b}{2}\right)$$

$$(f_c)_{\text{TE ímpar}} = \frac{(n-1)c}{b\sqrt{n_1^2 - n_2^2}}, \quad n = 1, 2, \dots$$



nota:

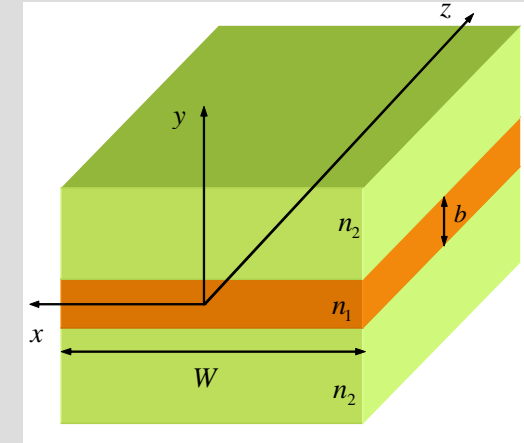
$$(f_c)_{\text{TE ímpar}} = (f_c)_{\text{TM ímpar}}$$

# Guias dieléctricos planares – resumo

MODOS		RELAÇÃO CARACTERÍSTICA	FREQUÊNCIA DE CORTE
PARES	TM	$\nu = -\left(\frac{n_2}{n_1}\right)^2 h_1 \cot\left(\frac{h_1 b}{2}\right)$	$f_c = \frac{\left(n - \frac{1}{2}\right)c}{b\sqrt{n_1^2 - n_2^2}}$
	TE	$\nu = -h_1 \cot\left(\frac{h_1 b}{2}\right)$	
ÍMPARES	TM	$\nu = \left(\frac{n_2}{n_1}\right)^2 h_1 \tan\left(\frac{h_1 b}{2}\right)$	$f_c = \frac{(n-1)c}{b\sqrt{n_1^2 - n_2^2}}$
	TE	$\nu = h_1 \tan\left(\frac{h_1 b}{2}\right)$	

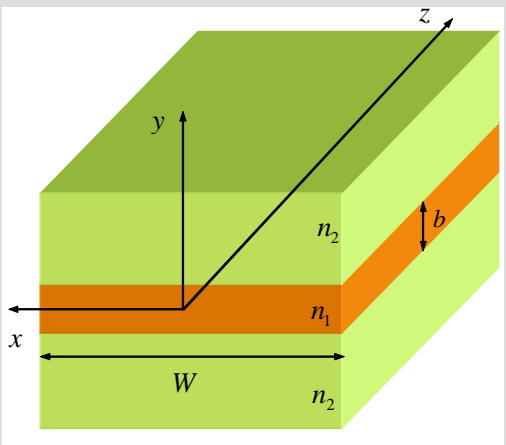
$$\nu = \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2}$$

$n = 1, 2, \dots$



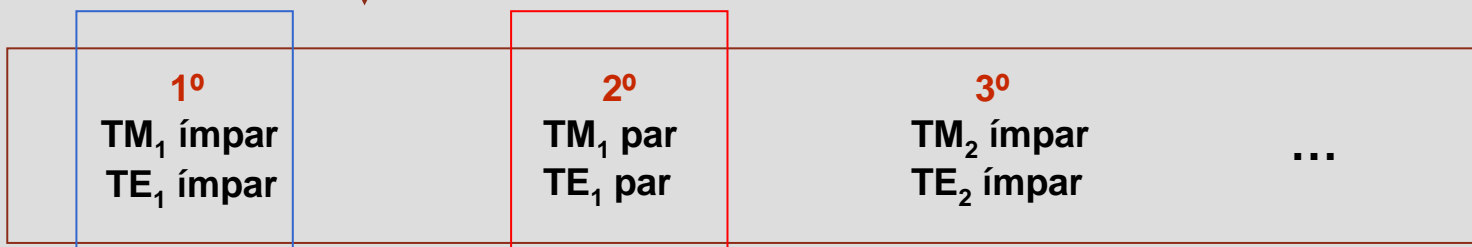


# Guias dieléctricos planares – modos dominantes



modos pares  $\rightarrow f_c = \frac{\left(n - \frac{1}{2}\right)c}{b\sqrt{n_1^2 - n_2^2}} \quad n = 1, 2, \dots$

modos ímpares  $\rightarrow f_c = \frac{(n-1)c}{b\sqrt{n_1^2 - n_2^2}}$

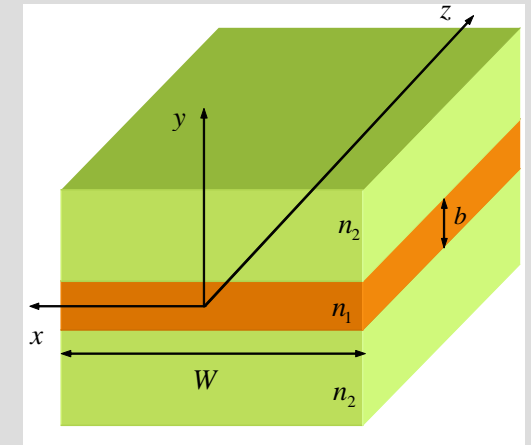
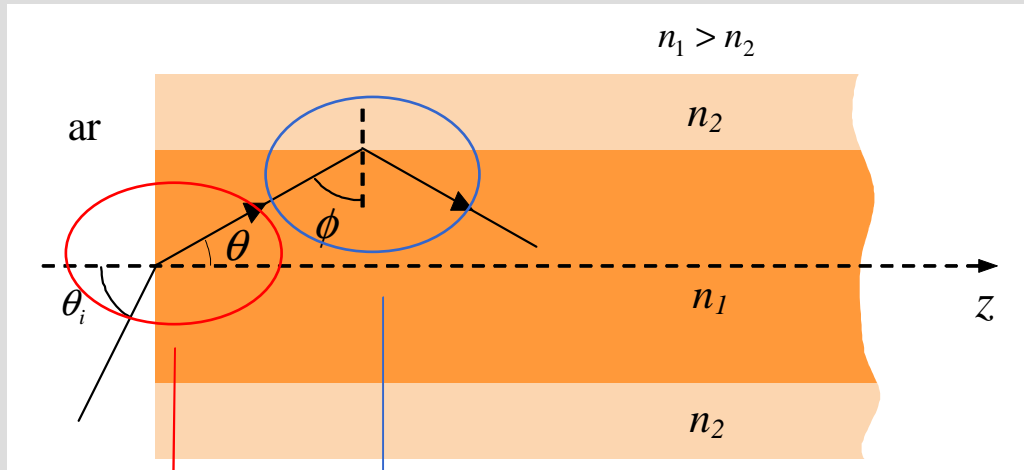


modos dominantes  
 $f_c = 0$   
 estão sempre presentes

$$f_c = \frac{c}{2b\sqrt{n_1^2 - n_2^2}}$$

regime “monomodo”:  $f < \frac{c}{2b\sqrt{n_1^2 - n_2^2}}$   
 exemplo  
 $n_2 = 1; n_1 = 2$   
 $b = 2 \text{ cm}$   $\rightarrow f < 2.17 \text{ GHz}$

# Guias dielétricos planares – reflexão interna total



$$n_1 \cos(\phi) = \sin(\theta_i)$$

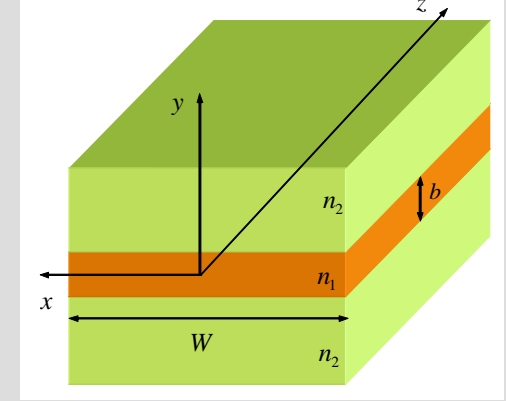
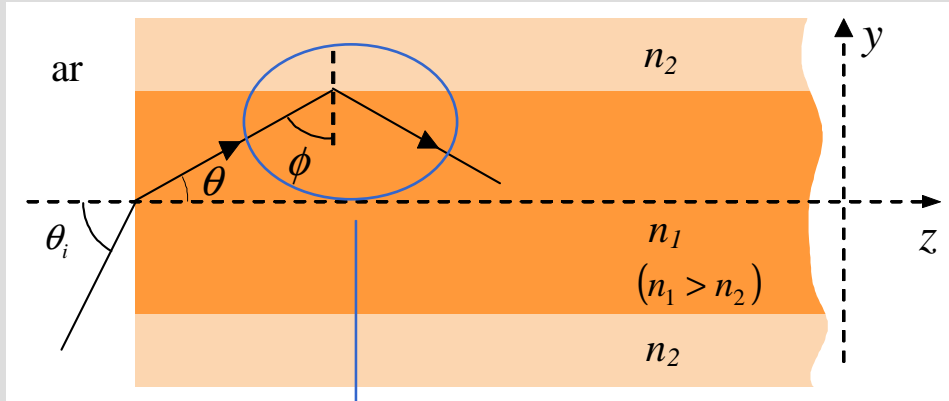
**reflexão interna total**  $\rightarrow \phi \geq \sin^{-1}\left(\frac{n_2}{n_1}\right) \rightarrow \sin \phi \geq \frac{n_2}{n_1} \rightarrow \cos \phi \leq \sqrt{1 - \left(\frac{n_2}{n_1}\right)^2}$

$$\sin(\theta_i) \leq \sqrt{n_1^2 - n_2^2}$$

## notas

1. abertura numérica (NA)  $\rightarrow \sqrt{n_1^2 - n_2^2}$
2. ângulo de aceitação  $\rightarrow \theta_A = \sin^{-1}(NA)$

# Reflexão interna total – campos no meio 2



$$n_1 \sin \phi = n_2 \sin \phi_t \rightarrow \sin \phi_t = \frac{n_1}{n_2} \sin \phi \rightarrow \cos(\theta_t) = \pm \sqrt{1 - \left(\frac{n_1}{n_2} \sin \phi\right)^2} = \pm j \sqrt{\left(\frac{n_1}{n_2} \sin \phi\right)^2 - 1}$$

$$\sin \phi \geq \frac{n_2}{n_1} \rightarrow \sin \phi_t \geq 1$$

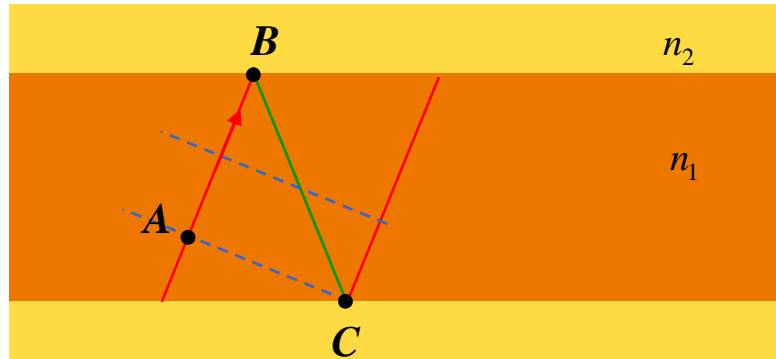
$$\cos \phi_t = -j \sqrt{\left(\frac{n_1}{n_2} \sin \phi\right)^2 - 1}$$

meio 2:

$$e^{-j\vec{k}_2 \cdot \vec{r}} = e^{-j|\vec{k}_2| \hat{a}_{nt} \cdot \vec{r}} = e^{-j|\vec{k}_2| (\cos \phi_t y + \sin \phi_t z)} = e^{-j|\vec{k}_2| n_1 \sin \phi / n_2 z} e^{\pm j|\vec{k}_2| \sqrt{(n_1 \sin \phi / n_2)^2 - 1} y}$$

$\hat{a}_{nt} = \cos \phi_t \hat{y} + \sin \phi_t \hat{z}$   
 onda a propagar-se segundo +z  
 decaimento exponencial

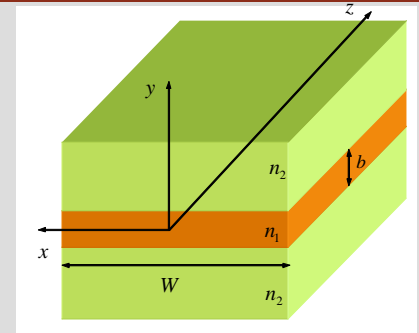
# Reflexão interna total – modos permitidos



--- frentes de onda

$C \rightarrow$  mesmo após a reflexão

$A$  e  $C$  estão na mesma frente de onda



$$\angle C - \angle A = \text{múltiplo de } 2\pi$$

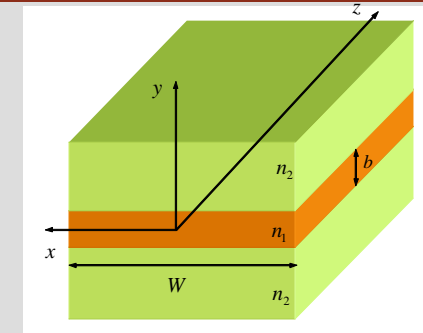
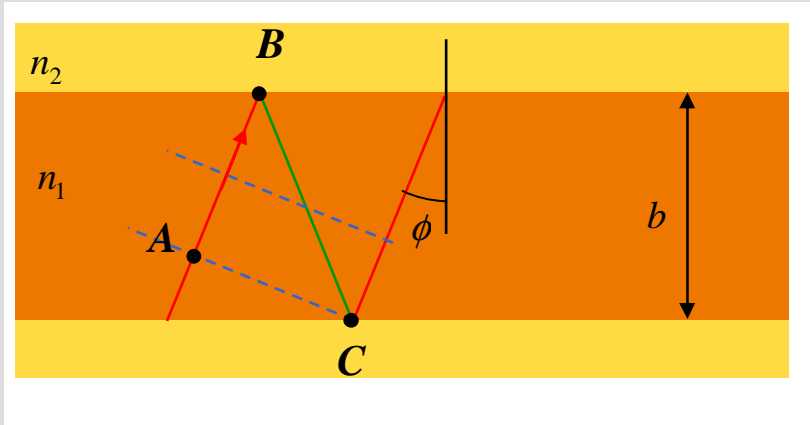
propagação ao longo de distância  $l \rightarrow$  onda adquire fase  $-k_1 l = -\frac{\omega}{c} n_1 l$

em cada reflexão  $\rightarrow$  onda adquire fase  $\angle \Gamma$

$$-\frac{\omega}{c} n_1 (l_{AB} + l_{BC}) + 2\angle \Gamma = 2n\pi$$

$$\angle C = \angle A - k_1 l_{AB} + \angle \Gamma - k_1 l_{BC} + \angle \Gamma = \angle A + 2\angle \Gamma - k_1 (l_{AB} + l_{BC})$$

# Reflexão interna total – modos permitidos



$$b = l_{BC} \cos(\phi) \longrightarrow l_{BC} = \frac{b}{\cos(\phi)}$$

$$l_{AB} = l_{BC} \cos(2\phi) = \frac{b}{\cos(\phi)} \cos(2\phi)$$

$$\longrightarrow l_{AB} + l_{BC} = \frac{b}{\cos(\phi)} (\cos(2\phi) + 1) = 2b \cos(\phi)$$

$$-\frac{\omega}{c} n_1 (l_{AB} + l_{BC}) + 2\angle\Gamma = 2n\pi$$

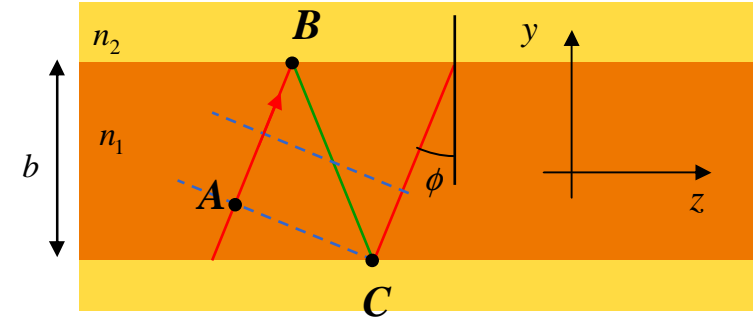
$$\longrightarrow -\frac{\omega}{c} n_1 b \cos(\phi) + \angle\Gamma = n\pi$$

# Reflexão interna total – fase do coeficiente de reflexão

incidência oblíqua:

$$\Gamma_{\perp} = \frac{n_1 \cos \phi_i - n_2 \cos \phi_t}{n_1 \cos \phi_i + n_2 \cos \phi_t}$$

$$\Gamma_{\parallel} = \frac{n_1 \cos \phi_t - n_2 \cos \phi_i}{n_1 \cos \phi_t + n_2 \cos \phi_i}$$



$$\cos \phi_t = -j \sqrt{\left(\frac{n_1}{n_2} \sin \phi\right)^2 - 1}$$

nota

polarização  $\perp \rightarrow \vec{E} = E_x \hat{x} \rightarrow E_z = 0 \rightarrow$  onda TE

polarização  $\parallel \rightarrow \vec{H} = H_x \hat{x} \rightarrow H_z = 0 \rightarrow$  onda TM

$$\Gamma_{\perp} = \frac{n_1 \cos(\phi) + j \sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_1 \cos(\phi) - j \sqrt{n_1^2 \sin^2(\phi) - n_2^2}}$$

$$\Gamma_{\parallel} = \frac{-n_2^2 \cos(\phi) - j n_1 \sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_2^2 \cos(\phi) - j n_1 \sqrt{n_1^2 \sin^2(\phi) - n_2^2}}$$

$$\angle \Gamma_{\perp} = 2 \tan^{-1} \left( \frac{\sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_1 \cos(\phi)} \right)$$

$\leftarrow \angle \Gamma_{TE}$

$$\angle \Gamma_{\parallel} = \pi + 2 \tan^{-1} \left( \frac{n_1 \sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_2^2 \cos(\phi)} \right)$$

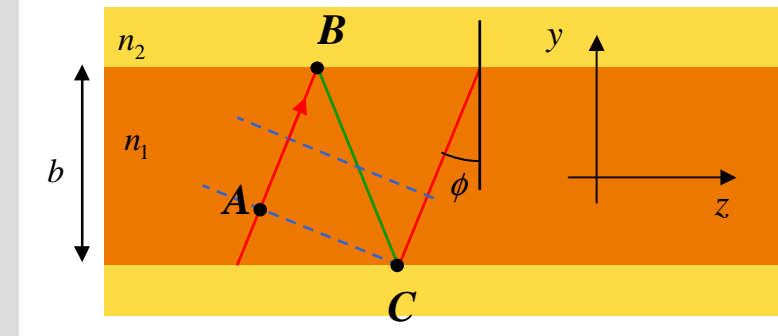
$\leftarrow \angle \Gamma_{TM}$

# Reflexão interna total – equação para os modos permitidos

$$-\frac{\omega}{c}n_1 b \cos(\phi) + \angle\Gamma = n\pi \quad n = 1, 2, \dots$$

$$\angle\Gamma_{TE} = 2 \tan^{-1} \left( \frac{\sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_1 \cos(\phi)} \right)$$

$$\angle\Gamma_{TM} = \pi + 2 \tan^{-1} \left( \frac{n_1 \sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_2^2 \cos(\phi)} \right)$$



$$-\frac{\omega}{c}n_1 b \cos \phi + 2 \tan^{-1} \left( \frac{\sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_1 \cos(\phi)} \right) = n\pi \quad , \quad \text{modos TE}$$

$$-\frac{\omega}{c}n_1 b \cos \phi + 2 \tan^{-1} \left( \frac{n_1 \sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_2^2 \cos(\phi)} \right) = (n-1)\pi \quad , \quad \text{modos TM}$$

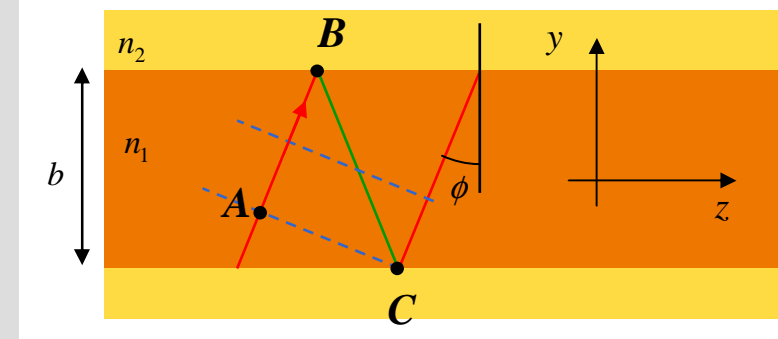
← só se propagam os modos que satisfazem estas equações

# Reflexão interna total – equações características

## óptica geométrica:

$$-\frac{\omega}{c}n_1b \cos \phi + 2 \tan^{-1} \left( \frac{\sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_1 \cos(\phi)} \right) = n\pi, \quad \text{modos TE}$$

$$-\frac{\omega}{c}n_1b \cos \phi + 2 \tan^{-1} \left( \frac{n_1 \sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_2^2 \cos(\phi)} \right) = (n-1)\pi, \quad \text{modos TM}$$



$n = 1, 2, \dots$

## análise modal:

$$\sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2} = -h_1 \cot\left(\frac{h_1 b}{2}\right) \quad \text{modos TE pares}$$

$$\sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2} = h_1 \tan\left(\frac{h_1 b}{2}\right) \quad \text{modos TE ímpares}$$

$$\sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2} = -\left(\frac{n_2}{n_1}\right)^2 h_1 \cot\left(\frac{h_1 b}{2}\right) \quad \text{modos TM pares}$$

$$\sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2} = \left(\frac{n_2}{n_1}\right)^2 h_1 \tan\left(\frac{h_1 b}{2}\right) \quad \text{modos TM ímpares}$$

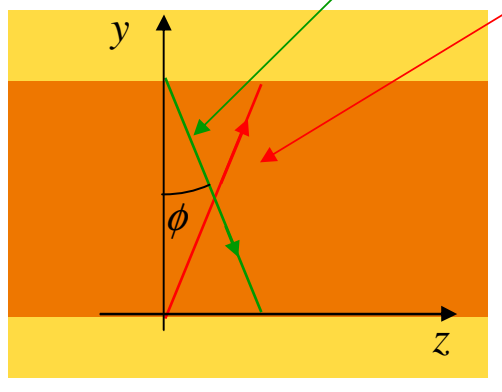


# Reflexão interna total – equivalência entre as duas abordagens

modos TE ímpares:  $E_x = -\frac{j\omega\mu_0}{h_1} A \cos(h_1 y) e^{-j\beta z}$  ← no núcleo

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$E_x = -\frac{j\omega\mu_0}{2h_1} A \left[ e^{-j(-h_1 y + \beta z)} + e^{-j(h_1 y + \beta z)} \right]$$



$$\pm h_1 y + \beta z = k_1 (\pm \cos \phi y + \sin \phi z)$$

$$h_1 = \frac{\omega}{c} n_1 \cos \phi$$

$$\beta = \frac{\omega}{c} n_1 \sin \phi$$

**análise modal**

$$v = -h_1 \cot\left(\frac{h_1 b}{2}\right) \quad \text{TE pares}$$

$$v = h_1 \tan\left(\frac{h_1 b}{2}\right) \quad \text{TE ímpares}$$

**óptica geométrica**

$$-\frac{\omega}{c} n_1 b \cos \phi + 2 \tan^{-1} \left( \frac{\sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_1 \cos(\phi)} \right) = n\pi$$

$$-h_1 b + 2 \tan^{-1} \left( \frac{\sqrt{\beta^2 - (\omega n_2 / c)^2}}{h_1} \right) = n\pi$$

**equação válida para modos TE pares e ímpares**

nota

modos TE pares:  $E_x = \frac{j\omega\mu_0}{h_1} B \sin(h_1 y) e^{-j\beta z} = \frac{j\omega\mu_0}{2h_1} B \left[ e^{-j(-h_1 y + \beta z)} - e^{-j(h_1 y + \beta z)} \right]$

# Reflexão interna total – equivalência entre as duas abordagens

$$-h_1 b + 2 \tan^{-1} \left( \frac{\sqrt{\beta^2 - \left(\frac{\omega}{c} n_2\right)^2}}{h_1} \right) = n\pi$$

$$\nu^2 = \beta^2 - \left(\frac{\omega}{c} n_2\right)^2$$

$$-h_1 b + 2 \tan^{-1} \left( \frac{\nu}{h_1} \right) = n\pi$$

$$\nu = h_1 \tan \left( n \frac{\pi}{2} + \frac{h_1 b}{2} \right)$$

$$\tan \left( n \frac{\pi}{2} + \theta \right) = \begin{cases} \tan \theta, & n = 2m \\ -\cot \theta, & n = 2m + 1 \end{cases}$$

$$\nu = \begin{cases} h_1 \tan \left( \frac{h_1 b}{2} \right), & n \text{ par} \\ -h_1 \cot \left( \frac{h_1 b}{2} \right), & n \text{ ímpar} \end{cases}$$

equação obtida usando óptica geométrica é equivalente às equações características da análise modal

**modos TE**  
**análise modal**

$$\nu = -h_1 \cot \left( \frac{h_1 b}{2} \right) \quad \text{pares}$$

$$\nu = h_1 \tan \left( \frac{h_1 b}{2} \right) \quad \text{ímpares}$$

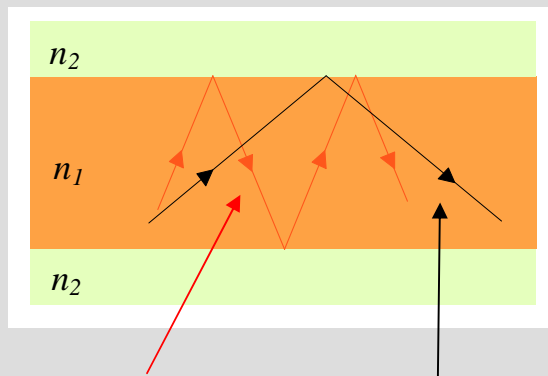
**óptica geométrica**

$$-\frac{\omega}{c} n_1 b \cos \phi + 2 \tan^{-1} \left( \frac{\sqrt{n_1^2 \sin^2(\phi) - n_2^2}}{n_1 \cos(\phi)} \right) = n\pi$$

**análise modal** → diferentes modos propagam-se com velocidades diferentes  $\left( v_g = \frac{1}{\sqrt{\mu \epsilon}} \sqrt{1 - \left( \frac{f_c}{f} \right)^2} \right)$

→ componentes do sinal propagadas nos diferentes modos são recebidas em instantes de tempo diferentes → **distorção do sinal**

**óptica geométrica** → modos representados por vectores que indicam a direcção de propagação de **ondas planas** → propagam-se no meio 1 com velocidade  $v = c/n_1$



distância maior a percorrer

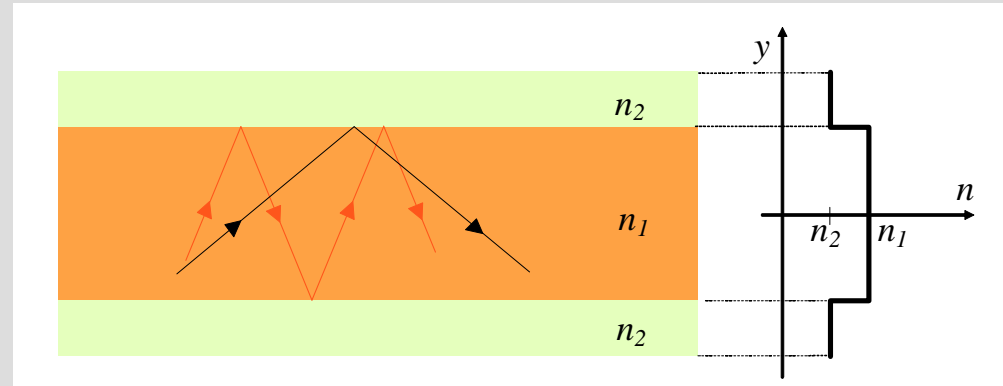
distância menor a percorrer

$$t = l/v$$

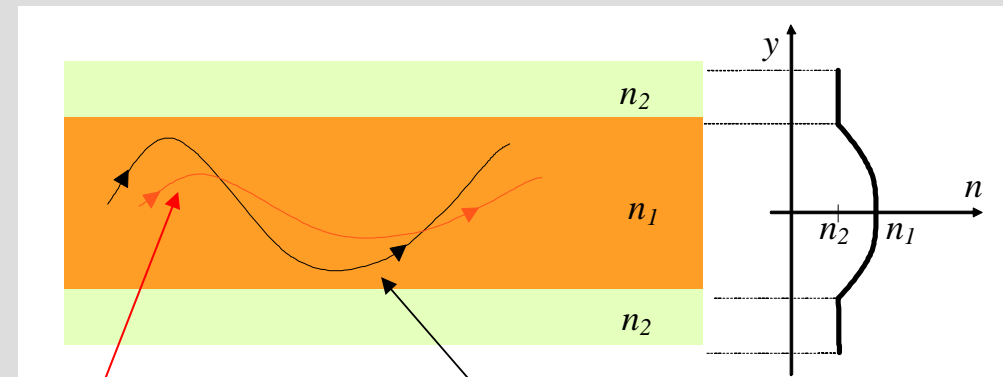
→ tempos diferentes → **distorção**

# Guias com índice de refração gradual

“step-index”



“graded-index”

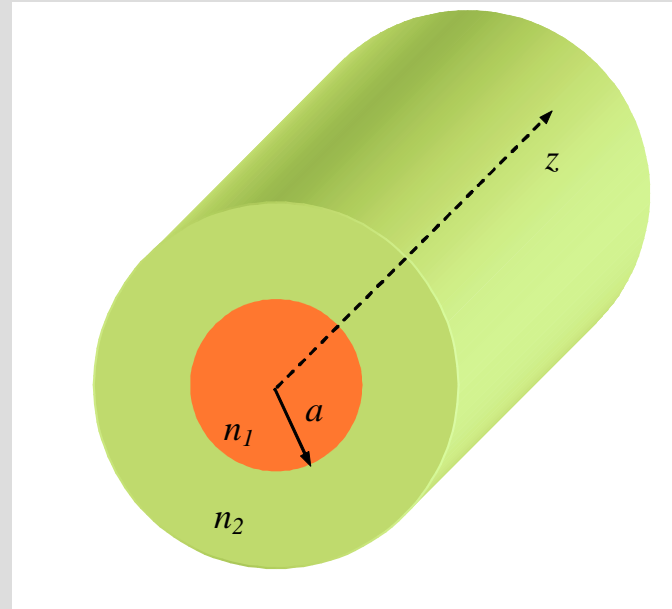


$v = c/n$  → velocidade é menor na região central

tempos (aproximadamente) iguais ←

distância e velocidade menores

distância e velocidade maiores



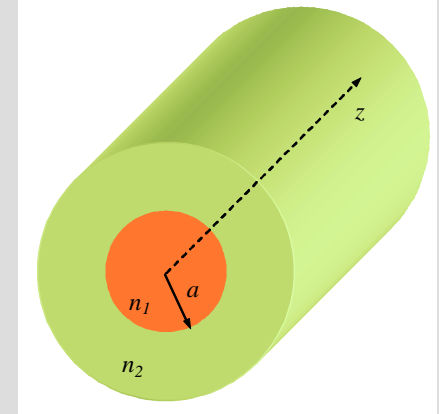
**materiais sem perdas**

**comprimento infinito → propagação segundo  $+z$**

**geometria circular → coordenadas cilíndricas**

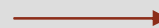
coordenadas cilíndricas

$$\vec{E} = (E_r^0 \hat{r} + E_\phi^0 \hat{\phi} + E_z^0 \hat{z}) e^{-\gamma z}$$
$$\vec{H} = (H_r^0 \hat{r} + H_\phi^0 \hat{\phi} + H_z^0 \hat{z}) e^{-\gamma z}$$



condições fronteira

$E_{\text{tan}}$  contínuo e  $H_{\text{tan}}$  contínuo



$E_z^0$  e  $E_\phi^0$  contínuos em  $r = a$   
 $H_z^0$  e  $H_\phi^0$  contínuos em  $r = a$

## 1. resolver

$$\nabla_{r\phi}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\nabla_{r\phi}^2 H_z^0 + h^2 H_z^0 = 0$$

$$\nabla_{r\phi}^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

$$h^2 = \gamma^2 + \omega^2 \mu \epsilon$$

$$h^2 = \begin{cases} \gamma^2 + \left( \frac{\omega}{c} n_1 \right)^2, & r \leq a \\ \gamma^2 + \left( \frac{\omega}{c} n_2 \right)^2, & r > a \end{cases}$$

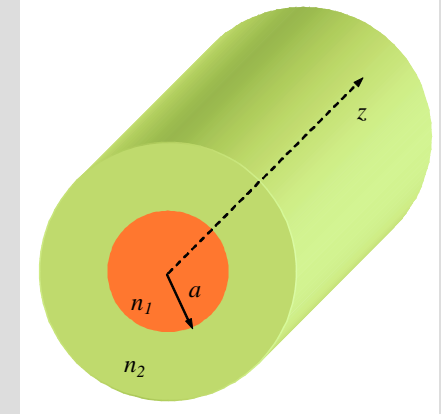
## 2. determinar (se $h \neq 0$ )

$$H_r^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial r} - \frac{j\omega\epsilon}{r} \frac{\partial E_z^0}{\partial \phi} \right)$$

$$H_\phi^0 = -\frac{1}{h^2} \left( \frac{\gamma}{r} \frac{\partial H_z^0}{\partial \phi} + j\omega\epsilon \frac{\partial E_z^0}{\partial r} \right)$$

$$E_r^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial r} + \frac{j\omega\mu}{r} \frac{\partial H_z^0}{\partial \phi} \right)$$

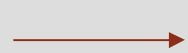
$$E_\phi^0 = -\frac{1}{h^2} \left( \frac{\gamma}{r} \frac{\partial E_z^0}{\partial \phi} - j\omega\mu \frac{\partial H_z^0}{\partial r} \right)$$



# Guias dieléctricos circulares – equação de onda

resolver

$$\begin{cases} \nabla_{r\phi}^2 E_z^0 + h^2 E_z^0 = 0 \\ \nabla_{r\phi}^2 H_z^0 + h^2 H_z^0 = 0 \end{cases}$$



$$\nabla_{r\phi}^2 \psi + h^2 \psi = 0$$

$$h^2 = \begin{cases} h_1^2, & r \leq a \\ h_2^2, & r > a \end{cases}$$

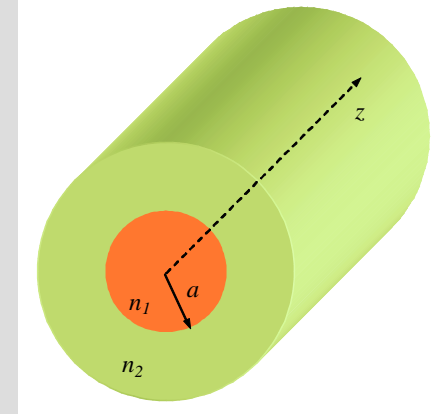
$$\psi(r, \phi) = R(r)\Phi(\phi)$$

$$\Phi(\phi) = Ae^{jn\phi}$$



$$r^2 \frac{d^2 R(r)}{dr^2} + r \frac{dR(r)}{dr} + (h^2 r^2 - n^2) R(r) = 0$$

equação diferencial de Bessel



**solução geral:**  $h^2 > 0 \rightarrow h \text{ real} \rightarrow R(r) = A J_n(hr)$  (se região de interesse incluir a origem)

funções de Bessel  
de 1ª espécie

$h^2 < 0 \rightarrow h = jv \rightarrow R(r) = B I_n(vr) + C K_n(vr)$

funções de Bessel  
modificadas de  
1ª espécie

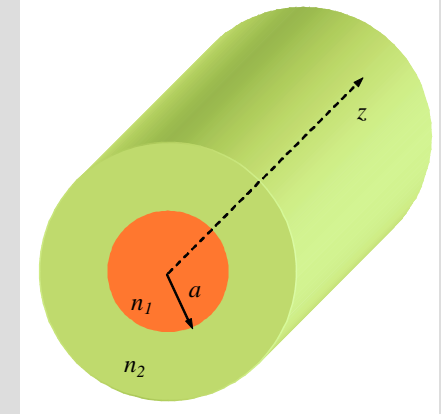
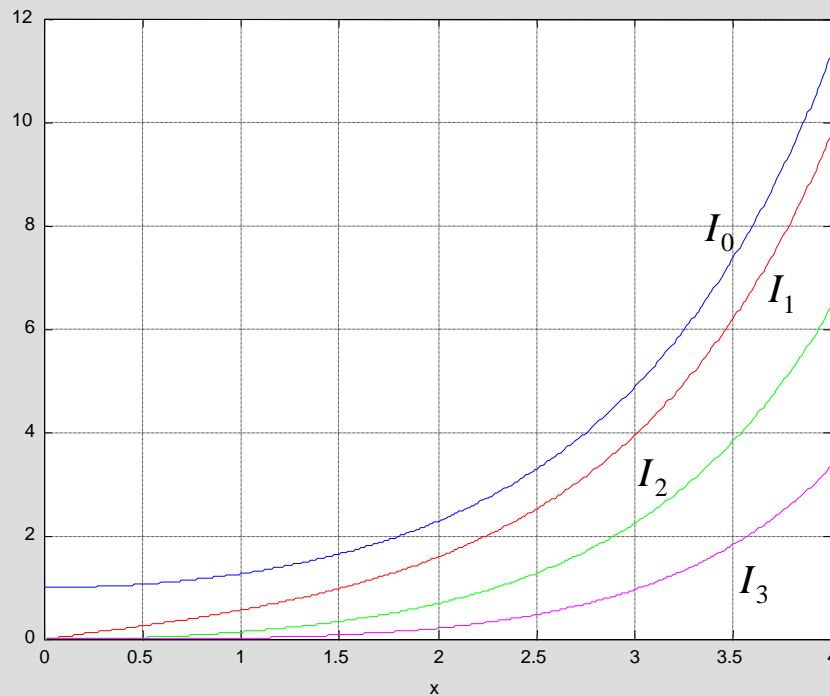
funções de Bessel  
modificadas de  
2ª espécie



# Funções de Bessel modificadas de 1ª espécie

para  $n$  inteiro

$$I_n(x) = j^{-n} J_n(jx) = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!(n+k)!}$$



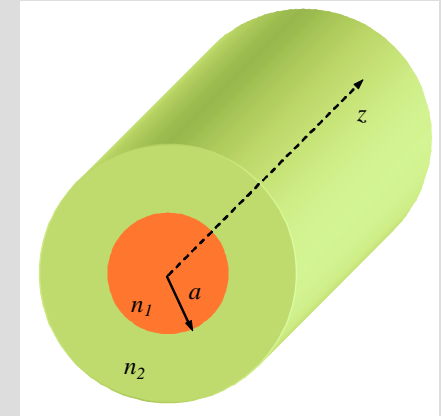
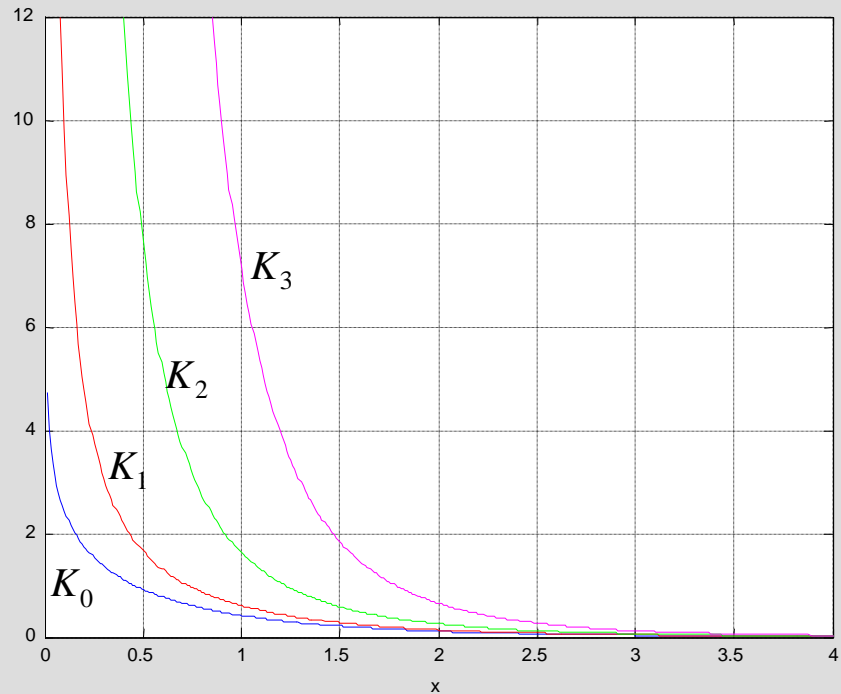
$$\lim_{x \rightarrow \infty} I_n(x) = \infty$$

$I_n$  não deve fazer parte da solução quando a região de interesse incluir o infinito

# Funções de Bessel modificadas de 2ª espécie

para  $n$  inteiro

$$K_n(x) = \lim_{p \rightarrow n} \frac{\pi}{2 \sin(p\pi)} [I_{-p}(x) - I_p(x)]$$



$$\lim_{x \rightarrow \infty} K_n(x) = 0$$

$$\lim_{x \rightarrow 0} K_n(x) = \infty$$

$K_n$  não deve fazer parte da solução quando a região de interesse incluir a origem

# Solução da equação de onda

$h$  real  $\rightarrow$

$$R(r) = AJ_n(hr)$$

$h = jv$   $\rightarrow$

$$R(r) = BI_n(vr) + CK_n(vr)$$

$$\lim_{r \rightarrow \infty} = \infty$$

$$\lim_{r \rightarrow 0} = \infty$$

$$\lim_{r \rightarrow \infty} = 0$$

onda guiada  $\rightarrow$

$h_1$  real

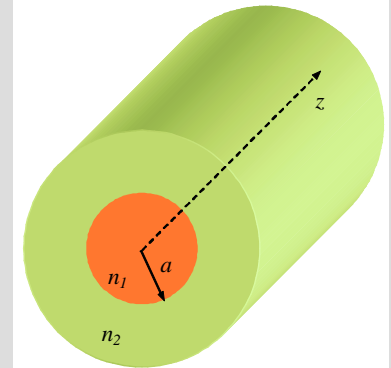
$h_2 = jv$

$$v = \sqrt{\left(\frac{\omega}{c}\right)^2 (n_1^2 - n_2^2) - h_1^2}$$

$$\beta = \sqrt{\left(\frac{\omega}{c}n_1\right)^2 - h_1^2} = \sqrt{\left(\frac{\omega}{c}n_2\right)^2 + v^2}$$

$$\psi(r, \phi) = \begin{cases} AJ_n(h_1 r)e^{jn\phi}, & r \leq a \\ BK_n(vr)e^{jn\phi}, & r > a \end{cases}$$

$$\frac{\omega}{c}n_1 > \beta > \frac{\omega}{c}n_2 \rightarrow n_1 > n_2$$



$$\psi(r, \phi) = R(r)\Phi(\phi)$$

$$\Phi(\phi) = Ae^{jn\phi}$$

# Componentes longitudinais

$$\psi(r, \phi) = \begin{cases} AJ_n(h_1 r)e^{jn\phi}, & r \leq a \\ BK_n(\nu r)e^{jn\phi}, & r > a \end{cases}$$

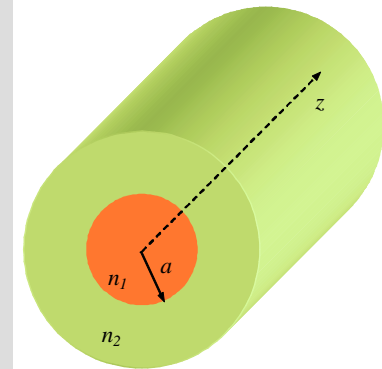


**núcleo**

$$E_z^0 = AJ_n(h_1 r)e^{jn\phi}$$
$$H_z^0 = BJ_n(h_1 r)e^{jn\phi}$$

**bainha**

$$E_z^0 = CK_n(\nu r)e^{jn\phi}$$
$$H_z^0 = DK_n(\nu r)e^{jn\phi}$$



$$\psi(r, \phi) = R(r)\Phi(\phi)$$
$$\Phi(\phi) = Ae^{jn\phi}$$

**nota:**

**modos TM**  $\rightarrow H_z^0 = 0$

**modos TE**  $\rightarrow E_z^0 = 0$

**modos HE e EH**  $\rightarrow E_z^0 \neq 0$  e  $H_z^0 \neq 0$       modos híbridos

## Componentes transversais – núcleo

$$E_z^0 = AJ_n(h_1 r)e^{jn\phi}$$

$$H_z^0 = BJ_n(h_1 r)e^{jn\phi}$$

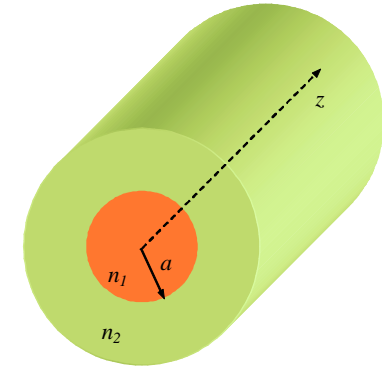
$$\downarrow h = h_1$$

$$H_r^0 = -\frac{1}{h_1^2} \left[ j\beta h_1 BJ'_n(h_1 r) + \frac{\omega \epsilon_1 n}{r} AJ_n(h_1 r) \right] e^{jn\phi}$$

$$H_\phi^0 = -\frac{1}{h_1^2} \left[ -\frac{\beta n}{r} BJ_n(h_1 r) + j\omega \epsilon_1 h_1 AJ'_n(h_1 r) \right] e^{jn\phi}$$

$$E_r^0 = -\frac{1}{h_1^2} \left[ j\beta h_1 AJ'_n(h_1 r) - \frac{\omega \mu_0 n}{r} BJ_n(h_1 r) \right] e^{jn\phi}$$

$$E_\phi^0 = -\frac{1}{h_1^2} \left[ -\frac{\beta n}{r} AJ_n(h_1 r) - j\omega \mu_0 h_1 BJ'_n(h_1 r) \right] e^{jn\phi}$$



$$H_r^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial r} - \frac{j\omega \epsilon}{r} \frac{\partial E_z^0}{\partial \phi} \right)$$

$$H_\phi^0 = -\frac{1}{h^2} \left( \frac{\gamma}{r} \frac{\partial H_z^0}{\partial \phi} + j\omega \epsilon \frac{\partial E_z^0}{\partial r} \right)$$

$$E_r^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial r} + \frac{j\omega \mu}{r} \frac{\partial H_z^0}{\partial \phi} \right)$$

$$E_\phi^0 = -\frac{1}{h^2} \left( \frac{\gamma}{r} \frac{\partial E_z^0}{\partial \phi} - j\omega \mu \frac{\partial H_z^0}{\partial r} \right)$$

## Componentes transversais – bainha

$$E_z^0 = CK_n(\nu r)e^{jn\phi}$$

$$H_z^0 = DK_n(\nu r)e^{jn\phi}$$

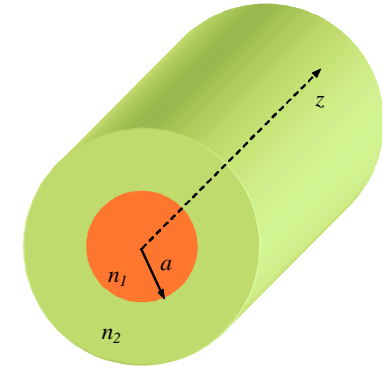
$$h = j\nu$$

$$H_r^0 = \frac{1}{\nu^2} \left[ j\beta\nu DK'_n(\nu r) + \frac{\omega\epsilon_2 n}{r} CK_n(\nu r) \right] e^{jn\phi}$$

$$H_\phi^0 = \frac{1}{\nu^2} \left[ -\frac{\beta n}{r} DK_n(\nu r) + j\omega\epsilon_2 \nu CK'_n(\nu r) \right] e^{jn\phi}$$

$$E_r^0 = \frac{1}{\nu^2} \left[ j\beta\nu CK'_n(\nu r) - \frac{\omega\mu_0 n}{r} DK_n(\nu r) \right] e^{jn\phi}$$

$$E_\phi^0 = \frac{1}{\nu^2} \left[ -\frac{\beta n}{r} CK_n(\nu r) - j\omega\mu_0 \nu DK'_n(\nu r) \right] e^{jn\phi}$$



$$H_r^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial r} - \frac{j\omega\epsilon}{r} \frac{\partial E_z^0}{\partial \phi} \right)$$

$$H_\phi^0 = -\frac{1}{h^2} \left( \frac{\gamma}{r} \frac{\partial H_z^0}{\partial \phi} + j\omega\epsilon \frac{\partial E_z^0}{\partial r} \right)$$

$$E_r^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial r} + \frac{j\omega\mu}{r} \frac{\partial H_z^0}{\partial \phi} \right)$$

$$E_\phi^0 = -\frac{1}{h^2} \left( \frac{\gamma}{r} \frac{\partial E_z^0}{\partial \phi} - j\omega\mu \frac{\partial H_z^0}{\partial r} \right)$$

# Aplicação das condições fronteira

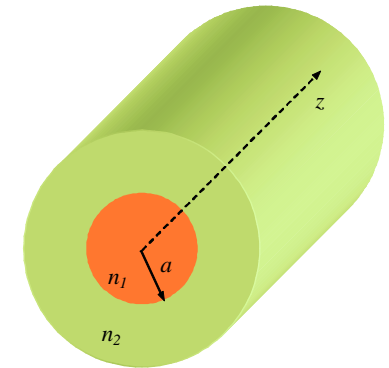
$$E_z^0 = \begin{cases} AJ_n(h_1 r)e^{jn\phi}, & \text{núcleo} \\ CK_n(\nu r)e^{jn\phi}, & \text{bainha} \end{cases} \xrightarrow{E_z^0 \text{ contínuo em } r=a} AJ_n(h_1 a) = CK_n(\nu a)$$

$$AJ_n(h_1 a) - CK_n(\nu a) = 0$$

$$BJ_n(h_1 a) - DK_n(\nu a) = 0$$

$$B \frac{\beta n}{h_1^2 a} J_n(h_1 a) - A \frac{j\omega \epsilon_1}{h_1} J'_n(h_1 a) + D \frac{\beta n}{\nu^2 a} K_n(\nu a) - C \frac{j\omega \epsilon_2}{\nu} K'_n(\nu a) = 0$$

$$A \frac{\beta n}{h_1^2 a} J_n(h_1 a) + B \frac{j\omega \mu_0}{h_1} J'_n(h_1 a) + C \frac{\beta n}{\nu^2 a} K_n(\nu a) + D \frac{j\omega \mu_0}{\nu} K'_n(\nu a) = 0$$



$E_z^0$  e  $E_\phi^0$  contínuos em  $r = a$

$H_z^0$  e  $H_\phi^0$  contínuos em  $r = a$

← de forma semelhante

## Aplicação das condições fronteira

$$AJ_n(h_1a) - CK_n(va) = 0$$

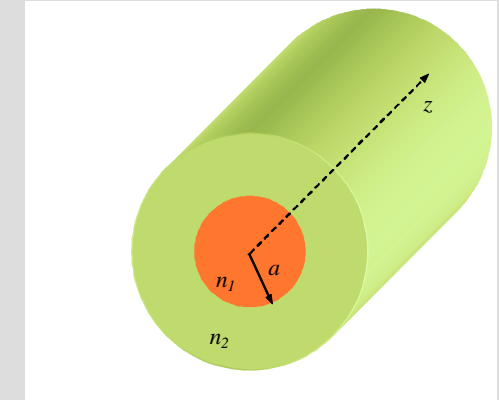
$$BJ_n(h_1a) - DK_n(va) = 0$$

$$B \frac{\beta n}{h_1^2 a} J_n(h_1a) - A \frac{j\omega\epsilon_1}{h_1} J'_n(h_1a) + D \frac{\beta n}{v^2 a} K_n(va) - C \frac{j\omega\epsilon_2}{v} K'_n(va) = 0$$

$$A \frac{\beta n}{h_1^2 a} J_n(h_1a) + B \frac{j\omega\mu_0}{h_1} J'_n(h_1a) + C \frac{\beta n}{v^2 a} K_n(va) + D \frac{j\omega\mu_0}{v} K'_n(va) = 0$$

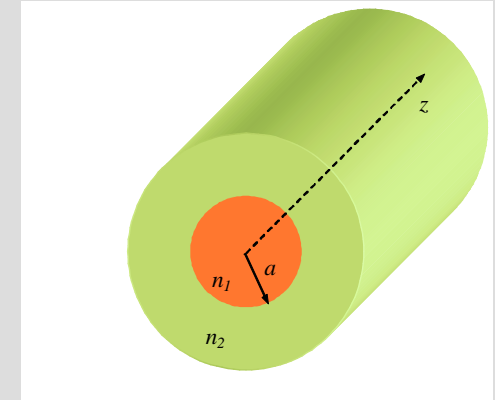
notação matricial

$$\begin{bmatrix} J_n(h_1a) & 0 & K_n(va) & 0 \\ 0 & J_n(h_1a) & 0 & K_n(va) \\ -\frac{j\omega\epsilon_1}{h_1} J'_n(h_1a) & \frac{\beta n}{h_1^2 a} J_n(h_1a) & -\frac{j\omega\epsilon_2}{v} K'_n(va) & \frac{\beta n}{v^2 a} K_n(va) \\ \frac{\beta n}{h_1^2 a} J_n(h_1a) & \frac{j\omega\mu_0}{h_1} J'_n(h_1a) & \frac{\beta n}{v^2 a} K_n(va) & \frac{j\omega\mu_0}{v} K'_n(va) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$





# Relação característica



$$\begin{bmatrix} J_n(h_1 a) & 0 & K_n(\nu a) & 0 \\ 0 & J_n(h_1 a) & 0 & K_n(\nu a) \\ -\frac{j\omega\epsilon_1}{h_1} J'_n(h_1 a) & \frac{\beta n}{h_1^2 a} J_n(h_1 a) & -\frac{j\omega\epsilon_2}{\nu} K'_n(\nu a) & \frac{\beta n}{\nu^2 a} K_n(\nu a) \\ \frac{\beta n}{h_1^2 a} J_n(h_1 a) & \frac{j\omega\mu_0}{h_1} J'_n(h_1 a) & \frac{\beta n}{\nu^2 a} K_n(\nu a) & \frac{j\omega\mu_0}{\nu} K'_n(\nu a) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

solução não trivial

$$\begin{vmatrix} J_n(h_1 a) & 0 & K_n(\nu a) & 0 \\ 0 & J_n(h_1 a) & 0 & K_n(\nu a) \\ -\frac{j\omega\epsilon_1}{h_1} J'_n(h_1 a) & \frac{\beta n}{h_1^2 a} J_n(h_1 a) & -\frac{j\omega\epsilon_2}{\nu} K'_n(\nu a) & \frac{\beta n}{\nu^2 a} K_n(\nu a) \\ \frac{\beta n}{h_1^2 a} J_n(h_1 a) & \frac{j\omega\mu_0}{h_1} J'_n(h_1 a) & \frac{\beta n}{\nu^2 a} K_n(\nu a) & \frac{j\omega\mu_0}{\nu} K'_n(\nu a) \end{vmatrix} = 0$$

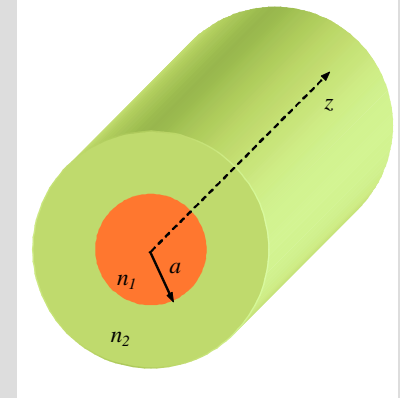
$$\left(\frac{\omega}{c}\right)^2 \left[ \frac{J'_n(h_1 a)}{h_1 J_n(h_1 a)} + \frac{K'_n(\nu a)}{\nu K_n(\nu a)} \right] \left[ n_1^2 \frac{J'_n(h_1 a)}{h_1 J_n(h_1 a)} + n_2^2 \frac{K'_n(\nu a)}{\nu K_n(\nu a)} \right] = \left(\frac{\beta n}{a}\right)^2 \left( \frac{1}{h_1^2} + \frac{1}{\nu^2} \right)^2$$

relação característica para  
modos TM, TE, HE e EH

# Relação característica – modos TM e TE

para  $n=0$

$$\begin{bmatrix} J_0(h_1 a) & 0 & K_0(\nu a) & 0 \\ 0 & J_0(h_1 a) & 0 & K_0(\nu a) \\ -\frac{j\omega\epsilon_1}{h_1} J'_0(h_1 a) & 0 & -\frac{j\omega\epsilon_2}{\nu} K'_0(\nu a) & 0 \\ 0 & \frac{j\omega\mu_0}{h_1} J'_0(h_1 a) & 0 & \frac{j\omega\mu_0}{\nu} K'_0(\nu a) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$



**modos TM**

$$\begin{bmatrix} J_0(h_1 a) & K_0(\nu a) \\ -\frac{\epsilon_1}{h_1} J'_0(h_1 a) & -\frac{\epsilon_2}{\nu} K'_0(\nu a) \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = 0$$

solução não trivial  $B=D=0$  possível  $\rightarrow H_z = 0$

$$\left[ n_1^2 \frac{J_1(h_1 a)}{h_1 J_0(h_1 a)} + n_2^2 \frac{K_1(\nu a)}{\nu K_0(\nu a)} \right] = 0$$

**modos TE**

$$\begin{bmatrix} J_0(h_1 a) & K_0(\nu a) \\ \frac{1}{h_1} J'_0(h_1 a) & \frac{1}{\nu} K'_0(\nu a) \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = 0$$

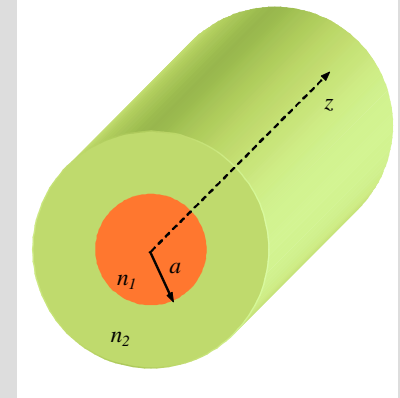
solução não trivial  $A=C=0$  possível  $\rightarrow E_z = 0$

$$\left[ \frac{J_1(h_1 a)}{h_1 J_0(h_1 a)} + \frac{K_1(\nu a)}{\nu K_0(\nu a)} \right] = 0$$

# Frequência de corte

condição de corte  $\rightarrow \nu = 0$

$n$	modo	corte
0	TE <sub>0p</sub> TM <sub>0p</sub>	$J_0(h_1 a) = 0$
1	HE <sub>1p</sub> EH <sub>1p</sub>	$J_1(h_1 a) = 0$
$\geq 2$	EH <sub>np</sub>	$J_n(h_1 a) = 0$
	HE <sub>np</sub>	$\left(\frac{n_1^2}{n_2^2} + 1\right) J_{n-1}(h_1 a) = \frac{h_1 a}{n-1} J_n(h_1 a)$



## notas

1. frequência de corte do modo HE<sub>11</sub> = 0
2. modos seguintes: TE<sub>01</sub> e TM<sub>01</sub> (frequência de corte associada ao 1º zero de J<sub>0</sub>  $\rightarrow$  2.4048)

# Frequência normalizada

frequência normalizada  
(parâmetro V)

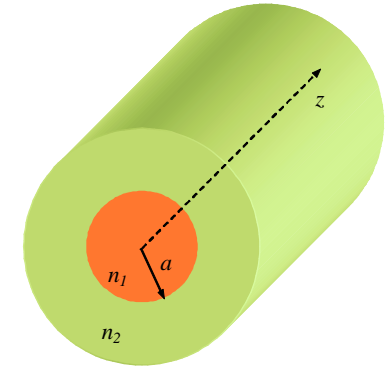
$$V^2 = (h_1^2 + \nu^2)a^2 = \left(\frac{\omega a}{c}\right)^2 (n_1^2 - n_2^2)$$

$$h_1^2 = \gamma^2 + \left(\frac{\omega}{c}n_1\right)^2$$

$$\nu^2 = -\gamma^2 - \left(\frac{\omega}{c}n_2\right)^2$$

comprimento de onda no vazio

$$V = \frac{2\pi a}{\lambda_0} \sqrt{n_1^2 - n_2^2}$$



$$HE_{11} \rightarrow f_c = 0$$

propagação dos modos:

$$TM_{01} \text{ e } TE_{01} \rightarrow h_1 a > 2.4048$$

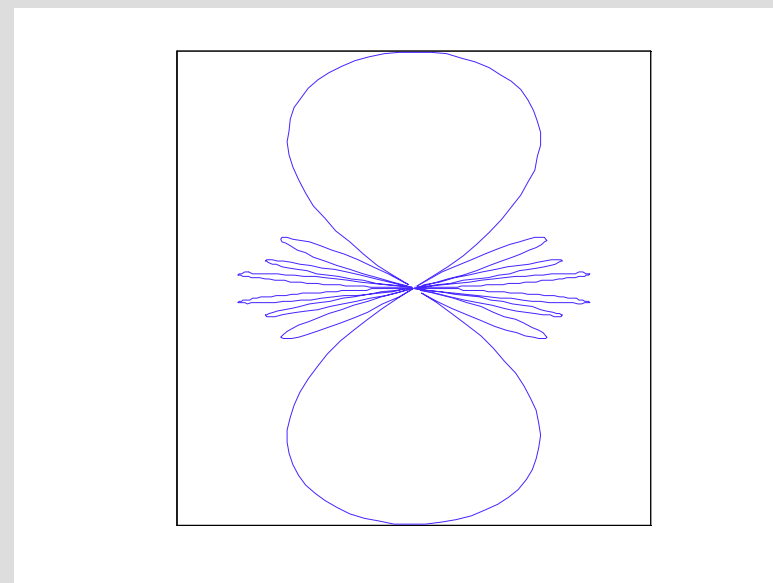
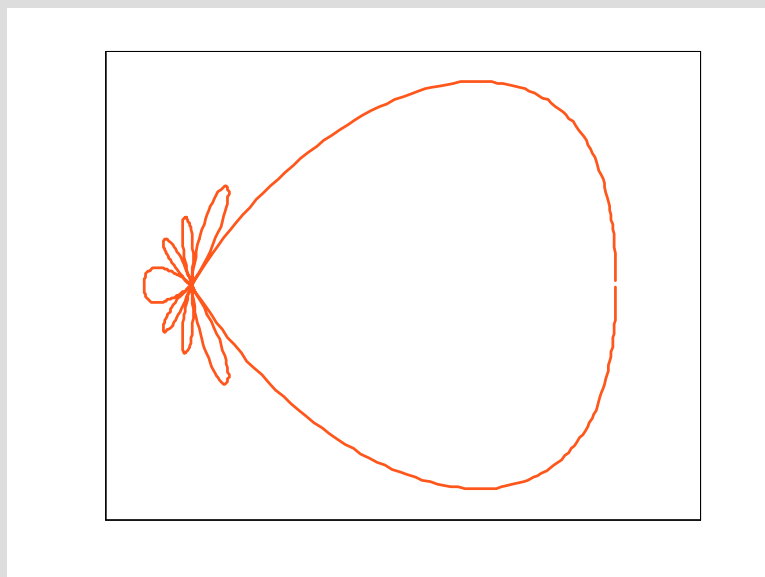
nota

corte  $\rightarrow \nu = 0 \rightarrow (V)_{\text{corte}} = (h_1 a)_{\text{corte}}$

regime monomodo  $\rightarrow V < 2.4048$

regime multimodo  $\rightarrow V > 2.4048$

# Antenas e Radiação



OE - MIEEC 2006/2007

## Meios LHI com fontes

equações de Maxwell para meios LHI com fontes ( $\rho \neq 0$  e  $\vec{J} \neq 0$ )

$$\begin{aligned} \nabla \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} \\ \nabla \times \vec{H} &= \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon} \\ \nabla \cdot \vec{H} &= 0 \end{aligned}$$

do mesmo modo

$$\nabla^2 \vec{H} - \mu \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = -\nabla \times \vec{J}$$

(eq. não homogénea)

$$\nabla \times (\nabla \times \vec{E}) = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H}) \longrightarrow \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} \left( \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right)$$

$$\nabla \left( \frac{\rho}{\varepsilon} \right) - \nabla^2 \vec{E} = -\mu \frac{\partial \vec{J}}{\partial t} - \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} - \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{J}}{\partial t} + \frac{1}{\varepsilon} \nabla(\rho)$$

(eq. não homogénea)

**nota**

$$\text{equação da continuidade: } \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

## Equação de onda para o vector potencial magnético

vector potencial magnético  $\rightarrow \vec{A} \longrightarrow \boxed{\vec{H} = \frac{1}{\mu} \nabla \times \vec{A}}$

potencial eléctrico  $\rightarrow V \longrightarrow \boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}}$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon}$$

$$\nabla \cdot \vec{H} = 0$$

$$\frac{1}{\mu} \nabla \times (\nabla \times \vec{A}) = \vec{J} - \epsilon \frac{\partial}{\partial t} \left( \nabla V + \frac{\partial \vec{A}}{\partial t} \right)$$

$$\nabla^2 \vec{A} - \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \nabla \left( \cancel{\nabla \cdot \vec{A} + \mu \epsilon \frac{\partial V}{\partial t}} \right)$$

**nota**

$$\text{condição de Lorentz : } \nabla \cdot \vec{A} + \mu \epsilon \frac{\partial V}{\partial t} = 0$$

$$\boxed{\nabla^2 \vec{A} - \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}}$$

## Equação de onda para o potencial eléctrico

$$\begin{array}{l}
 \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \\
 \nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \\
 \boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon}} \\
 \nabla \cdot \vec{H} = 0
 \end{array}
 \quad \xrightarrow{\quad} \quad
 \begin{array}{l}
 \boxed{\vec{H} = \frac{1}{\mu} \nabla \times \vec{A}} \\
 \boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}}
 \end{array}$$
  

$$\nabla \cdot \left( -\nabla V - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\epsilon}$$
  

$$-\nabla^2 V - \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = \frac{\rho}{\epsilon}$$
  

$$\nabla \cdot \vec{A} = -\mu \epsilon \frac{\partial V}{\partial t}$$
  

$$\boxed{\nabla^2 V - \mu \epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}}$$



## Soluções para os potenciais – caso estático

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

$$\frac{\partial}{\partial t} = 0$$

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

**equações de Poisson**

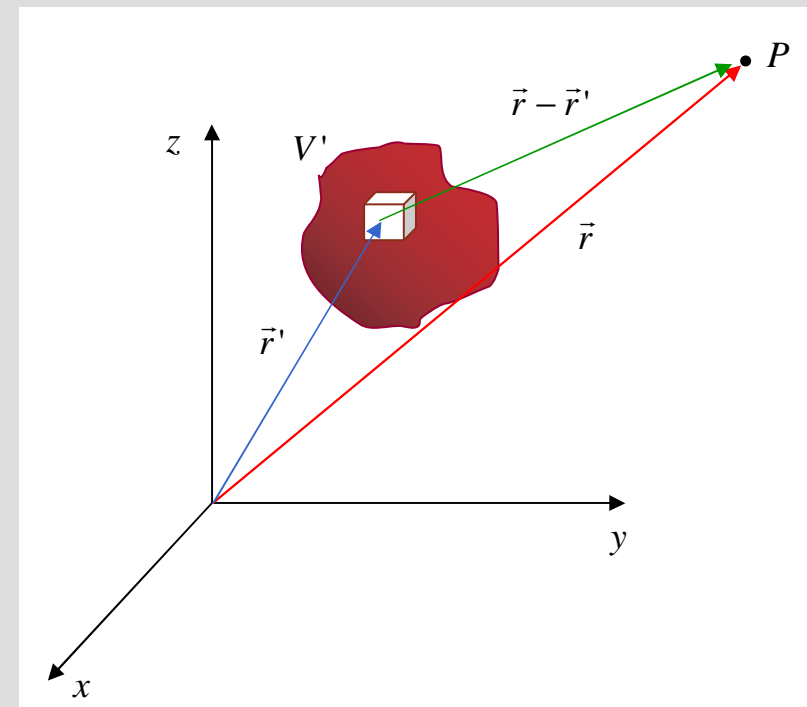
$$V(\vec{r}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\vec{r}')}{R} dv'$$

$$A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}')}{R} dv'$$

$$R = |\vec{r} - \vec{r}'|$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$



## Soluções para os potenciais – caso dinâmico

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

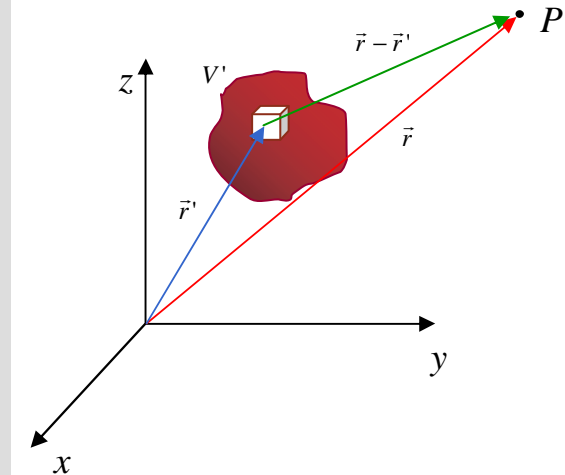
$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\vec{r}', t - R/v)}{R} dv'$$

$$A(\vec{r}, t) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}', t - R/v)}{R} dv'$$

**potenciais retardados**

potencial depende de carga em instante anterior

$R/v \rightarrow$  tempo que demora alteração na carga a propagar-se até  $P$



caso estático  $R = |\vec{r} - \vec{r}'|$

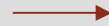
$$V(\vec{r}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\vec{r}')}{R} dv'$$

$$A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}')}{R} dv'$$

## Potenciais retardados – variação harmónica no tempo

$$\vec{J}(\vec{r}', t) = \vec{J}(\vec{r}') \cos(\omega t)$$

$$\rho(\vec{r}', t) = \rho(\vec{r}') \cos(\omega t)$$



$$\vec{J}(\vec{r}', t - R/v) = \vec{J}(\vec{r}') \cos[\omega(t - R/v)]$$

$$\rho(\vec{r}', t - R/v) = \rho(\vec{r}') \cos[\omega(t - R/v)]$$

$$\beta = \omega/v$$

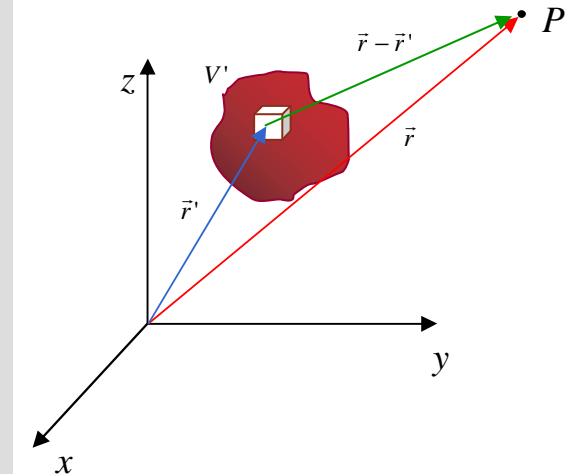
$$\vec{J}(\vec{r}', t - R/v) = \vec{J}(\vec{r}') \cos[\omega t - \beta R]$$

$$\rho(\vec{r}', t - R/v) = \rho(\vec{r}') \cos[\omega t - \beta R]$$

notação fasorial

$$V(\vec{r}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\vec{r}') e^{-j\beta R}}{R} dv'$$

$$A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$$



potenciais retardados  $R = |\vec{r} - \vec{r}'|$

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\vec{r}', t - R/v)}{R} dv'$$

$$A(\vec{r}, t) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}', t - R/v)}{R} dv'$$

## Determinação dos campos radiados

$$A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\vec{r}') e^{-j\beta R}}{R} dv'$$

$$\vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$\vec{E} = -\nabla V - j\omega \vec{A}$$

$$\nabla \times \vec{H} = \vec{J} + j\omega\epsilon \vec{E}$$

$$\nabla \times \vec{E} = -j\omega\mu \vec{H}$$

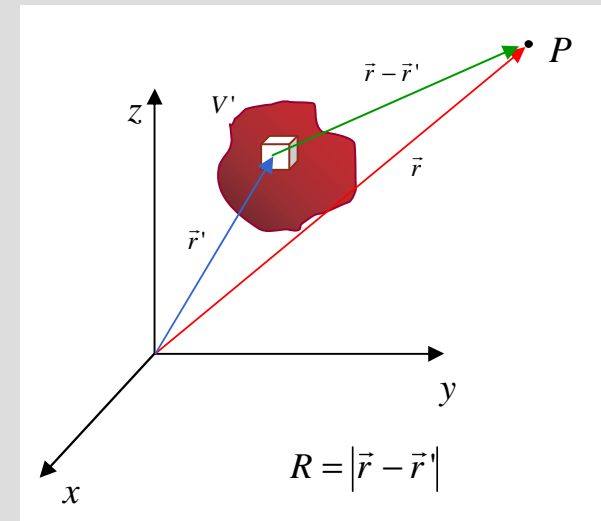
### determinação dos campos radiados

1. determinação de  $A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$

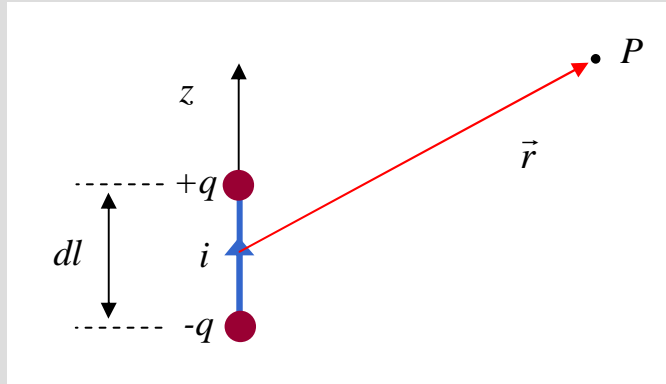
2. determinação de  $\vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$

3. determinação de  $\vec{E} = \frac{1}{j\omega\epsilon} (\nabla \times \vec{H} - \vec{J}) = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$

nos pontos onde  $\vec{J} = 0$



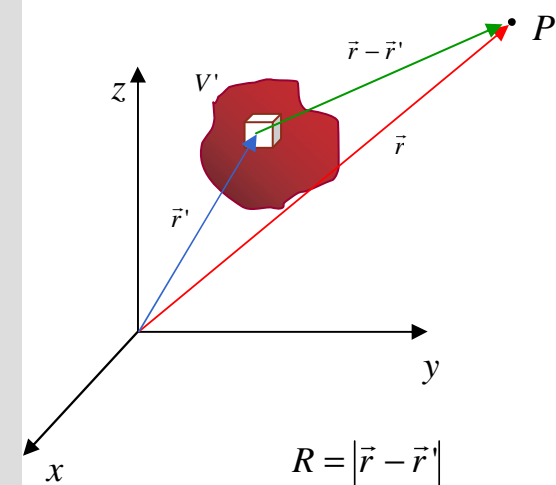
# O dipolo eléctrico elementar



$$q = Q \cos(\omega t)$$

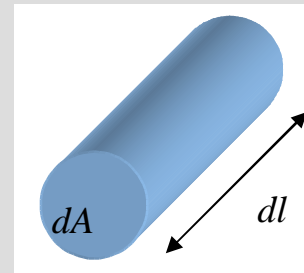
$$i = dq/dt$$

$$I = j\omega Q$$



$$1. \quad A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv' = \frac{\mu}{4\pi} \frac{e^{-j\beta r}}{r} \int_{V'} \vec{J}(\vec{r}') dv'$$

$\vec{r}' = 0 \rightarrow R = r$



$$1. \quad A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$$

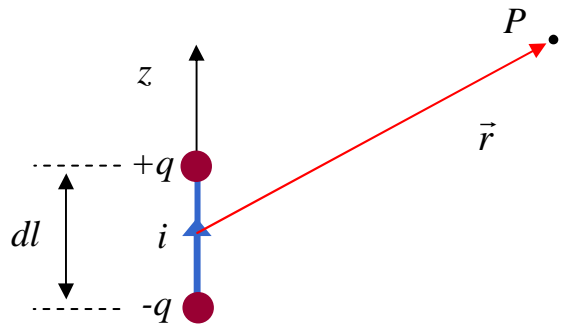
$$2. \quad \vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$3. \quad \vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$$

$$A(\vec{r}) = \frac{\mu_0}{4\pi} Idl \frac{e^{-j\beta r}}{r} \hat{z}$$

$$\int_{V'} \vec{J}(\vec{r}') dv' = \hat{z} \int_{V'} J dv' = \hat{z} \int_{dl} dz \int_{dA} J dx dy = Idl \hat{z}$$

# O dipolo eléctrico elementar – o potencial magnético



$$1. \quad A(\vec{r}) = \frac{\mu_0}{4\pi} Idl \frac{e^{-j\beta r}}{r} \hat{z}$$

nota:  $r = \sqrt{x^2 + y^2 + z^2}$  →

$$A(\vec{r}) = \frac{\mu_0}{4\pi} Idl \frac{e^{-j\beta\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2 + y^2 + z^2}} \hat{z}$$

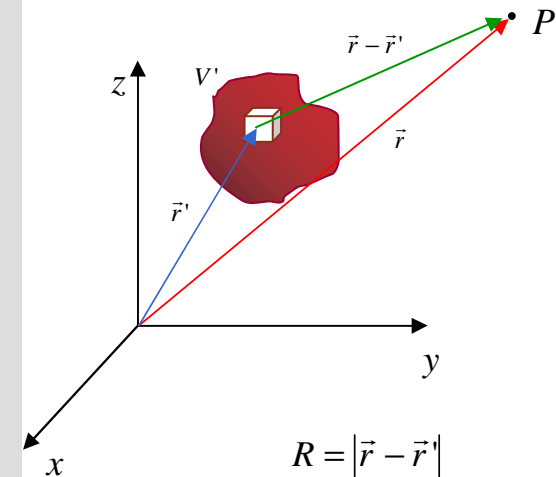
é conveniente utilizar **coordenadas esféricas**

$\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$  →

$$A_r = A_z \cos \theta = \frac{\mu_0}{4\pi} Idl \frac{e^{-j\beta r}}{r} \cos \theta$$

$$A_\theta = -A_z \sin \theta = -\frac{\mu_0}{4\pi} Idl \frac{e^{-j\beta r}}{r} \sin \theta$$

$$A_\phi = 0$$



1.  $A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$
2.  $\vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$
3.  $\vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$

## O dipolo eléctrico elementar – campo magnético

$$2. \quad \vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A} = \frac{1}{\mu_0} \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r \sin \theta A_\phi \end{vmatrix}$$

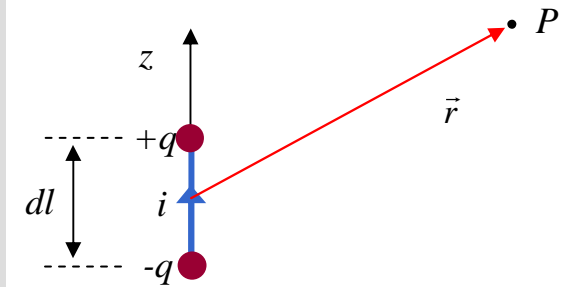
$$A_r = \frac{\mu_0}{4\pi} Idl \frac{e^{-j\beta r}}{r} \cos \theta$$

$$A_\theta = -\frac{\mu_0}{4\pi} Idl \frac{e^{-j\beta r}}{r} \sin \theta$$

$$A_\phi = 0$$

$$\frac{\partial}{\partial \phi} = 0$$

$$\vec{H} = -\frac{Idl}{4\pi} \beta^2 \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} \right] e^{-j\beta r} \hat{\phi}$$



$$1. \quad A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$$

$$2. \quad \vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$3. \quad \vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$$

## O dipolo eléctrico elementar – campo eléctrico

$$3. \quad \vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$$

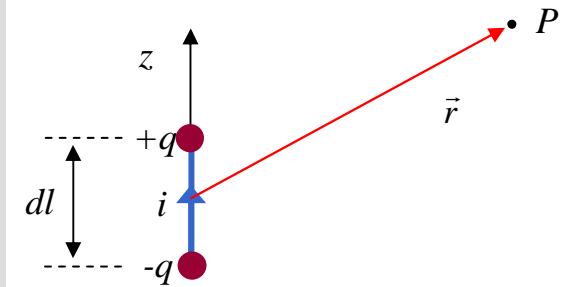
$$\vec{E} = \frac{1}{j\omega\epsilon_0} \nabla \times \vec{H} = \frac{1}{j\omega\epsilon_0} \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r \sin \theta H_\phi \end{vmatrix}$$

$$H_\phi = -\frac{Idl}{4\pi} \beta^2 \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} \right] e^{-j\beta r}$$

$$H_r = H_\theta = 0$$

$$\frac{\partial}{\partial \phi} = 0$$

$$\vec{E} = -\frac{Idl\beta^2\eta_0}{4\pi} \left\{ 2 \cos \theta \left[ \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] \hat{r} + \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] \hat{\theta} \right\} e^{-j\beta r}$$



$$1. \quad A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$$

$$2. \quad \vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$3. \quad \vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$$



## O dipolo eléctrico elementar – campo próximo

campo próximo:  $\beta r \ll 1 \xrightarrow{\beta = 2\pi/\lambda} r \ll \frac{\lambda}{2\pi}$

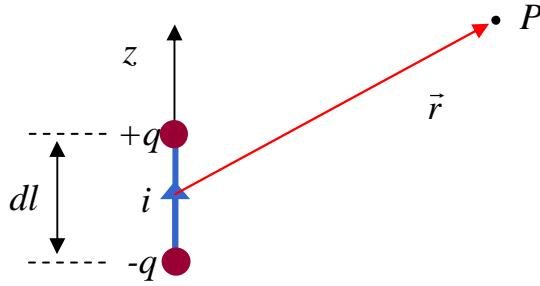
$$e^{-j\beta r} = 1 - j\beta r + (-j\beta r)^2 + (-j\beta r)^3 + \dots \cong 1$$

$$1 + \frac{1}{-j\beta r} \cong \frac{1}{-j\beta r}$$

$$H_\phi = \frac{Idl}{4\pi r^2} \sin \theta$$

$$E_r = \frac{Idl}{j\omega\epsilon_0 4\pi r^3} 2 \cos \theta$$

$$E_\theta = \frac{Idl}{j\omega\epsilon_0 4\pi r^3} \sin \theta$$



The diagram shows a vertical dipole with charges  $+q$  and  $-q$  separated by distance  $dl$ . A point  $P$  is located at a distance  $r$  from the center of the dipole, at an angle  $\theta$  from the vertical  $z$ -axis. A red vector  $\vec{r}$  points from the center to  $P$ . A blue arrow labeled  $i$  indicates the direction of current flow from  $-q$  to  $+q$ .

$$H_\phi = -\frac{Idl}{4\pi} \beta^2 \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} \right] e^{-j\beta r}$$

$$E_r = -\frac{Idl\beta^2\eta_0}{4\pi} 2 \cos \theta \left[ \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] e^{-j\beta r}$$

$$E_\theta = -\frac{Idl\beta^2\eta_0}{4\pi} \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] e^{-j\beta r}$$

**nota**

para um dipolo:  $p = Qdl = \frac{I}{j\omega} dl \longrightarrow$

$$E_r = \frac{p}{4\pi\epsilon_0 r^3} 2 \cos \theta$$

$$E_\theta = \frac{p}{4\pi\epsilon_0 r^3} \sin \theta$$

mesmas expressões que para o caso estático

## O dipolo eléctrico elementar – campo distante

campo distante:  $\beta r \gg 1 \xrightarrow{\beta = 2\pi/\lambda} r \gg \frac{\lambda}{2\pi}$

$$1 + \frac{1}{-j\beta r} \cong 1$$

$$H_\phi = j \frac{Idl}{4\pi} \beta \sin \theta \frac{e^{-j\beta r}}{r}$$

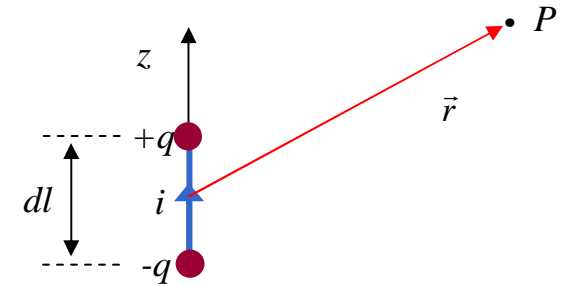
$$E_r = \frac{Idl\eta_0\beta^2}{4\pi} 2 \cos \theta \frac{e^{-j\beta r}}{\beta^2 r^2}$$

$$E_\theta = j \frac{Idl\eta_0\beta^2}{4\pi} \sin \theta \frac{e^{-j\beta r}}{\beta r}$$



$$\vec{E} = j \frac{Idl\eta_0}{4\pi} \beta \sin \theta \frac{e^{-j\beta r}}{r} \hat{\theta}$$

$$\vec{H} = j \frac{Idl}{4\pi} \beta \sin \theta \frac{e^{-j\beta r}}{r} \hat{\phi}$$



$$H_\phi = -\frac{Idl}{4\pi} \beta^2 \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} \right] e^{-j\beta r}$$

$$E_r = -\frac{Idl\beta^2\eta_0}{4\pi} 2 \cos \theta \left[ \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] e^{-j\beta r}$$

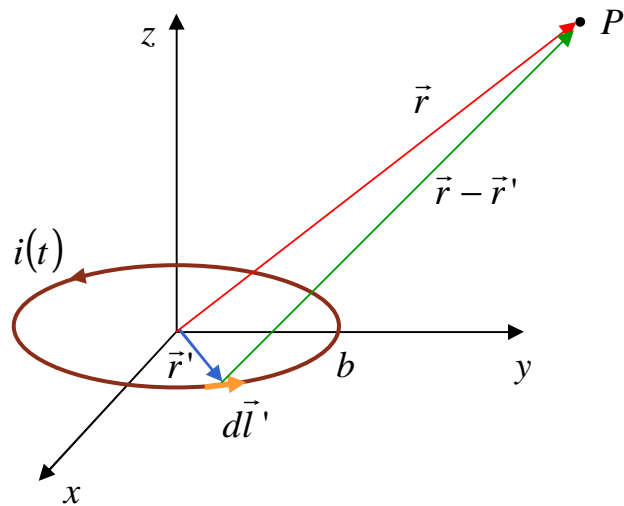
$$E_\theta = -\frac{Idl\beta^2\eta_0}{4\pi} \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] e^{-j\beta r}$$

### notas

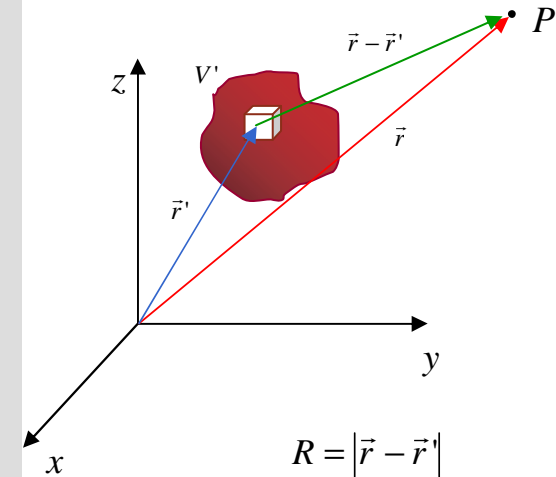
1.  $\vec{E}$  e  $\vec{H}$  estão em quadratura no espaço e em fase no tempo
2.  $\frac{E_\theta}{H_\phi} = \eta_0$

campo distante tem propriedades de onda plana

# O dipolo magnético elementar



$$i(t) = I \cos(\omega t)$$



$$1. \quad A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv' = \frac{\mu_0 I}{4\pi} \oint \frac{e^{-j\beta R}}{R} d\vec{l}'$$

anel pequeno  $\rightarrow b \ll r \rightarrow e^{-j\beta R} = e^{-j\beta r} e^{-j\beta(R-r)} \cong e^{-j\beta r} [1 - j\beta(R-r)]$

$$\rightarrow A(\vec{r}) = \frac{\mu_0 I}{4\pi} e^{-j\beta r} \left\{ (1 + j\beta r) \oint \frac{d\vec{l}'}{R} - j\beta \oint d\vec{l}' \right\} = \frac{\mu_0 I}{4\pi} e^{-j\beta r} (1 + j\beta r) \oint \frac{d\vec{l}'}{R}$$

1.  $A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$
2.  $\vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$
3.  $\vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$

## O dipolo magnético elementar – potencial magnético

$$A(\vec{r}) = \frac{\mu_0 I}{4\pi} e^{-j\beta r} (1 + j\beta r) \oint \frac{d\vec{l}'}{R}$$

$$\begin{aligned} d\vec{l}' &= b d\phi' \hat{\phi} \\ \hat{\phi} &= -\sin\phi' \hat{x} + \cos\phi' \hat{y} \end{aligned}$$

$$d\vec{l}' = b d\phi' (-\sin\phi' \hat{x} + \cos\phi' \hat{y})$$

$$\vec{r}' = b \cos\phi' \hat{x} + b \sin\phi' \hat{y}$$

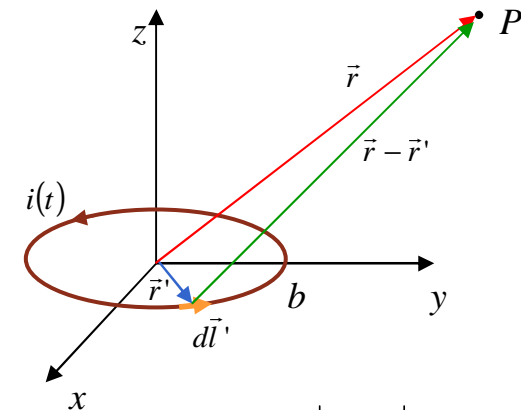
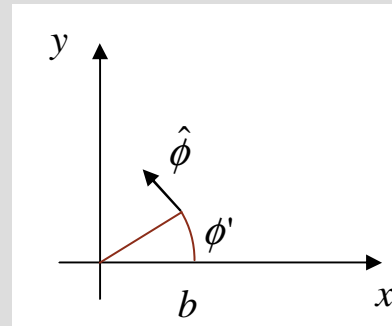
$$\vec{r} = r\hat{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

considerando  $P$  no plano  $yz$  (por razões de simetria,  $\partial A(\vec{r})/\partial\phi = 0$ )

$$\vec{r} = y\hat{y} + z\hat{z}$$

$$R = |\vec{r} - \vec{r}'| = \sqrt{(b \cos\phi')^2 + (y - b \sin\phi')^2 + z^2} = \sqrt{r^2 + b^2 - 2by \sin\phi'} \cong \sqrt{r^2 - 2br \sin\theta \sin\phi'} \cong r - b \sin\theta \sin\phi'$$

$$\begin{aligned} r &\gg b \\ y &= r \sin\theta \end{aligned}$$



$$R = |\vec{r} - \vec{r}'|$$

$$1. \quad A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$$

$$2. \quad \vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$3. \quad \vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$$

## O dipolo magnético elementar – potencial magnético

$$A(\vec{r}) = \frac{\mu_0 I}{4\pi} e^{-j\beta r} (1 + j\beta r) \oint \frac{d\vec{l}'}{R}$$

$$R \cong r - b \sin \theta \sin \phi' \quad \longrightarrow \quad \frac{1}{R} \cong \frac{1}{r} \left( 1 + \frac{b}{r} \sin \theta \sin \phi' \right)$$

$$A(\vec{r}) = \frac{\mu_0 I}{4\pi} e^{-j\beta r} (1 + j\beta r) \left\{ \frac{1}{r} \oint d\vec{l}' + \frac{b}{r} \sin \theta \oint \sin \phi' d\vec{l}' \right\}$$

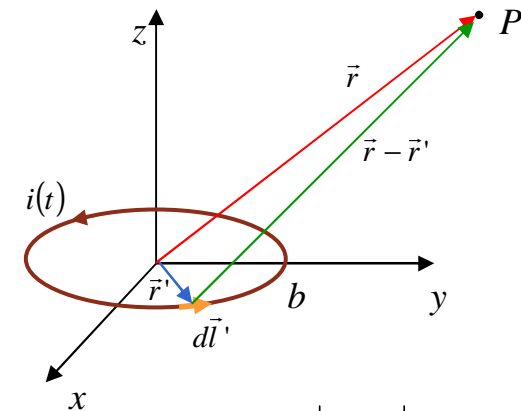
$$d\vec{l}' = b d\phi' (-\sin \phi' \hat{x} + \cos \phi' \hat{y})$$

$$A(\vec{r}) = \frac{\mu_0 I}{4\pi} b^2 \sin \theta \frac{e^{-j\beta r}}{r} (1 + j\beta r) \int_0^{2\pi} \sin \phi' (-\sin \phi' \hat{x} + \cos \phi' \hat{y}) d\phi'$$

$$A(\vec{r}) = -\frac{\mu_0 I}{4} b^2 \sin \theta \frac{e^{-j\beta r}}{r^2} (1 + j\beta r) \hat{x} \quad \longleftarrow \quad \text{para } P \text{ no plano } yz$$

para ponto genérico

$$A(\vec{r}) = \frac{\mu_0 I}{4} b^2 \sin \theta \frac{e^{-j\beta r}}{r^2} (1 + j\beta r) \hat{\phi}$$



$$R = |\vec{r} - \vec{r}'|$$

$$1. \quad A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$$

$$2. \quad \vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$3. \quad \vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$$

## O dipolo magnético elementar – campo magnético

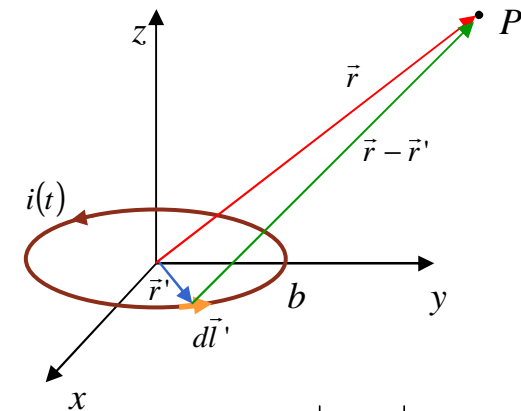
$$2. \quad \vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A} = \frac{1}{\mu_0} \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r \sin \theta A_\phi \end{vmatrix}$$

$$A_\phi = \frac{\mu_0 I}{4} b^2 \sin \theta \frac{e^{-j\beta r}}{r^2} (1 + j\beta r)$$

$$A_r = A_\theta = 0$$

$$\frac{\partial}{\partial \phi} = 0$$



$$R = |\vec{r} - \vec{r}'|$$

$$1. \quad A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$$

$$2. \quad \vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$3. \quad \vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$$

$$\vec{H} = -\frac{j\omega\mu_0 I b^2}{4\eta_0} \beta^2 \left\{ 2 \cos \theta \left[ \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] \hat{r} + \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] \hat{\theta} \right\} e^{-j\beta r}$$

## O dipolo magnético elementar – campo eléctrico

$$3. \quad \vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$$

$$\vec{E} = \frac{1}{j\omega\epsilon_0} \nabla \times \vec{H} = \frac{1}{j\omega\epsilon_0} \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r \sin \theta H_\phi \end{vmatrix}$$

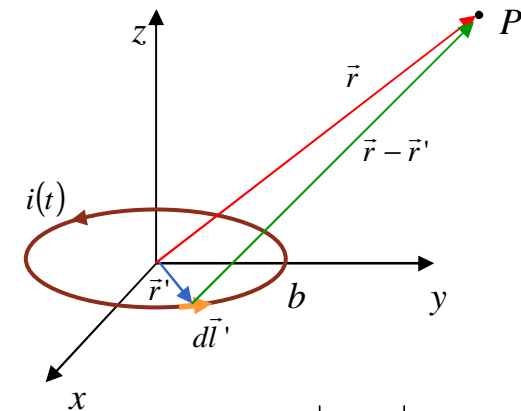
$$H_r = -\frac{j\omega\mu_0 I b^2}{4\eta_0} \beta^2 2 \cos \theta \left[ \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] e^{-j\beta r}$$

$$H_\theta = -\frac{j\omega\mu_0 I b^2}{4\eta_0} \beta^2 \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] e^{-j\beta r}$$

$$H_\phi = 0$$

$$\frac{\partial}{\partial \phi} = 0$$

$$\vec{E} = \frac{j\omega\mu_0 I b^2}{4} \beta^2 \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} \right] e^{-j\beta r} \hat{\phi}$$



$$R = |\vec{r} - \vec{r}'|$$

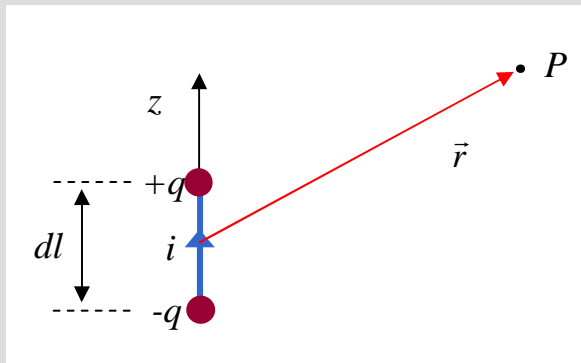
$$1. \quad A(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}') e^{-j\beta R}}{R} dv'$$

$$2. \quad \vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$3. \quad \vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}$$

# O dipolo eléctrico e magnético elementares

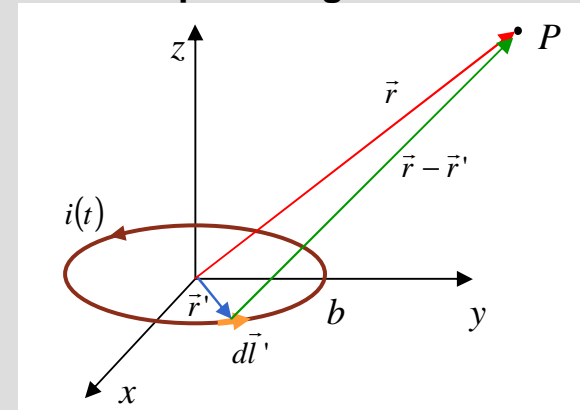
**dipolo eléctrico**



$$\vec{E}_e = -\frac{Idl\beta^2\eta_0}{4\pi} \left\{ 2\cos\theta \left[ \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] \hat{r} + \sin\theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] \hat{\theta} \right\} e^{-j\beta r}$$

$$\vec{H}_e = -\frac{Idl}{4\pi} \beta^2 \sin\theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} \right] e^{-j\beta r} \hat{\phi}$$

**dipolo magnético**



$$\vec{E}_m = \frac{j\omega\mu_0 Ib^2}{4} \beta^2 \sin\theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} \right] e^{-j\beta r} \hat{\phi}$$

$$\vec{H}_m = -\frac{j\omega\mu_0 Ib^2}{4\eta_0} \beta^2 \left\{ 2\cos\theta \left[ \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] \hat{r} + \sin\theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] \hat{\theta} \right\} e^{-j\beta r}$$

→ se  $dl = j\beta\pi b^2$  →

$$\vec{E}_e = \eta_0 \vec{H}_m \quad \text{e} \quad \vec{H}_e = -\frac{\vec{E}_m}{\eta_0}$$

→ **dipolos eléctrico e magnético são dispositivos duais**



## Potência média radiada

$$P_r = \oint_A \vec{S}_{med} \cdot d\vec{a}$$

onde  $\vec{S}_{med} = \frac{1}{2} \text{Re}\{\vec{E} \times \vec{H}^*\}$

A → superfície fechada

campos em coordenadas esféricas → considerar superfície de uma esfera

$$d\vec{a} = r^2 \sin \theta d\theta d\phi \hat{r}$$

$$P_r = \int_0^{2\pi} \int_0^{\pi} S_{med} r^2 \sin \theta d\theta d\phi$$

$$P_r = \int_0^{2\pi} \int_0^{\pi} (\vec{S}_{med} \cdot \hat{r}) r^2 \sin \theta d\theta d\phi$$

$$\vec{S}_{med} = S_{med} \hat{r}$$

# Intensidade de radiação

$$P_r = \int_0^{2\pi} \int_0^\pi S_{med} r^2 \sin \theta d\theta d\phi$$

$$U = r^2 S_{med}$$

$$P_r = \int_0^{2\pi} \int_0^\pi U \sin \theta d\theta d\phi$$

$$d\Omega = \sin \theta d\theta d\phi$$

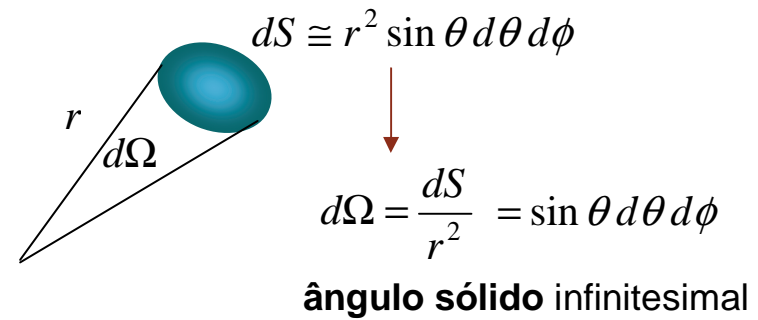
$$P_r = \oint U d\Omega$$

$$U = r^2 S_{med}$$

**intensidade de radiação**

representa a potência média por unidade de ângulo sólido

unidade de  $U \rightarrow \text{W/sr}$



**exemplo:**

ângulo sólido de uma esfera  $\rightarrow \Omega = \frac{4\pi r^2}{r^2} = 4\pi$

**nota:**

unidade de  $\Omega \rightarrow \text{sr}$  (estereo-radiano)

## Resistência de radiação

**resistência de radiação ( $R_r$ )** → resistência que dissipa potência igual à potência radiada quando é atravessada pela corrente máxima na antena:

$$P_r = \frac{1}{2} R_r I^2$$

onde  $I$  é a corrente máxima na antena

$$R_r = \frac{2 \oint \vec{S}_{med} \cdot d\vec{a}}{I^2}$$

**nota:**

$R_r$  elevada para potências radiadas elevadas e correntes baixas

## Potência radiada – dipolo eléctrico elementar

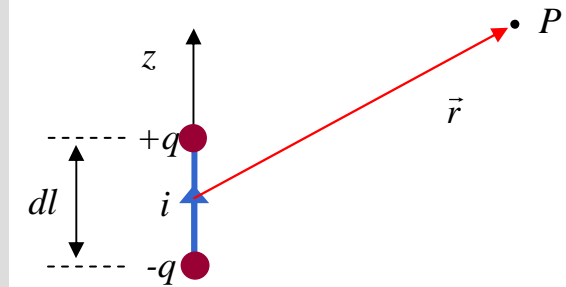
$$P_r = \int_0^{2\pi} \int_0^{\pi} S_{med} r^2 \sin \theta d\theta d\phi$$

$$\vec{S}_{med} = \frac{1}{2} \text{Re}\{\vec{E} \times \vec{H}^*\}$$

$$= \frac{1}{2} \text{Re}\left\{ j \frac{Idl\eta_0}{4\pi} \beta \sin \theta \frac{e^{-j\beta r}}{r} (-j) \frac{Idl}{4\pi} \beta \sin \theta \frac{e^{+j\beta r}}{r} (\hat{\theta} \times \hat{\phi}) \right\}$$

$$= \frac{\eta_0}{2} \left( \frac{Idl}{4\pi r} \beta \sin \theta \right)^2 \hat{r}$$

$$P_r = \frac{\eta_0}{2} \left( \frac{Idl\beta}{4\pi} \right)^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3 \theta d\theta = \frac{I^2 dl^2 \beta^2 \eta_0}{12\pi}$$



campo distante:

$$\vec{E} = j \frac{Idl\eta_0}{4\pi} \beta \sin \theta \frac{e^{-j\beta r}}{r} \hat{\theta}$$

$$\vec{H} = j \frac{Idl}{4\pi} \beta \sin \theta \frac{e^{-j\beta r}}{r} \hat{\phi}$$

notas:

1. intensidade de radiação  $\rightarrow U = \frac{\eta_0}{2} \left( \frac{Idl}{4\pi} \beta \sin \theta \right)^2$

2. resistência de radiação  $\rightarrow R_r = \frac{2P_r}{I^2} = \frac{2dl^2 \beta^2 \eta_0}{12\pi}$

$\xrightarrow[\eta_0 = 120\pi]{\beta = 2\pi/\lambda}$

$R_r = 80\pi^2 \left( \frac{dl}{\lambda} \right)^2$

## Diagramas de radiação

**diagrama (ou padrão) de radiação** → descreve a variação da amplitude do campo distante com a **direcção** (para um  $r$  fixo)

↑  
variação com  $\theta$  e com  $\phi$   
↓

normalmente apresenta-se esta variação em 2 gráficos separados

**diagrama de plano – E** → amplitude normalizada do campo (face ao seu valor de pico)  
em função de  $\theta$  para um  $\phi$  constante

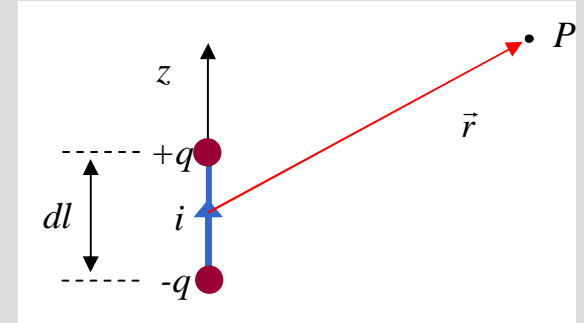
**diagrama de plano – H** → amplitude normalizada do campo em função de  $\phi$  para  $\theta = \frac{\pi}{2}$   
variação no plano  $xy$

## Diagramas de radiação – dipolo eléctrico

dipolo eléctrico (campo distante)  $\rightarrow$

$$\vec{E} = j \frac{Idl\eta_0}{4\pi} \beta \sin \theta \frac{e^{-j\beta r}}{r} \hat{\theta}$$

$$\vec{H} = j \frac{Idl}{4\pi} \beta \sin \theta \frac{e^{-j\beta r}}{r} \hat{\phi}$$



para  $r$  fixo:

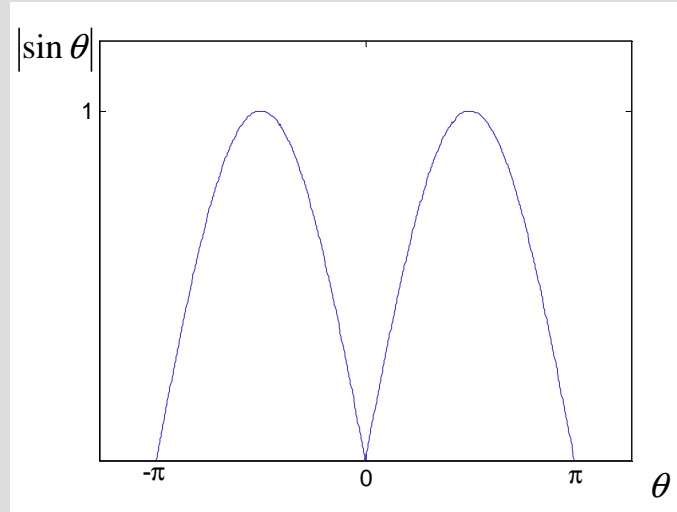
$$|\vec{E}|_{MAX} = \frac{Idl\eta_0\beta}{4\pi r} |\sin \theta|_{MAX} = \frac{Idl\eta_0\beta}{4\pi r}$$

$$\frac{|\vec{E}|}{|\vec{E}|_{MAX}} = |\sin \theta| = \frac{|\vec{H}|}{|\vec{H}|_{MAX}}$$

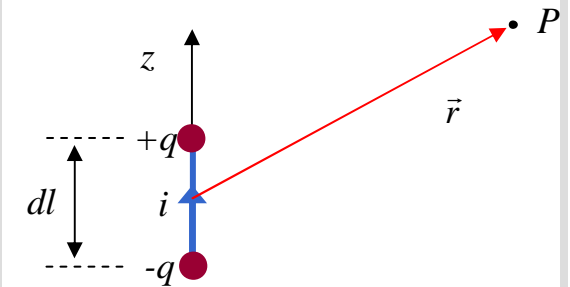
$\longrightarrow$  diagramas de radiação dependem de  $|\sin \theta|$

## Diagrama de plano – E para o dipolo eléctrico

diagrama de plano – E:



↑  
**gráfico cartesiano:** variável independente no eixo dos  $xx$   
variável dependente no eixo dos  $yy$



**diagrama de plano – E:**

amplitude normalizada do campo  
em função de  $\theta$  para  $\phi = \text{const.}$

**dipolo eléctrico:**

$$\frac{|\vec{E}|}{|\vec{E}|_{MAX}} = |\sin \theta| = \frac{|\vec{H}|}{|\vec{H}|_{MAX}}$$

é habitual apresentar diagramas de radiação em **gráficos polares:** variável independente em ângulo  
variável dependente em distância à origem

# Diagrama de plano – E para o dipolo eléctrico

diagrama de plano – E:

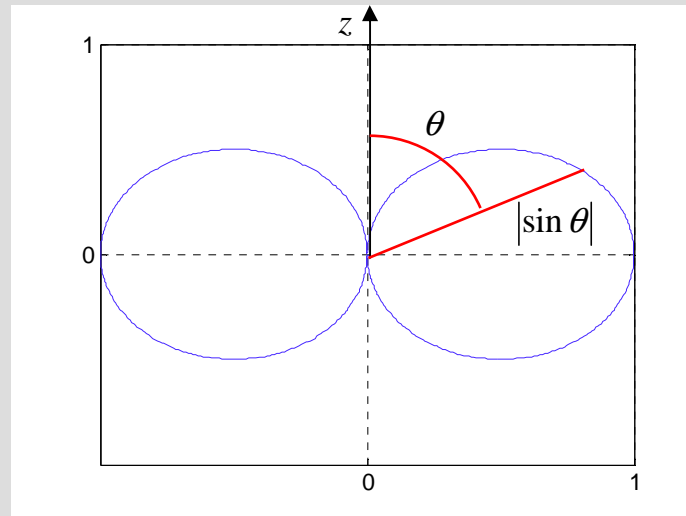


gráfico polar

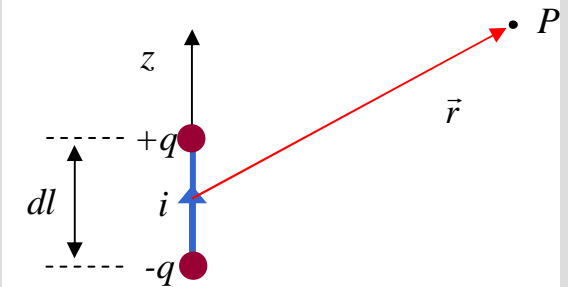
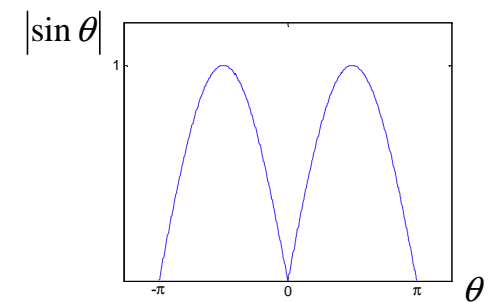


diagrama de plano – E:

amplitude normalizada do campo em função de  $\theta$  para  $\phi = \text{const.}$

dipolo eléctrico:

$$\frac{|\vec{E}|}{|\vec{E}|_{MAX}} = |\sin \theta| = \frac{|\vec{H}|}{|\vec{H}|_{MAX}}$$





# Diagrama de plano – H para o dipolo eléctrico

diagrama de plano – H:

$$\theta = \frac{\pi}{2} \rightarrow |\sin \theta| = 1$$

gráfico cartesiano

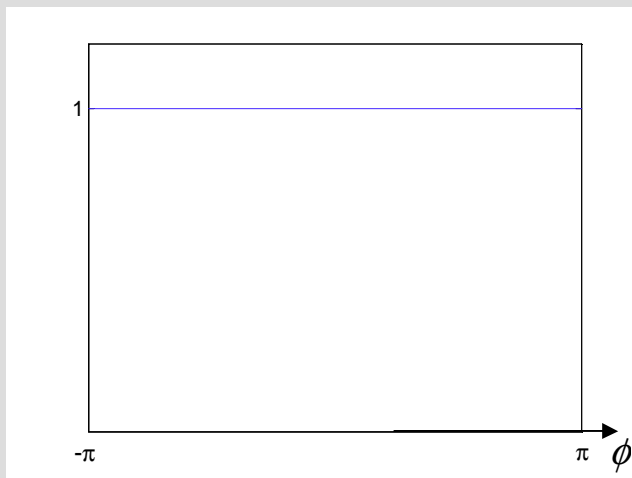


gráfico polar

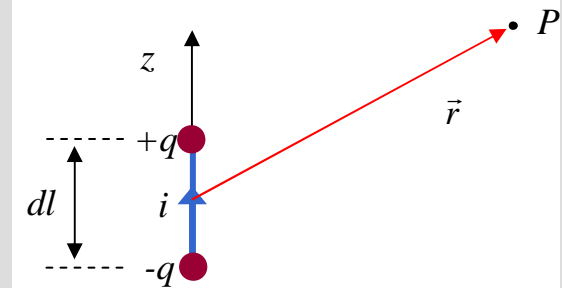
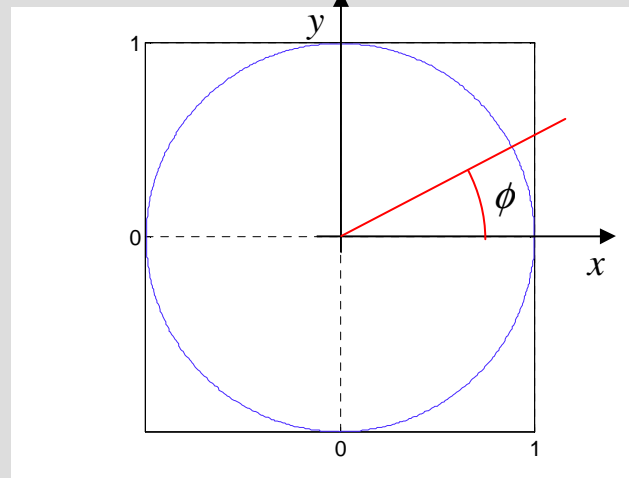


diagrama de plano – H:

amplitude normalizada do campo  
em função de  $\phi$  para  $\theta = \frac{\pi}{2}$

dipolo eléctrico:

$$\frac{|\vec{E}|}{|\vec{E}|_{MAX}} = |\sin \theta| = \frac{|\vec{H}|}{|\vec{H}|_{MAX}}$$

# Diagrama de radiação típico

diagrama de plano – H:

diagrama de plano – E:

amplitude normalizada do campo em função de  $\theta$  para  $\phi = \text{const.}$

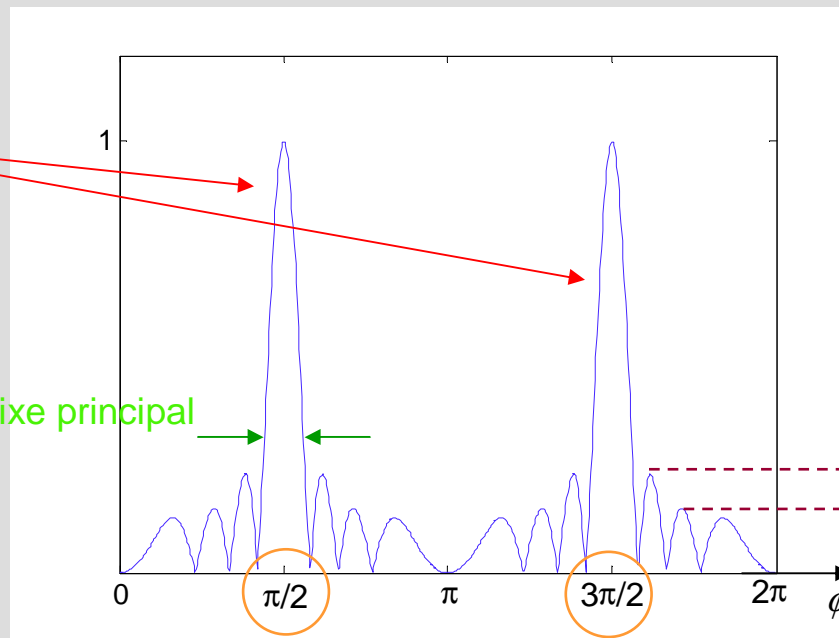
diagrama de plano – H:

amplitude normalizada do campo em função de  $\phi$  para  $\theta = \frac{\pi}{2}$

gráfico cartesiano

feixe principal

largura do feixe principal



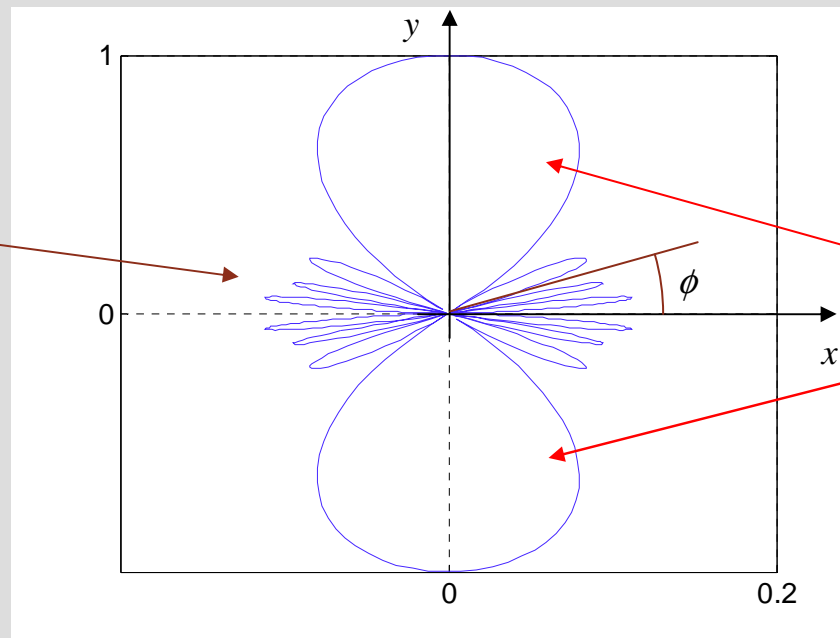
nível dos lóbulos laterais

direcção do feixe principal

## Diagrama de radiação típico

diagrama de plano – H:

gráfico polar



lóbulos laterais

diagrama de plano – E:

amplitude normalizada do campo  
em função de  $\theta$  para  $\phi = \text{const.}$

diagrama de plano – H:

amplitude normalizada do campo  
em função de  $\phi$  para  $\theta = \frac{\pi}{2}$

feixe  
principal

## parâmetros característicos

### 1. largura do feixe principal

- normalmente é a largura entre os pontos de metade da potência máxima
- pode ser a largura entre os primeiros zeros

### 2. nível dos lóbulos laterais

- amplitude dos lóbulos laterais
- habitualmente interessa apenas a amplitude do lóbulo mais próximo do feixe principal

### 3. directividade

- mede a capacidade global da antena radiar potência numa dada direcção
- pode ser calculado a partir do **ganho direccionado**, o qual depende da intensidade de radiação

## Ganho direccional e directividade

### ganho direccional

$$G_D(\theta, \phi) = \frac{4\pi U(\theta, \phi)}{P_r}$$

nota:

$$G_D(\theta, \phi) = \frac{U(\theta, \phi)}{\frac{1}{4\pi} \oint U(\theta, \phi) d\Omega}$$

← o ganho direccional é a razão entre a intensidade de radiação numa dada direcção e a intensidade de radiação média

### directividade

$$D = (G_D)_{MAX}$$

## Ganho direccional e directividade do dipolo eléctrico

**ganho direccional:** 
$$G_D(\theta, \phi) = \frac{4\pi U(\theta, \phi)}{P_r} = \frac{3}{2} \sin^2 \theta$$

**directividade** 
$$D = (G_D)_{MAX} = G_D \Big|_{\theta=\frac{\pi}{2}} = \frac{3}{2}$$

### dipolo eléctrico

$$P_r = \frac{I^2 dl^2 \beta^2 \eta_0}{12\pi}$$

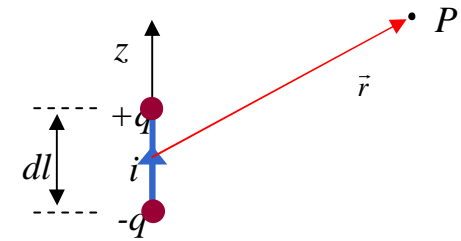
$$U = \frac{\eta_0}{2} \left( \frac{Idl}{4\pi} \beta \sin \theta \right)^2$$

# Antenas finas lineares

**dipolo eléctrico** → baixa resistência de radiação  
baixa directividade

← não é radiador eficiente!

**dipolo eléctrico:**



$$R_r = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2$$

$$D = \frac{3}{2}$$

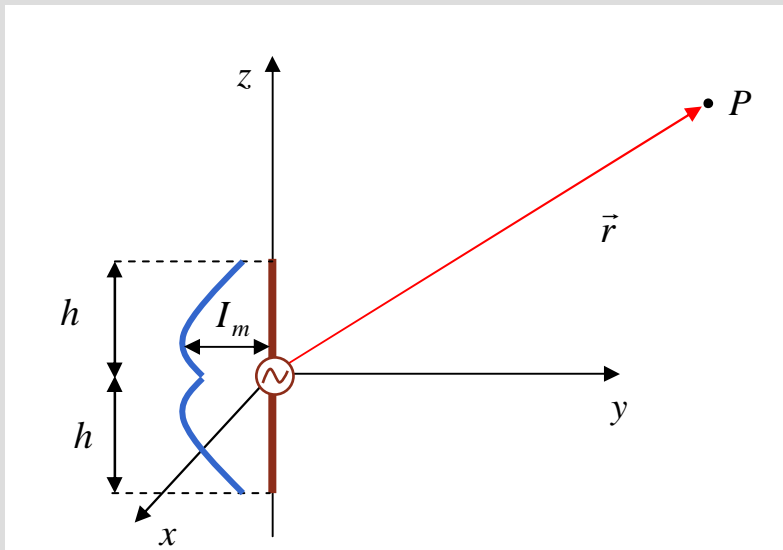
**antena fina linear:** comprimento comparável ao comprimento de onda

necessário considerar **distribuição de corrente na antena**

- distribuição exacta não tem solução analítica
- corrente nas extremidades tem que ser zero

**admitir:**

$$I(z') = I_m \sin[\beta(h - |z'|)] = \begin{cases} I_m \sin[\beta(h - z')], & z' > 0 \\ I_m \sin[\beta(h + z')], & z' < 0 \end{cases}$$



# Antenas finas lineares – campo eléctrico elemental

princípio da sobreposição:  $\vec{E} = \int d\vec{E}$  e  $\vec{H} = \int d\vec{H}$

admitir:

$$d\vec{E} = dE_\theta \hat{\theta}$$

$$d\vec{H} = dH_\phi \hat{\phi}$$

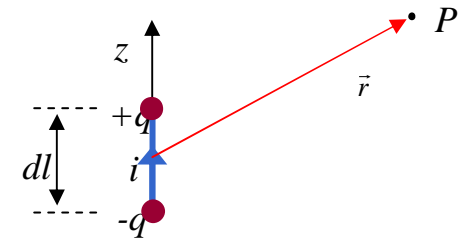
$$\frac{dE_\theta}{dH_\phi} = \eta_0$$

e

$$dE_\theta = j \frac{I(z') dz'}{4\pi R} \left( \frac{e^{-j\beta R}}{R} \right) \eta_0 \beta \sin \theta$$

onde  $R = |\vec{r} - \vec{r}'|$

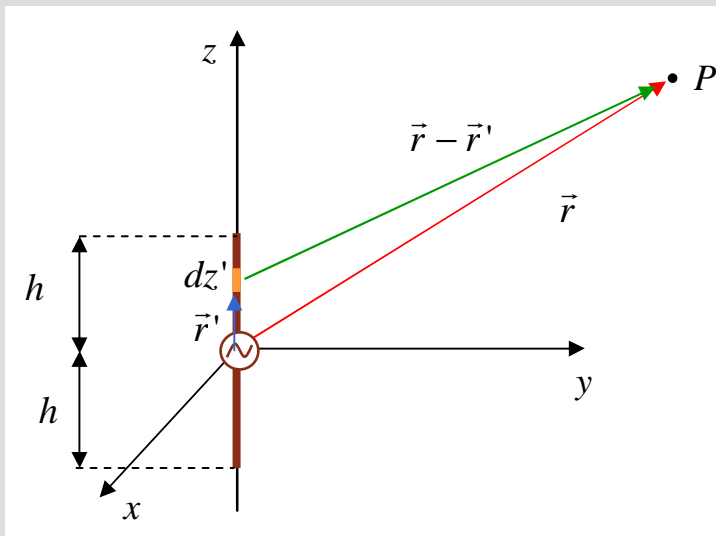
dipolo eléctrico:



campo distante:

$$\vec{E} = j \frac{Idl}{4\pi} \left( \frac{e^{-j\beta r}}{r} \right) \eta_0 \beta \sin \theta \hat{\theta}$$

$$\vec{H} = j \frac{Idl}{4\pi} \left( \frac{e^{-j\beta r}}{r} \right) \beta \sin \theta \hat{\phi}$$



$$\vec{r}' = z' \hat{z}$$

$$\vec{r} = r \hat{r} = x \hat{x} + y \hat{y} + z \hat{z}$$

$$R = \sqrt{x^2 + y^2 + (z - z')^2} = \sqrt{r^2 + (z')^2 - 2zz'} \cong r - z' \cos \theta$$

$$r \gg z'$$

$$z = r \cos \theta$$



# Antenas finas lineares – campo eléctrico

$$R \cong r - z' \cos \theta \longrightarrow \begin{cases} e^{-j\beta R} \cong e^{-j\beta r} e^{j\beta z' \cos \theta} \\ \frac{1}{R} \cong \frac{1}{r} \end{cases}$$

$$dE_{\theta} = j \frac{I(z') dz'}{4\pi} \left( \frac{e^{-j\beta r}}{r} \right) e^{j\beta z' \cos \theta} \eta_0 \beta \sin \theta$$

$$E_{\theta} = \int_{-h}^h dE_{\theta}$$

$$E_{\theta} = j \frac{\eta_0}{4\pi} \left( \frac{e^{-j\beta r}}{r} \right) \beta \sin \theta \int_{-h}^h I(z') e^{j\beta z' \cos \theta} dz'$$

comprimento efectivo  $l_e(\theta)$

**antena fina linear:**

$$dE_{\theta} = j \frac{I(z') dz'}{4\pi} \left( \frac{e^{-j\beta R}}{R} \right) \eta_0 \beta \sin \theta$$

$$\frac{dE_{\theta}}{dH_{\phi}} = \eta_0$$

## notas

1.  $H_{\phi} = \int_{-h}^h dH_{\phi} = \frac{1}{\eta_0} \int_{-h}^h dE_{\theta} = \frac{E_{\theta}}{\eta_0}$
2.  $E_{\theta} = j \frac{\eta_0}{4\pi} \left( \frac{e^{-j\beta r}}{r} \right) \beta l_e(\theta)$

# Antenas finas lineares – campo eléctrico

$$E_{\theta} = j \frac{\eta_0}{4\pi} \left( \frac{e^{-j\beta r}}{r} \right) \beta \sin \theta \int_{-h}^h I(z') e^{j\beta z' \cos \theta} dz'$$

$$I(z') = \begin{cases} I_m \sin[\beta(h - z')], & z' > 0 \\ I_m \sin[\beta(h + z')], & z' < 0 \end{cases} \quad \text{função par}$$

$$e^{j\beta z' \cos \theta} = \cos(\beta z' \cos \theta) + j \sin(\beta z' \cos \theta)$$

função par                  função ímpar

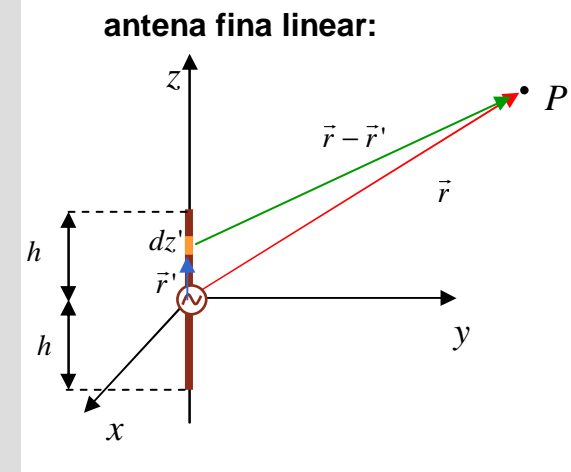
$$E_{\theta} = j \frac{\eta_0}{4\pi} \left( \frac{e^{-j\beta r}}{r} \right) \beta \sin \theta \cdot 2 \int_0^h I(z') \cos(\beta z' \cos \theta) dz' = j \frac{\eta_0 I_m}{2\pi} \left( \frac{e^{-j\beta r}}{r} \right) \beta \sin \theta \int_0^h \sin[\beta(h - z')] \cos(\beta z' \cos \theta) dz'$$

$$E_{\theta} = j \frac{\eta_0 I_m}{2\pi} \left( \frac{e^{-j\beta r}}{r} \right) \frac{\cos(\beta h \cos \theta) - \cos(\beta h)}{\sin \theta} \quad F(\theta)$$

$$\eta_0 = 120\pi \Omega$$

$$E_{\theta} = j60 I_m \left( \frac{e^{-j\beta r}}{r} \right) F(\theta)$$

$$H_{\phi} = j \frac{I_m}{2\pi} \left( \frac{e^{-j\beta r}}{r} \right) F(\theta)$$



# Antenas finas lineares – diagramas de radiação

campo normalizado: 
$$\frac{|\vec{E}|}{|\vec{E}|_{MAX}} = \frac{|\vec{H}|}{|\vec{H}|_{MAX}} = |F(\theta)|$$



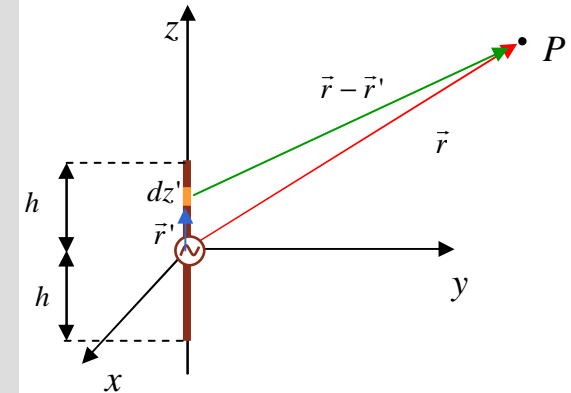
$|F(\theta)|$  define os diagramas de radiação

diagrama de plano – H  $\rightarrow \left| F\left(\theta = \frac{\pi}{2}\right) \right| = |1 - \cos(\beta h)| = \text{const.}$



gráfico polar é **circunferência**,  
tal como para o dipolo eléctrico elementar

antena fina linear:



campos distantes:

$$E_{\theta} = j60 I_m \left( \frac{e^{-j\beta r}}{r} \right) F(\theta)$$

$$H_{\phi} = j \frac{I_m}{2\pi} \left( \frac{e^{-j\beta r}}{r} \right) F(\theta)$$

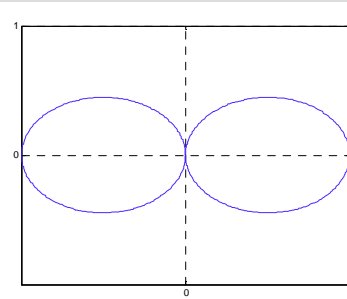
$$F(\theta) = \frac{\cos(\beta h \cos \theta) - \cos(\beta h)}{\sin \theta}$$

# Antenas finas lineares – diagramas de radiação

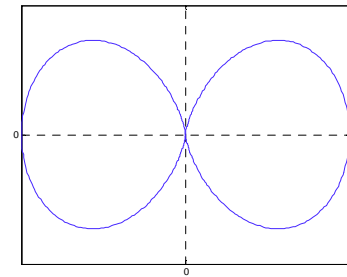
diagrama de plano – E  $\rightarrow |F(\theta)| = \left| \frac{\cos(\beta h \cos \theta) - \cos(\beta h)}{\sin \theta} \right|$

exemplos

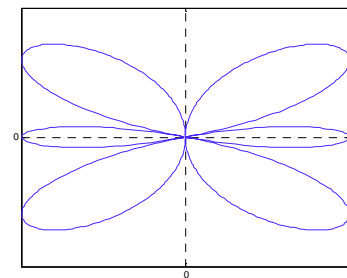
1.  $2h = \frac{\lambda}{2} \rightarrow \beta h = \frac{2\pi}{\lambda} \frac{\lambda}{4} = \frac{\pi}{2}$



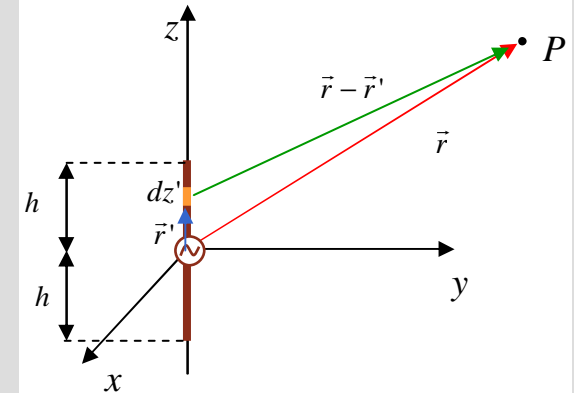
2.  $2h = \lambda \rightarrow \beta h = \pi$



3.  $2h = \frac{3\lambda}{2} \rightarrow \beta h = \frac{3\pi}{2}$



antena fina linear:



campos distantes:

$$E_{\theta} = j60 I_m \left( \frac{e^{-j\beta r}}{r} \right) F(\theta)$$

$$H_{\phi} = j \frac{I_m}{2\pi} \left( \frac{e^{-j\beta r}}{r} \right) F(\theta)$$

$$F(\theta) = \frac{\cos(\beta h \cos \theta) - \cos(\beta h)}{\sin \theta}$$

# Antena de meio comprimento de onda – potência radiada

$$2h = \frac{\lambda}{2} \longrightarrow F(\theta) = \frac{\cos\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta}$$

potência média radiada

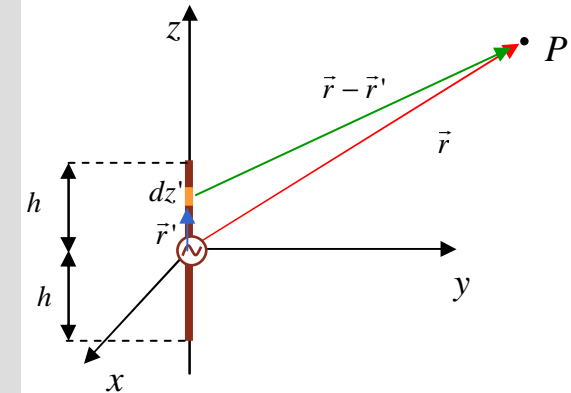
$$P_r = \oint_A \vec{S}_{med} \cdot d\vec{a} = \int_0^{2\pi} \int_0^{\pi} S_{med} r^2 \sin \theta d\theta d\phi$$

$$\vec{S}_{med} = \frac{1}{2} \text{Re}\{\vec{E} \times \vec{H}^*\} \longrightarrow S_{med} = \frac{1}{2} \frac{60 I_m^2}{2\pi r^2} F^2(\theta) = \frac{15 I_m^2}{\pi r^2} F^2(\theta)$$

$$P_r = 30 I_m^2 \int_0^{\pi} F^2(\theta) \sin \theta d\theta = 30 I_m^2 \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} d\theta = 36.54 I_m^2 \text{ W}$$

integral calculado por métodos numéricos

antena fina linear:



campos distantes:

$$E_{\theta} = j60 I_m \left( \frac{e^{-j\beta r}}{r} \right) F(\theta)$$

$$H_{\phi} = j \frac{I_m}{2\pi} \left( \frac{e^{-j\beta r}}{r} \right) F(\theta)$$

$$F(\theta) = \frac{\cos(\beta h \cos \theta) - \cos(\beta h)}{\sin \theta}$$

## Antena de meio comprimento de onda – características

### resistência de radiação

$$P_r = \frac{1}{2} R_r I_m^2 \quad \xrightarrow{P_r = 36.54 I_m^2 \text{ W}} \quad R_r = \frac{2P_r}{I_m^2} = 73.1 \Omega$$

### intensidade de radiação

$$U = r^2 S_{med} \quad \xrightarrow{S_{med} = \frac{15 I_m^2}{\pi r^2} F^2(\theta)} \quad U = \frac{15 I_m^2}{\pi} \left( \frac{\cos\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} \right)^2$$

$$F(\theta) = \frac{\cos\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta}$$

### ganho direccional

$$G_D(\theta, \phi) = \frac{4\pi U(\theta, \phi)}{P_r} \quad \xrightarrow{} \quad G_D(\theta, \phi) = 1.64 \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta}$$

### directividade

$$D = (G_D)_{MAX} \quad \xrightarrow{} \quad D = G_D\left(\theta = \frac{\pi}{2}\right) = 1.64$$

### dipolo eléctrico elementar

$$R_r = 80\pi^2 \left( \frac{dl}{\lambda} \right)^2$$

$R_r$  pequeno porque  $dl \ll \lambda$

se  $dl = 0.01\lambda \rightarrow R_r = 0.08 \Omega$

$$G_D = \frac{3}{2} \sin^2 \theta$$

$$D = 1.5$$

- grupos de antenas semelhantes, dispostas em diversas configurações, e com amplitudes e diferenças de fase apropriados para radiação com determinadas características
- a largura e a direcção do feixe principal, os níveis dos lóbulos laterais e a directividade são as características normalmente consideradas importantes
- a radiação originada por um agrupamento de antenas é obtida aplicando o **princípio da sobreposição**

## Agrupamentos de duas antenas

- para uma antena →  
(caso geral)

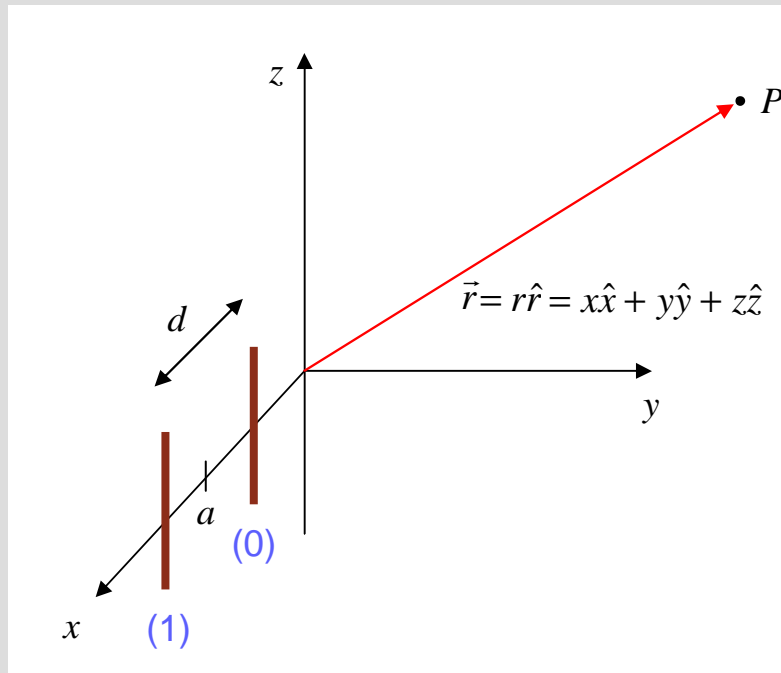
$$\vec{E} = E_m \left( \frac{e^{-j\beta R}}{R} \right) F(\theta, \phi) \hat{\theta}$$

campo distante de antena fina:

$$\vec{E} = j60 I_m \left( \frac{e^{-j\beta R}}{R} \right) F(\theta) \hat{\theta}$$

onde  $R = |\vec{r} - \vec{r}'|$

- agrupamento de duas antenas iguais, separadas de  $d$  e esfasadas de  $\xi$



antena (0):

$$\vec{E}_0 = E_{m0} \left( \frac{e^{-j\beta R_0}}{R_0} \right) F_0(\theta, \phi) \hat{\theta}$$

$$R_0 = |\vec{r} - \vec{r}_0'|$$

$$\vec{r}_0' = \left( a - \frac{d}{2} \right) \hat{x}$$

antena (1):

$$\vec{E}_1 = E_{m1} \left( \frac{e^{-j\beta R_1}}{R_1} \right) F_1(\theta, \phi) e^{j\xi} \hat{\theta}$$

$$R_1 = |\vec{r} - \vec{r}_1'|$$

$$\vec{r}_1' = \left( a + \frac{d}{2} \right) \hat{x}$$

antenas iguais:

$$E_{m0} = E_{m1} = E_m$$

$$F_0(\theta, \phi) = F_1(\theta, \phi) = F(\theta, \phi)$$



## Agrupamentos de duas antenas

**antena (0):**  $\vec{E}_0 = E_m \left( \frac{e^{-j\beta R_0}}{R_0} \right) F(\theta, \phi) \hat{\theta}$

$$R_0 = |\vec{r} - \vec{r}_0| = \sqrt{\left(x - a + \frac{d}{2}\right)^2 + y^2 + z^2} = \sqrt{r^2 + \left(a - \frac{d}{2}\right)^2 - 2x\left(a - \frac{d}{2}\right)}$$

$$\cong \sqrt{r^2 - 2x\left(a - \frac{d}{2}\right)} \cong r - \left(a - \frac{d}{2}\right) \sin \theta \cos \phi$$

$r \gg a, \frac{d}{2}$

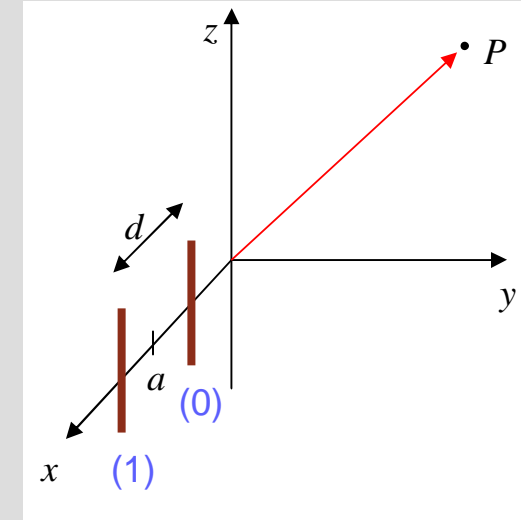
$x = r \sin \theta \cos \phi$

$\frac{1}{R_0} \cong \frac{1}{r}$

$$\vec{E}_0 = E_m \left( \frac{e^{-j\beta r}}{r} \right) e^{j\beta \left(a - \frac{d}{2}\right) \sin \theta \cos \phi} F(\theta, \phi) \hat{\theta}$$

**antena (1):**  $R_1 = |\vec{r} - \vec{r}_1| \cong r - \left(a + \frac{d}{2}\right) \sin \theta \cos \phi$

$$\vec{E}_1 = E_m \left( \frac{e^{-j\beta r}}{r} \right) e^{j\beta \left(a + \frac{d}{2}\right) \sin \theta \cos \phi} e^{j\xi} F(\theta, \phi) \hat{\theta}$$



## Agrupamentos de duas antenas – campo eléctrico

antena (0): 
$$\vec{E}_0 = E_m \left( \frac{e^{-j\beta r}}{r} \right) e^{j\beta \left( a - \frac{d}{2} \right) \sin \theta \cos \phi} F(\theta, \phi) \hat{\theta}$$

antena (1): 
$$\vec{E}_1 = E_m \left( \frac{e^{-j\beta r}}{r} \right) e^{j\beta \left( a + \frac{d}{2} \right) \sin \theta \cos \phi} e^{j\xi} F(\theta, \phi) \hat{\theta}$$

princípio da sobreposição

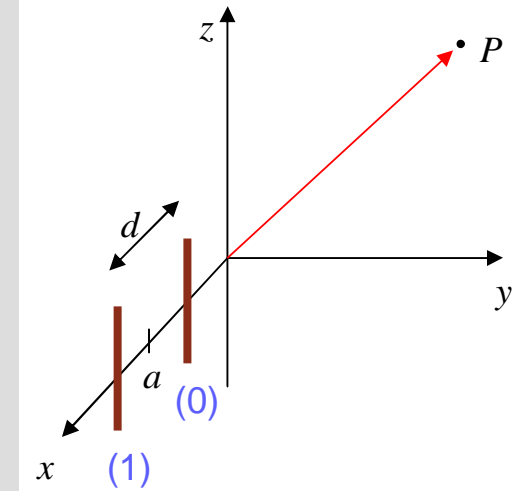
$$\vec{E} = \vec{E}_0 + \vec{E}_1 = E_m \left( \frac{e^{-j\beta r}}{r} \right) \left( e^{j\beta \left( a - \frac{d}{2} \right) \sin \theta \cos \phi} + e^{j\beta \left( a + \frac{d}{2} \right) \sin \theta \cos \phi} e^{j\xi} \right) F(\theta, \phi) \hat{\theta}$$

$$= E_m \left( \frac{e^{-j\beta r}}{r} \right) e^{j\beta a \sin \theta \cos \phi} e^{j\frac{\xi}{2}} \left( e^{-j\beta \frac{d}{2} \sin \theta \cos \phi} e^{-j\frac{\xi}{2}} + e^{j\beta \frac{d}{2} \sin \theta \cos \phi} e^{j\frac{\xi}{2}} \right) F(\theta, \phi) \hat{\theta}$$

$$= E_m \left( \frac{e^{-j\beta r}}{r} \right) e^{j\beta a \sin \theta \cos \phi} e^{j\frac{\xi}{2}} \left( e^{-j\frac{\psi}{2}} + e^{j\frac{\psi}{2}} \right) F(\theta, \phi) \hat{\theta} = 2E_m \left( \frac{e^{-j\beta r}}{r} \right) e^{j\beta a \sin \theta \cos \phi} e^{j\frac{\xi}{2}} \cos\left(\frac{\psi}{2}\right) F(\theta, \phi) \hat{\theta}$$

$$\psi = \beta d \sin \theta \cos \phi + \xi$$

$$|\vec{E}| = \frac{2|E_m|}{r} |F(\theta, \phi)| \left| \cos\left(\frac{\psi}{2}\right) \right|$$



## Factor de elemento e factor de grupo

$$|\vec{E}| = \frac{2|E_m|}{r} \cdot |F(\theta, \phi)| \cdot \left| \cos\left(\frac{\psi}{2}\right) \right|$$

→ depende da geometria do agregado e das fases relativas de cada antena

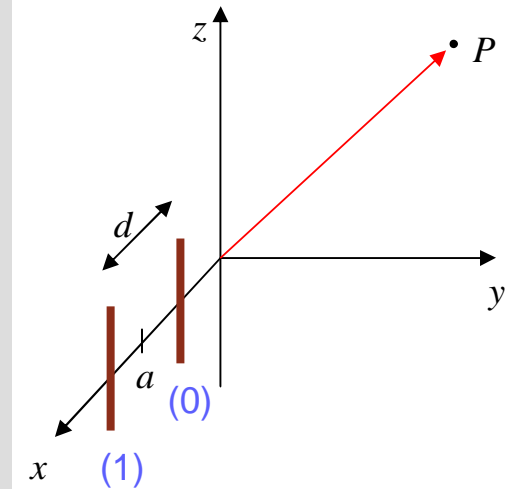
**factor de grupo normalizado**

depende de cada antena

**factor de elemento**

seja  $|A(\psi)|$  o factor de grupo normalizado

$|\vec{E}|$  é proporcional ao produto de  $|F(\theta, \phi)|$  por  $|A(\psi)|$

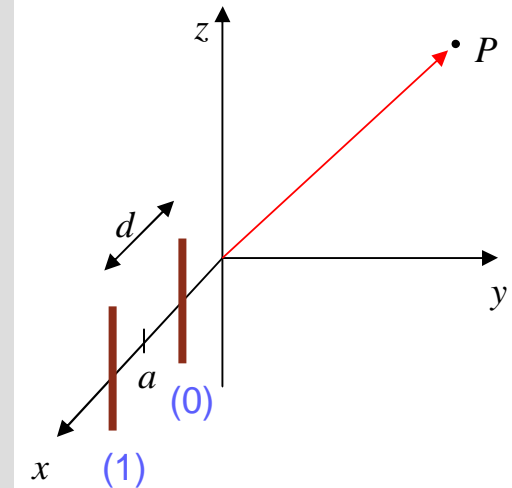


$$\psi = \beta d \sin \theta \cos \phi + \xi$$

## Agregado de duas antenas de $\lambda/2$

agregado de duas antenas de  $\lambda/2 \rightarrow F(\theta, \phi) = F(\theta) = \frac{\cos\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta}$

$$\vec{E} = \frac{2|E_m|}{r} \cdot \left| \frac{\cos\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} \right| \cdot \left| \cos\left(\frac{\psi}{2}\right) \right|$$



$$|\vec{E}| = \frac{2|E_m|}{r} \cdot |F(\theta, \phi)| \cdot |A(\psi)|$$

$$|A(\psi)| = \left| \cos\left(\frac{\psi}{2}\right) \right|$$

$$\psi = \beta d \sin \theta \cos \phi + \xi$$

### notas

1.  $\psi = \beta d \sin \theta \cos \phi + \xi \rightarrow$  diagrama de plano – E depende de  $|F(\theta)|$  e de  $|A(\psi)|$

2.  $\phi = \pm \pi/2 \rightarrow \psi = \xi \rightarrow$  diagrama de plano – E igual ao de um elemento

3.  $\theta = \pi/2 \rightarrow \boxed{\psi = \beta d \cos \phi + \xi}$   
 $\boxed{F\left(\theta = \frac{\pi}{2}, \phi\right) = 1} \rightarrow$  diagrama de plano – H definido pelo factor de grupo

# Agregado de duas antenas de $\lambda/2$ – exemplos

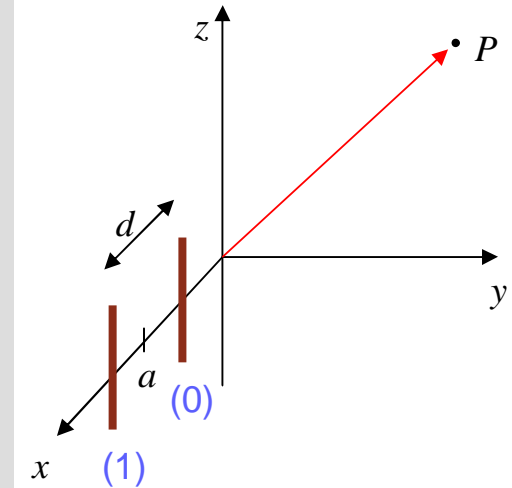
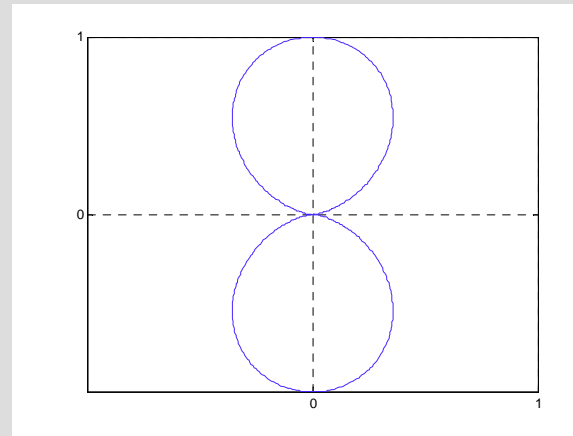
## caso 1

$$d = \frac{\lambda}{2} \quad e \quad \xi = 0$$

$$\psi = \pi \cos \phi$$

$$|A(\psi)| = \left| \cos \left( \frac{\pi}{2} \cos \phi \right) \right|$$

diagrama de plano – H



$$|\vec{E}| = \frac{2|E_m|}{r} \cdot |F(\theta, \phi)| \cdot |A(\psi)|$$

$$|A(\psi)| = \left| \cos \left( \frac{\psi}{2} \right) \right|$$

$$\psi = \beta d \sin \theta \cos \phi + \xi$$

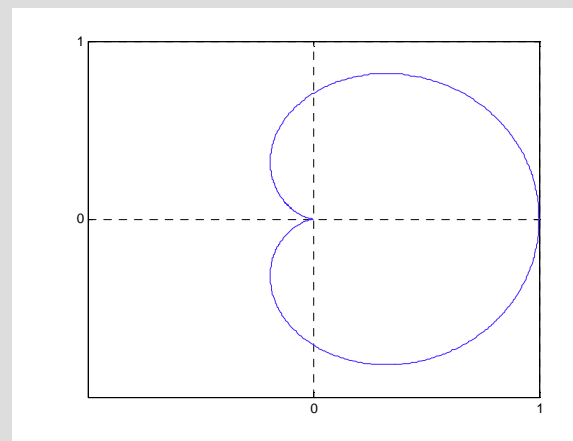
## caso 2

$$d = \frac{\lambda}{4} \quad e \quad \xi = -\frac{\pi}{2}$$

$$\psi = \frac{\pi}{2} (\cos \phi - 1)$$

$$|A(\psi)| = \left| \cos \left[ \frac{\pi}{4} (\cos \phi - 1) \right] \right|$$

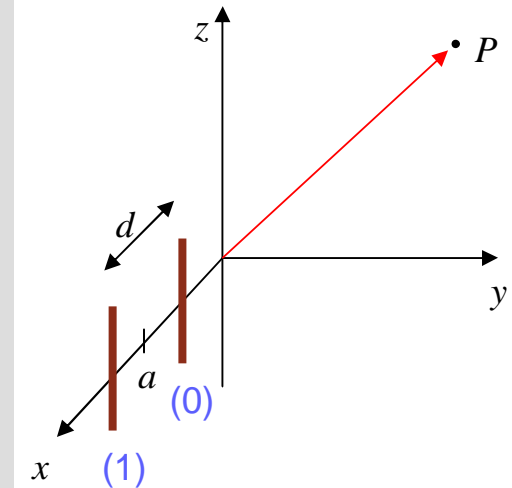
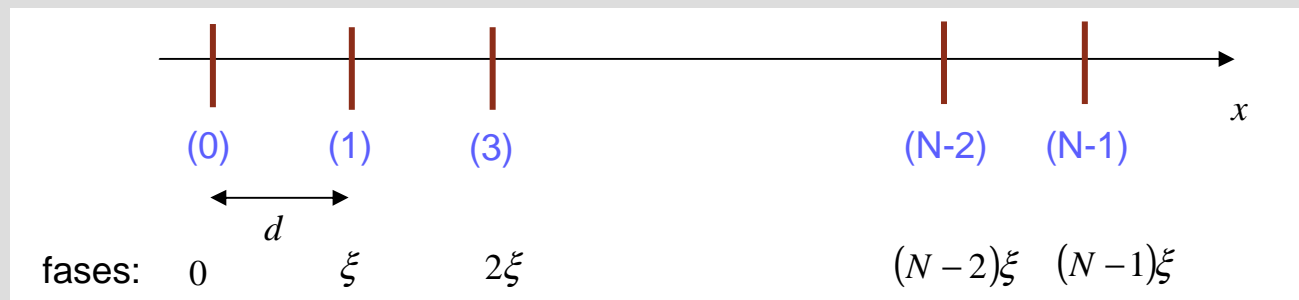
diagrama de plano – H



## Agrupamentos lineares uniformes

- antenas idênticas, igualmente espaçadas ao longo de uma direcção
- antenas alimentadas por correntes de igual amplitude e com iguais diferenças de fase entre antenas consecutivas

### agregado de N antenas



$$|\vec{E}| = \frac{2|E_m|}{r} \cdot |F(\theta, \phi)| \cdot |A(\psi)|$$

$$|A(\psi)| = \left| \cos\left(\frac{\psi}{2}\right) \right|$$

$$\psi = \beta d \sin \theta \cos \phi + \xi$$

antenas iguais  $\longrightarrow$  diagramas de radiação proporcionais a  $|F(\theta, \phi)| \cdot |A(\psi)|$

onde  $\psi = \beta d \sin \theta \cos \phi + \xi$

## Agrupamentos lineares uniformes – factor de grupo

$$|A(\psi)| = \frac{1}{N} \left| 1 + e^{j\psi} + e^{j2\psi} + \dots + e^{j(N-1)\psi} \right|$$

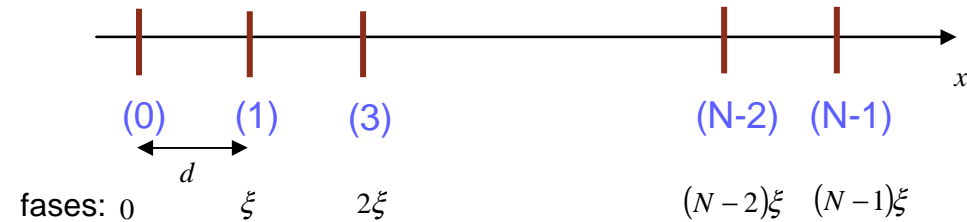
progressão geométrica

$$|A(\psi)| = \frac{1}{N} \left| \frac{1 - e^{jN\psi}}{1 - e^{j\psi}} \right|$$

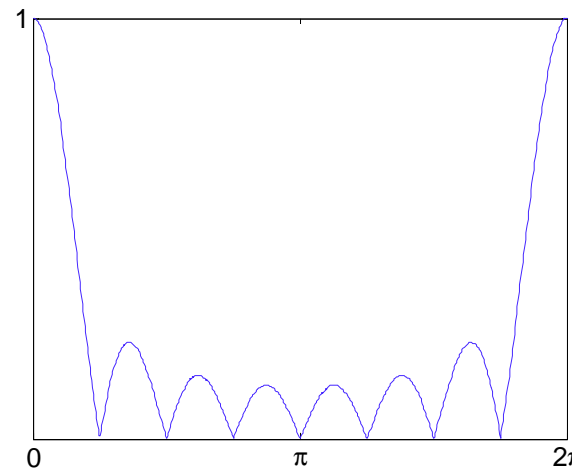
$$|A(\psi)| = \frac{1}{N} \left| \frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} \right|$$

$N = 8$

agregado de N antenas



diagramas de radiação proporcionais a  $|F(\theta, \phi)| \cdot |A(\psi)|$   
 $\psi = \beta d \sin \theta \cos \phi + \xi$



# Agrupamentos lineares uniformes – diagrama de plano – H

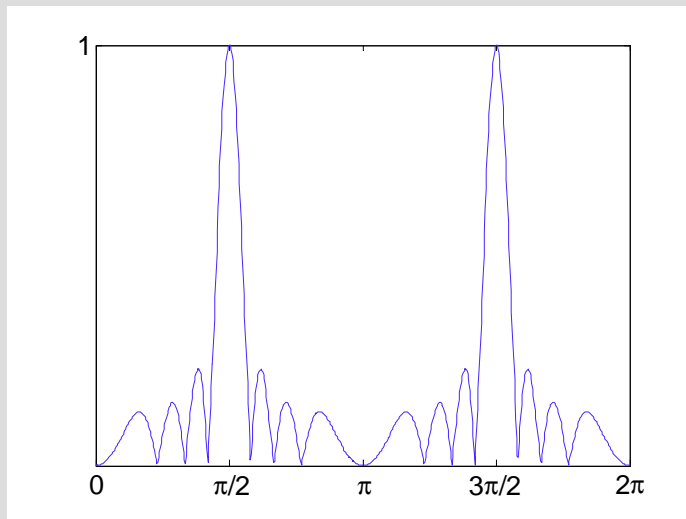
diagrama de plano – H  $\left( \theta = \frac{\pi}{2} \right)$

agrupamento de 8 antenas lineares finas

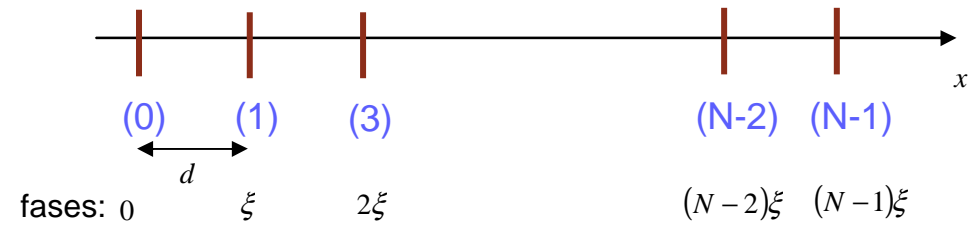
$$d = \frac{\lambda}{2} \quad \text{e} \quad \xi = 0$$

↓

$$\psi = \pi \cos \phi$$



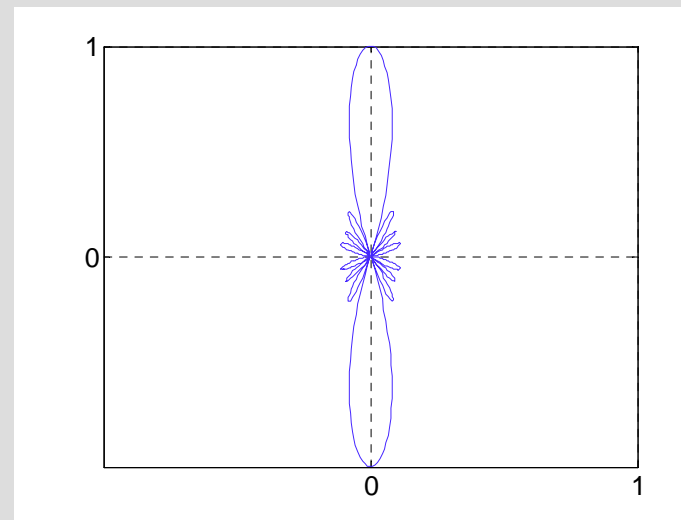
agregado de N antenas



diagramas de radiação proporcionais a  $|F(\theta, \phi)| \cdot |A(\psi)|$

$$\psi = \beta d \sin \theta \cos \phi + \xi$$

antenas finas lineares  $\rightarrow F\left(\frac{\pi}{2}\right) = 1$





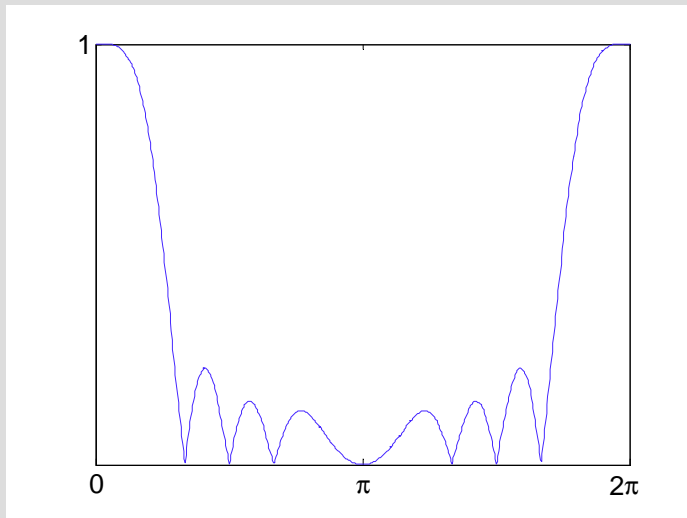
# Agrupamentos lineares uniformes – diagrama de plano – H

diagrama de plano – H  $\left( \theta = \frac{\pi}{2} \right)$

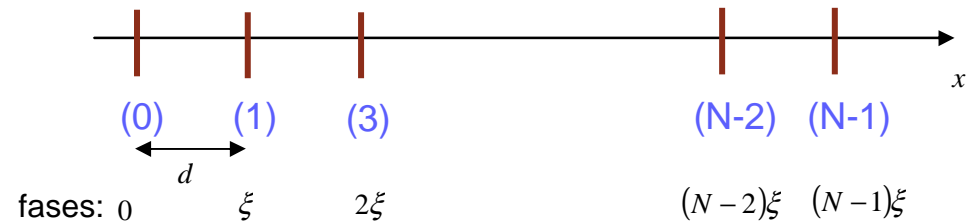
agrupamento de 8 antenas lineares finas

$$d = \frac{\lambda}{4} \quad \text{e} \quad \xi = -\frac{\pi}{2}$$

$$\psi = \frac{\pi}{2}(\cos \phi - 1)$$



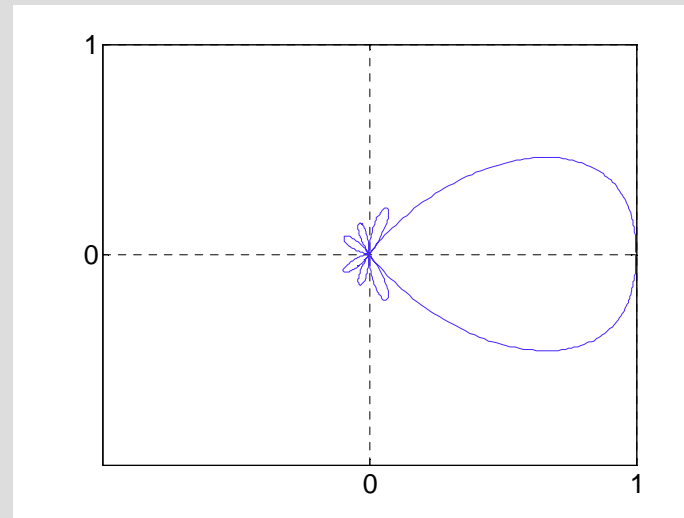
agregado de N antenas



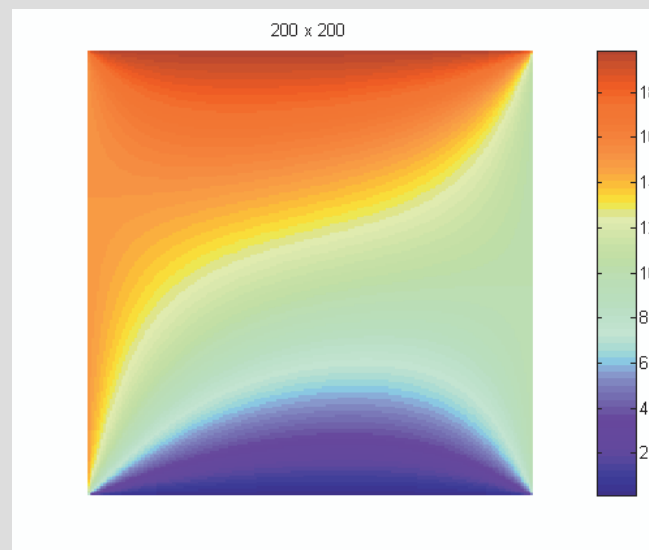
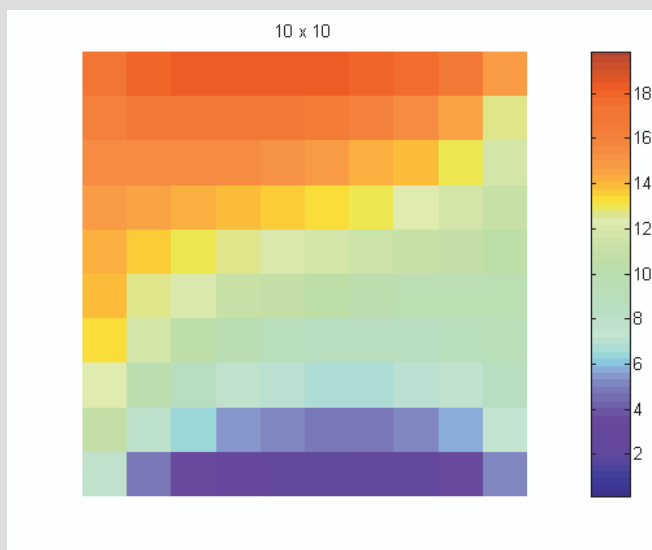
diagramas de radiação proporcionais a  $|F(\theta, \phi)| \cdot |A(\psi)|$

$$\psi = \beta d \sin \theta \cos \phi + \xi$$

antenas finas lineares  $\rightarrow F\left(\frac{\pi}{2}\right) = 1$



# Métodos Numéricos no Traçado de Campos



OE - MIEEC 2006/2007

Diferenças finitas

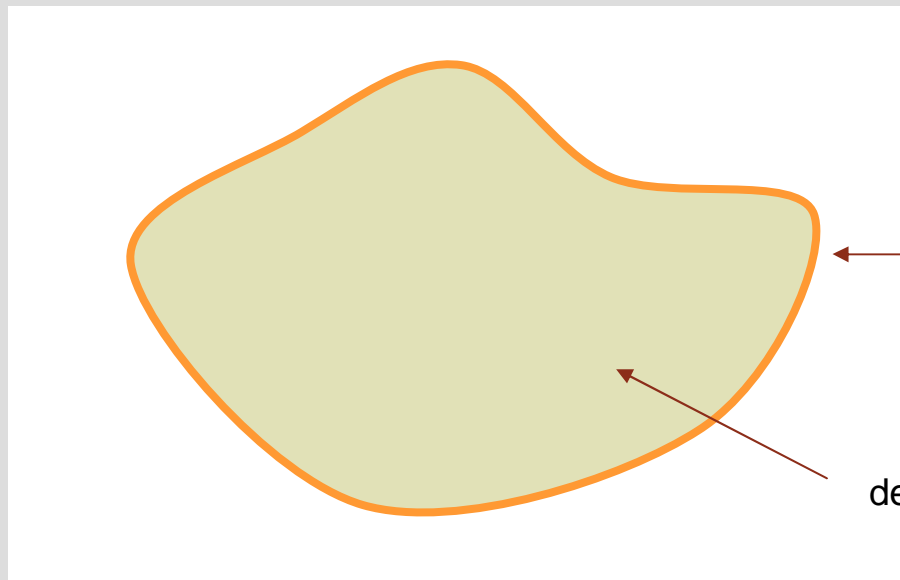
Elementos finitos

→ aplicação à determinação de potenciais electrostáticos

# Equação de Poisson

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

← descreve o potencial eléctrico em situação estacionária



← fronteira com potencial conhecido  
(ou restrições sobre campo eléctrico  
normal ou tangencial)

← densidade de carga  $\rho$

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

← equação diferencial parcial

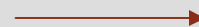
## **solução da equação**

**analítica** → problemas com simetria

**numérica** → caso geral

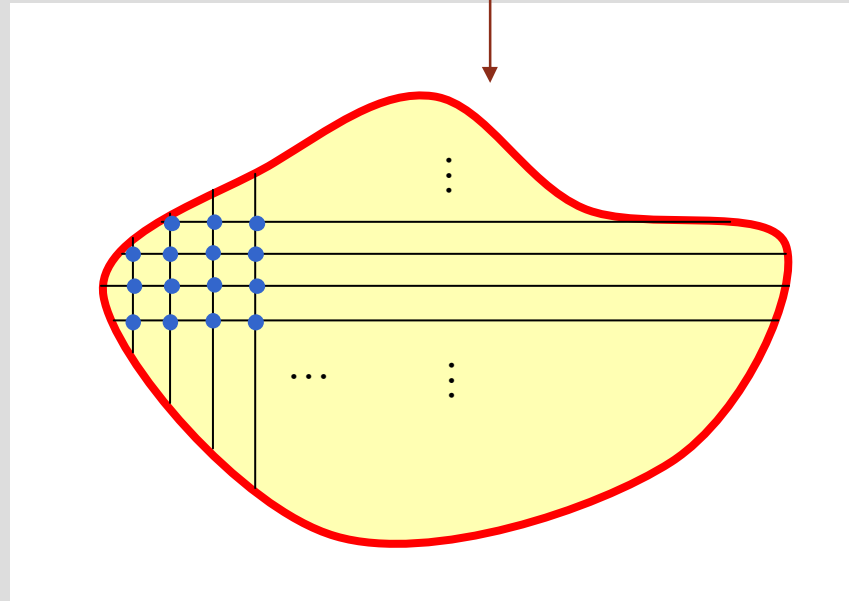
**caso bi-dimensional** →

$$V = V(x, y)$$

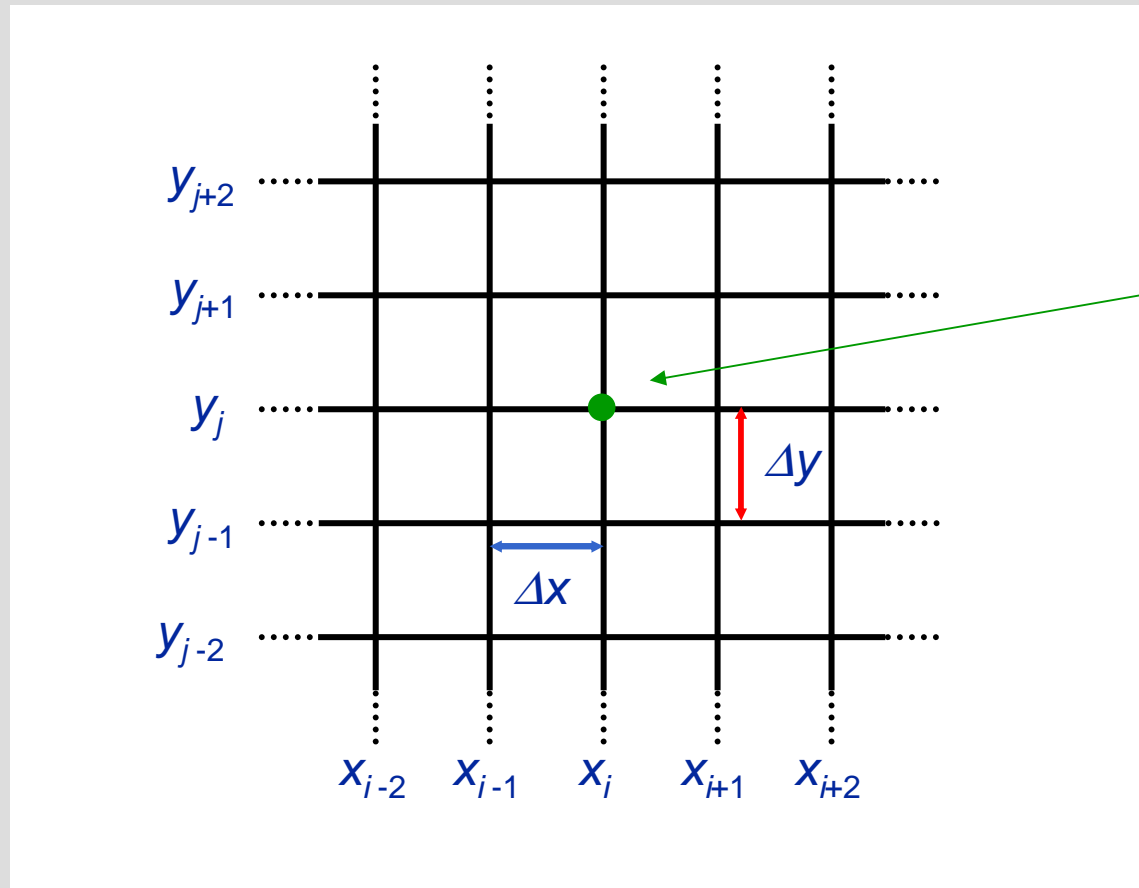


$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\rho(x, y)}{\epsilon}$$

equação diferencial parcial  $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\rho(x, y)}{\epsilon}$  é transformada  
numa **equação às diferenças** através de uma **discretização do espaço**



# Método das diferenças finitas



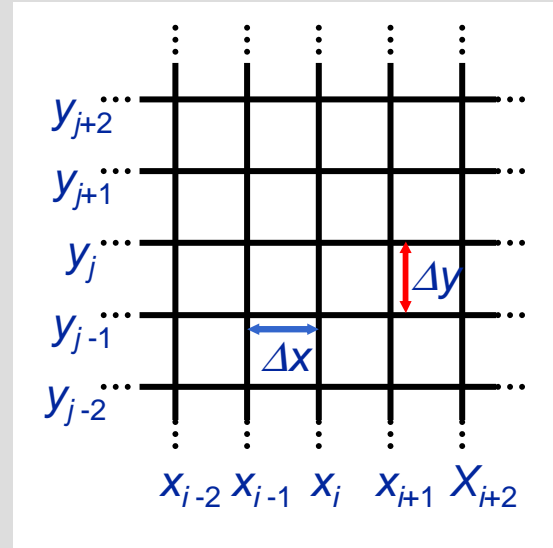
$$V_{i,j} = V(x_i, y_j)$$

valores a determinar

## primeiras derivadas

$$\left. \frac{\partial V}{\partial x} \right|_{x=x_i, y=y_j} \cong \frac{V(x_i + \Delta x, y_j) - V(x_i - \Delta x, y_j)}{2\Delta x} = \frac{V_{i+1,j} - V_{i-1,j}}{2\Delta x}$$

$$\left. \frac{\partial V}{\partial y} \right|_{x=x_i, y=y_j} \cong \frac{V(x_i, y_j + \Delta y) - V(x_i, y_j - \Delta y)}{2\Delta y} = \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta y}$$



## segundas derivadas

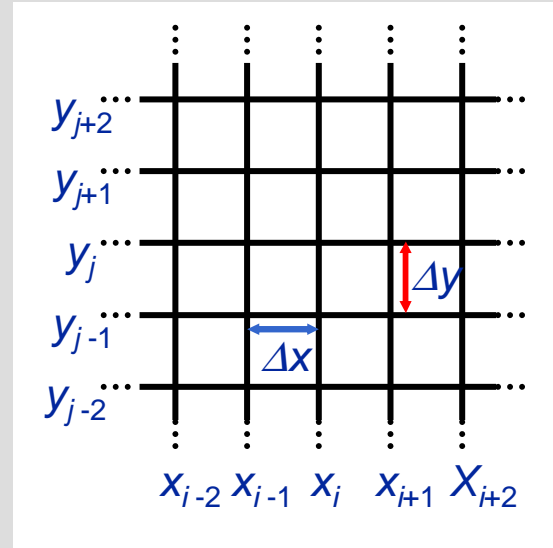
$$\left. \frac{\partial^2 V}{\partial x^2} \right|_{x=x_i, y=y_j} \cong \frac{\left. \frac{\partial V}{\partial x} \right|_{x=x_i+\Delta x/2, y=y_j} - \left. \frac{\partial V}{\partial x} \right|_{x=x_i-\Delta x/2, y=y_j}}{\Delta x} \cong \frac{V(x_i + \frac{\Delta x}{2} + \frac{\Delta x}{2}, y_j) - V(x_i + \frac{\Delta x}{2} - \frac{\Delta x}{2}, y_j)}{\Delta x} - \frac{V(x_i - \frac{\Delta x}{2} + \frac{\Delta x}{2}, y_j) - V(x_i - \frac{\Delta x}{2} - \frac{\Delta x}{2}, y_j)}{\Delta x}$$



## primeiras derivadas

$$\left. \frac{\partial V}{\partial x} \right|_{x=x_i, y=y_j} \cong \frac{V(x_i + \Delta x, y_j) - V(x_i - \Delta x, y_j)}{2\Delta x} = \frac{V_{i+1,j} - V_{i-1,j}}{2\Delta x}$$

$$\left. \frac{\partial V}{\partial y} \right|_{x=x_i, y=y_j} \cong \frac{V(x_i, y_j + \Delta y) - V(x_i, y_j - \Delta y)}{2\Delta y} = \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta y}$$



## segundas derivadas

$$\left. \frac{\partial^2 V}{\partial x^2} \right|_{x=x_i, y=y_j} \cong \frac{\left. \frac{\partial V}{\partial x} \right|_{x=x_i+\Delta x/2, y=y_j} - \left. \frac{\partial V}{\partial x} \right|_{x=x_i-\Delta x/2, y=y_j}}{\Delta x} \cong \frac{\frac{V(x_i + \Delta x, y_j) - V(x_i, y_j)}{\Delta x} - \frac{V(x_i, y_j) - V(x_i - \Delta x, y_j)}{\Delta x}}{\Delta x} = \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{\Delta x^2}$$

$$\left. \frac{\partial^2 V}{\partial y^2} \right|_{x=x_i, y=y_j} \cong \frac{\left. \frac{\partial V}{\partial y} \right|_{x=x_i, y=y_j+\Delta y/2} - \left. \frac{\partial V}{\partial y} \right|_{x=x_i, y=y_j-\Delta y/2}}{\Delta y} \cong \frac{\frac{V(x_i, y_j + \Delta y) - V(x_i, y_j)}{\Delta y} - \frac{V(x_i, y_j) - V(x_i, y_j - \Delta y)}{\Delta y}}{\Delta y} = \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta y^2}$$

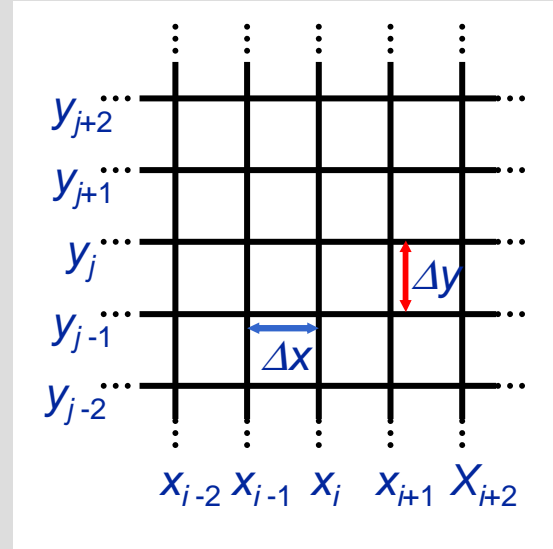
# Equação às diferenças

equação de Poisson: 
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\rho(x, y)}{\epsilon}$$

$$\frac{\partial^2 V}{\partial x^2} \Big|_{x=x_i, y=y_j} \approx \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{\Delta x^2}$$

$$\frac{\partial^2 V}{\partial y^2} \Big|_{x=x_i, y=y_j} \approx \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta y^2}$$

$\Delta x = \Delta y = h$



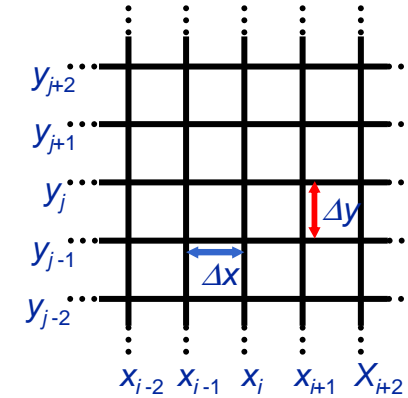
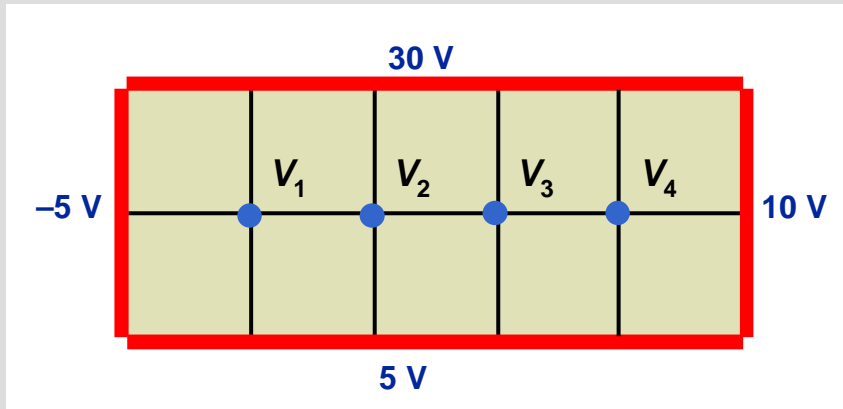
$$V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = -\frac{h^2 \rho_{i,j}}{\epsilon}$$

← uma equação para cada ponto  $(x_i, y_j)$

**equação (linear) às diferenças**

# Diferenças finitas – Exemplo 1

Considere a região da figura onde  $\rho = 0$



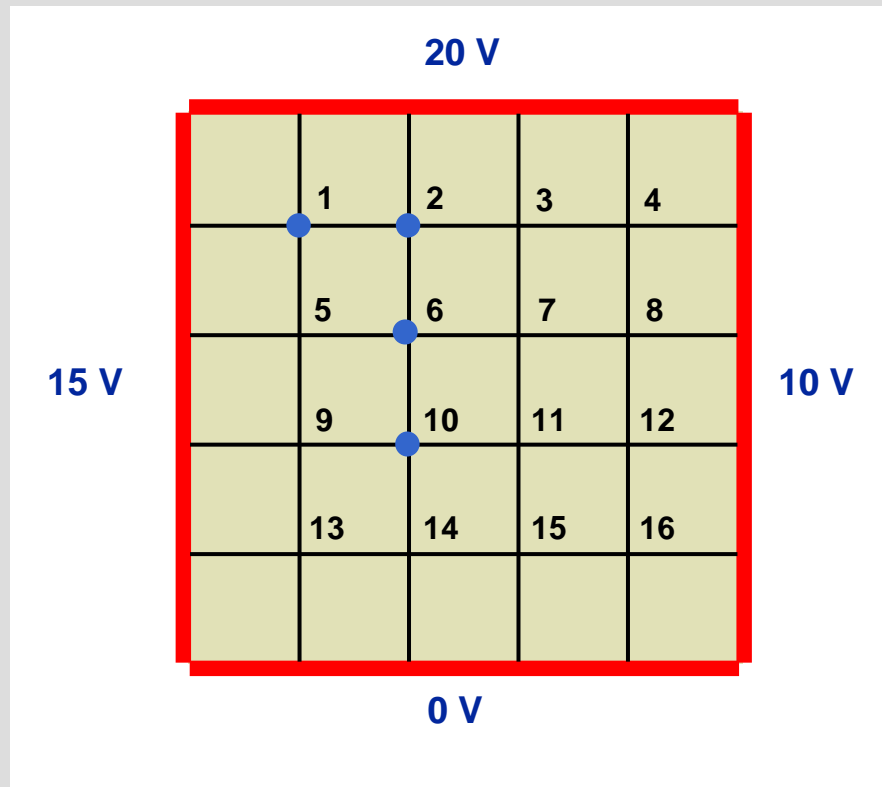
para cada ponto  $(x_i, y_j)$

$$V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = -\frac{h^2 \rho_{i,j}}{\epsilon}$$

$$\begin{aligned} -5 + 30 + V_2 + 5 - 4V_1 &= 0 \\ V_1 + 30 + V_3 + 5 - 4V_2 &= 0 \\ V_2 + 30 + V_4 + 5 - 4V_3 &= 0 \\ V_3 + 30 + 10 + 5 - 4V_4 &= 0 \end{aligned}$$

## Diferenças finitas – Exemplo 2

Considere a região da figura onde  $\rho = 0$



para cada ponto  $(x_i, y_j)$

$$V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = -\frac{h^2 \rho_{i,j}}{\epsilon}$$

$$\text{nó 1} \rightarrow -4V_1 + V_2 + V_5 = -15 - 20$$

$$\text{nó 2} \rightarrow V_1 - 4V_2 + V_3 + V_6 = -20$$

$$\text{nó 6} \rightarrow V_2 + V_5 - 4V_6 + V_7 + V_{10} = 0$$

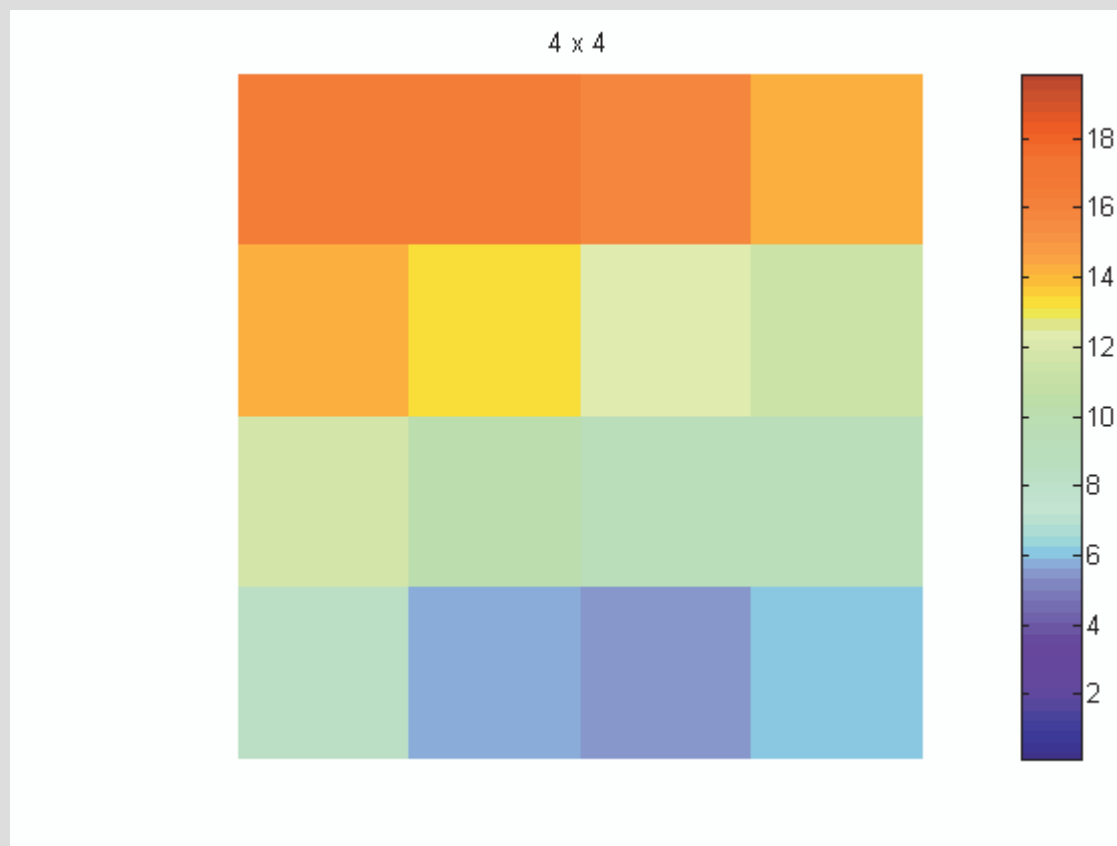
$$\text{nó 10} \rightarrow V_6 + V_9 - 4V_{10} + V_{11} + V_{14} = 0$$

# Diferenças finitas – Exemplo 2

Sistema de equações a resolver para 4x4 pontos

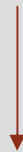
$$\begin{bmatrix}
 -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4
 \end{bmatrix}
 \begin{bmatrix}
 V_1 \\
 V_2 \\
 V_3 \\
 V_4 \\
 V_5 \\
 V_6 \\
 V_7 \\
 V_8 \\
 V_9 \\
 V_{10} \\
 V_{11} \\
 V_{12} \\
 V_{13} \\
 V_{14} \\
 V_{15} \\
 V_{16}
 \end{bmatrix}
 =
 \begin{bmatrix}
 -35 \\
 -20 \\
 -20 \\
 -30 \\
 -15 \\
 0 \\
 0 \\
 -10 \\
 -15 \\
 0 \\
 0 \\
 -10 \\
 -15 \\
 0 \\
 0 \\
 -10
 \end{bmatrix}$$

## Solução do sistema de equações



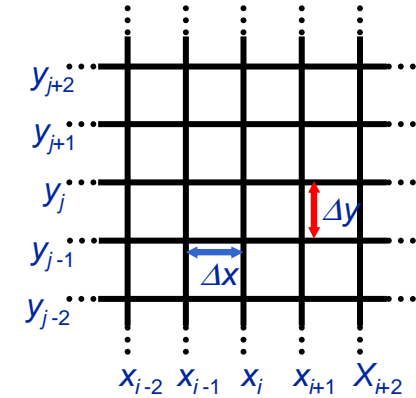
# Diferenças finitas – Sistema de equações

equação de Poisson: 
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\rho(x, y)}{\epsilon}$$



sistema de equações: 
$$V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = -\frac{h^2 \rho_{i,j}}{\epsilon}$$

→ uma incógnita  $V_{i,j}$  e uma equação para cada ponto  $(x_i, y_j)$



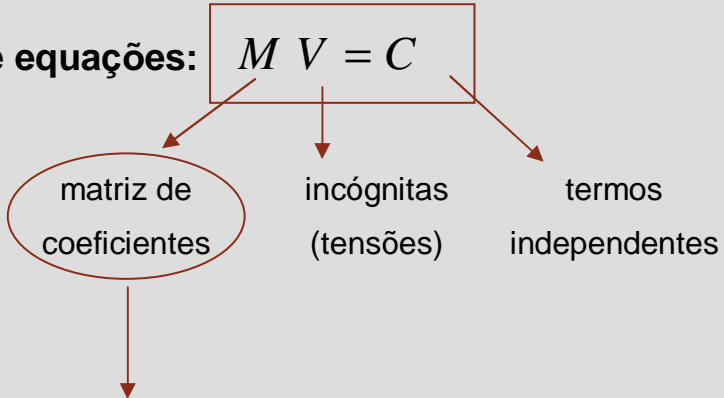
$\Delta x = \Delta y = h$

a solução do sistema de equações é uma aproximação da solução da equação de Poisson →  $V_{i,j} \cong V(x_i, y_j)$

a aproximação será tanto melhor quanto menor for o espaçamento entre os pontos da grelha → sistemas de elevada dimensão

Sistema de equações:

$$M V = C$$



**matriz esparsa** (grande parte dos elementos são nulos)

-4	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$V_1$	-35
1	-4	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$V_2$	-20
0	1	-4	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	$V_3$	-20
0	0	1	-4	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	$V_4$	-30
1	0	0	0	-4	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	$V_5$	-15
0	1	0	0	1	-4	1	0	0	1	0	0	0	0	0	0	0	0	0	0	$V_6$	0
0	0	1	0	0	1	-4	1	0	0	1	0	0	0	0	0	0	0	0	0	$V_7$	0
0	0	0	1	0	0	1	-4	0	0	0	1	0	0	0	0	0	0	0	0	$V_8$	-10
0	0	0	0	1	0	0	0	-4	1	0	0	1	0	0	0	0	0	0	0	$V_9$	-15
0	0	0	0	0	1	0	0	1	-4	1	0	0	1	0	0	0	0	0	0	$V_{10}$	0
0	0	0	0	0	0	1	0	0	1	-4	1	0	0	1	0	0	0	0	0	$V_{11}$	0
0	0	0	0	0	0	0	1	0	0	1	-4	0	0	0	0	1	0	0	0	$V_{12}$	-10
0	0	0	0	0	0	0	0	0	1	0	0	-4	1	0	0	0	0	0	0	$V_{13}$	-15
0	0	0	0	0	0	0	0	0	0	1	0	0	1	-4	1	0	0	0	0	$V_{14}$	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	-4	1	0	0	0	$V_{15}$	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	-4	1	0	0	$V_{16}$	-10

## Métodos de resolução

Eliminação gaussiana

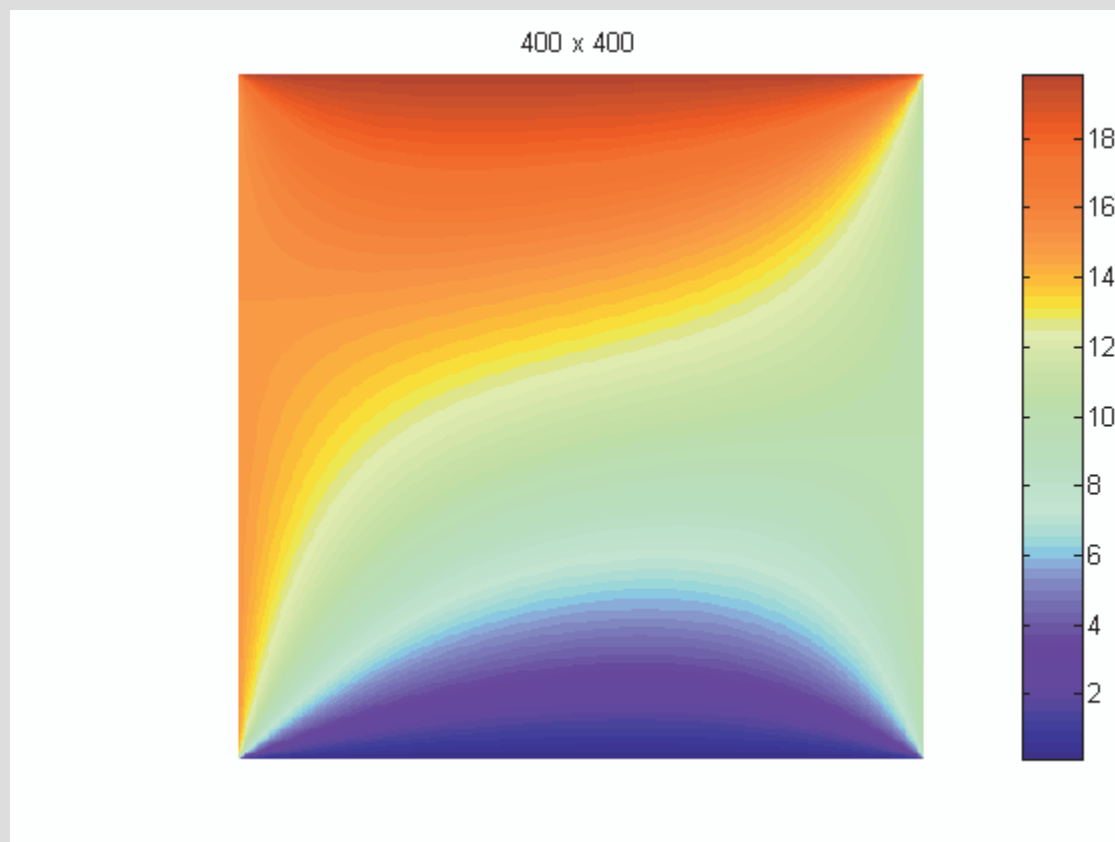
→ Não tira partido da esparsidade da matriz  
Pouco eficiente para elevados números de pontos  
Obtém solução num número definido de passos

Método iterativo (Jacobi)

→ Tira partido da esparsidade da matriz  
Eficiente para elevados números de pontos  
Solução é obtida por aproximações sucessivas



## Solução do problema para grelhas de diferentes espaçamentos



Equação para o nó  $i,j \rightarrow$

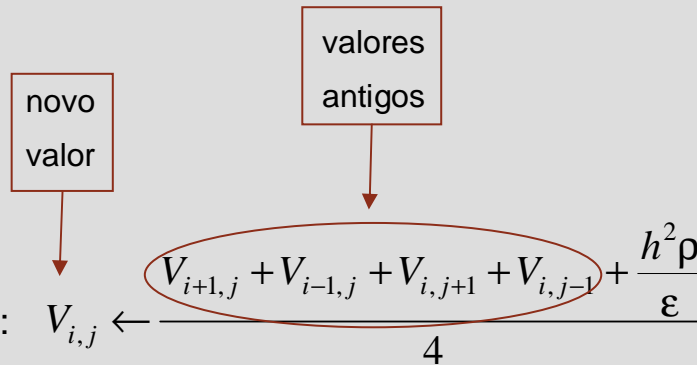
$$V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = -\frac{h^2 \rho}{\epsilon}$$

$$V_{i,j} = \frac{V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} + \frac{h^2 \rho}{\epsilon}}{4}$$

## Método iterativo

1. Inicializar valores das tensões  $V_{i,j}$
2. Repetir em cada iteração

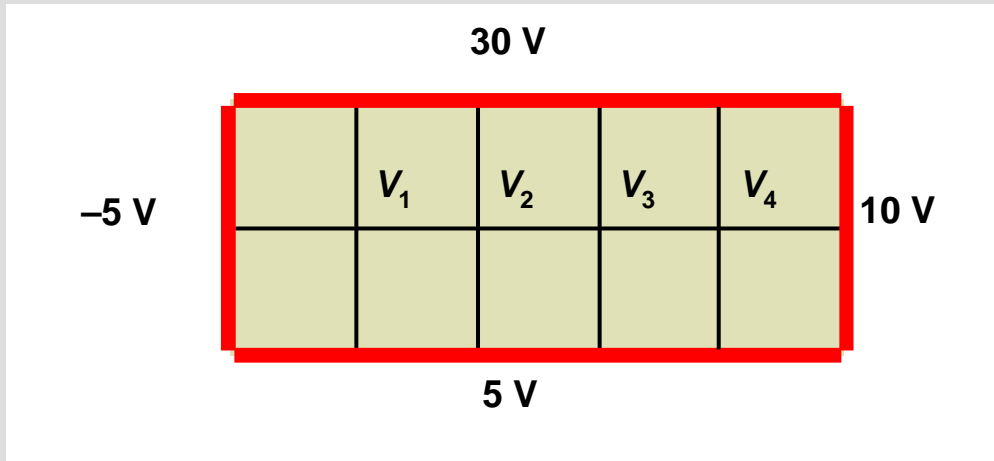
2.1. Actualizar valores das tensões:



2.2. Verificar **critério de paragem**

- Número máximo de iterações
- Pequenas variações das tensões
- ....

# Diferenças finitas – Exemplo 1 – método iterativo



## Sistema de equações

$$\begin{cases} -5 + 30 + V_2 + 5 - 4V_1 = 0 \\ V_1 + 30 + V_3 + 5 - 4V_2 = 0 \\ V_2 + 30 + V_4 + 5 - 4V_3 = 0 \\ V_3 + 30 + 10 + 5 - 4V_4 = 0 \end{cases}$$

## Relações para o método iterativo

$$\begin{aligned} V_1 &= \frac{V_2 + 30}{4} \\ V_2 &= \frac{V_1 + V_3 + 35}{4} \\ V_3 &= \frac{V_2 + V_4 + 35}{4} \\ V_4 &= \frac{V_3 + 45}{4} \end{aligned}$$

## Diferenças finitas – Exemplo 1 – Método iterativo

Iteração	$V_1$	$V_2$	$V_3$	$V_4$
0	0.000	0.000	0.000	0.000
1	7.500	8.750	8.750	11.250
2	9.688	12.813	13.750	13.438
3	10.703	14.609	15.313	14.688
4	11.152	15.254	16.074	15.078
5	11.313	15.557	16.333	15.269
6	11.389	15.662	16.456	15.333
7	11.415	15.711	16.499	15.364
8	11.428	15.729	16.519	15.375
9	11.432	15.737	16.526	15.380
10	11.434	15.739	16.529	15.381
11	11.435	15.741	16.530	15.382
12	11.435	15.741	16.531	15.383
13	11.435	15.741	16.531	15.383

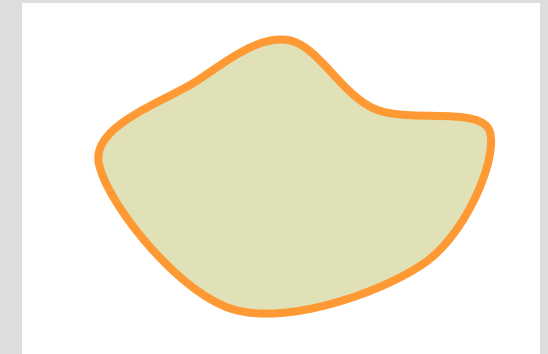
$$V_1 = \frac{V_2 + 30}{4}$$

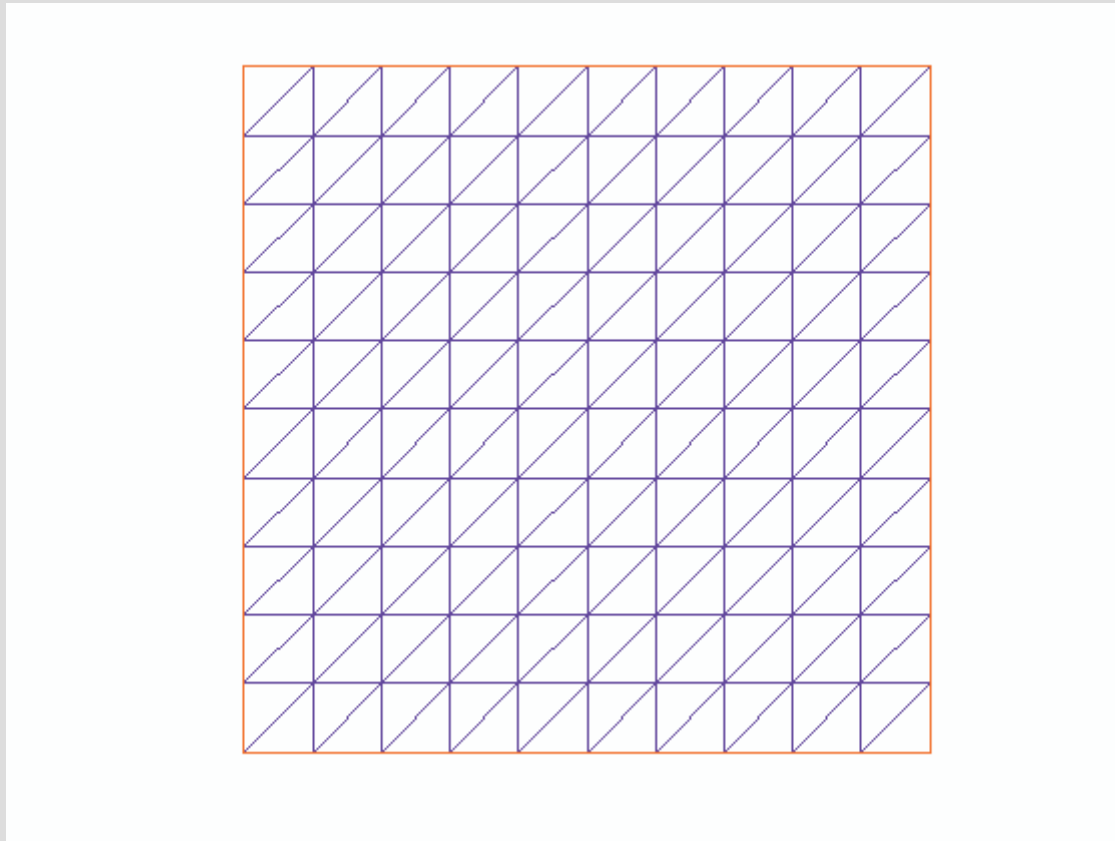
$$V_2 = \frac{V_1 + V_3 + 35}{4}$$

$$V_3 = \frac{V_2 + V_4 + 35}{4}$$

$$V_4 = \frac{V_3 + 45}{4}$$

- Divisão da região  $D$  em elementos (pequenas áreas de geometria bem definida)
- Solução do problema  $V(x,y)$  é obtida como **combinação linear** de **funções base**
- Cada função base tem uma expressão simples em cada elemento
- A resolução da equação diferencial  $\nabla^2 V = -\frac{\rho}{\epsilon}$  é substituída por um problema de minimização equivalente
- Coeficientes da **combinação linear** obtidos por resolução do problema de minimização

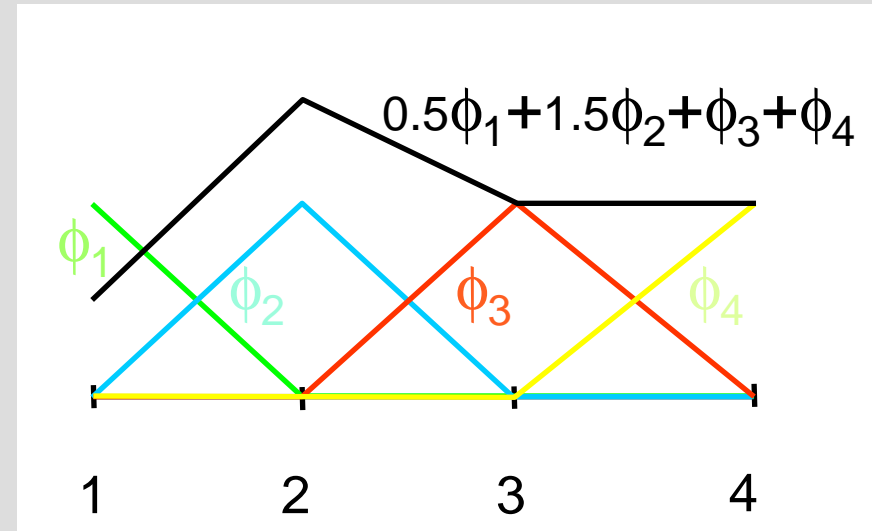




Divisão da região  $D$  em triângulos

## Cada função base

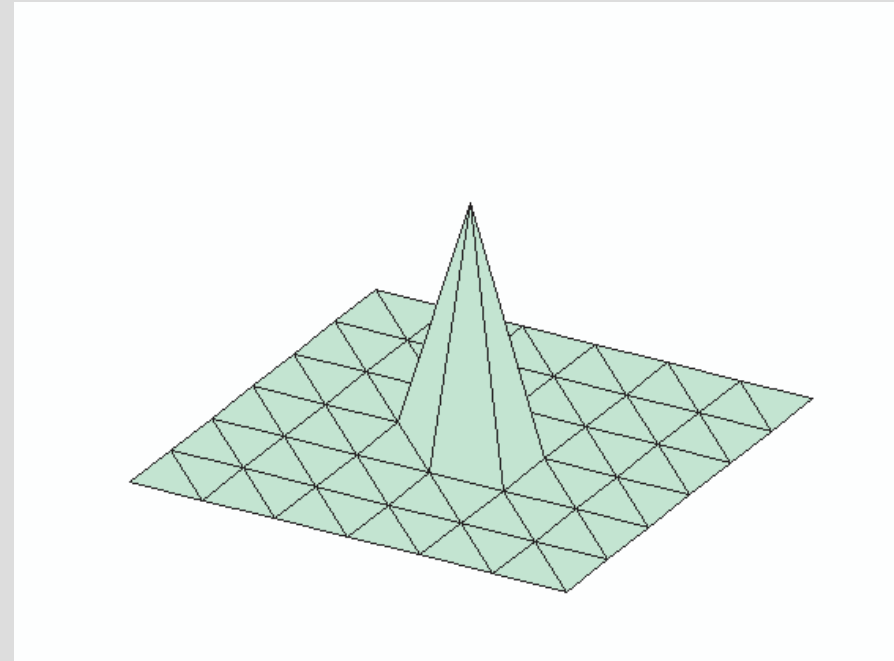
- Toma o valor 1 num ponto
- Toma o valor 0 em todos os outros
- É da forma  $a+bx$  em cada segmento



Combinação linear de funções base → função rectilínea por segmentos

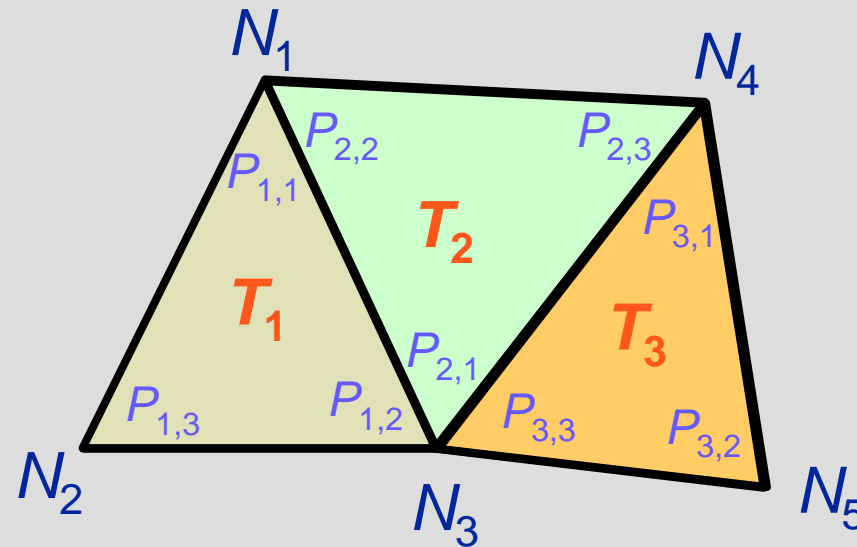
### Cada função base $\phi_j(x, y)$

- Toma o valor 1 num dos vértices
- Toma o valor 0 em todos os outros
- É da forma  $a+bx+cy$  em cada triângulo



Solução do problema  $\longrightarrow$  
$$V(x, y) = \sum_{j=1}^m \gamma_j \phi_j(x, y)$$





Numeração **global** de vértices:  $N_1, N_2, N_3, \dots$

Numeração **local** de vértices no triângulo  $T_i$ :  $P_{i,1}, P_{i,2}, P_{i,3}$

## Funções base no triângulo $T_i$

Em cada triângulo apenas as 3 funções base correspondentes aos seus vértices são não nulas

Funções base no triângulo  $T_i$  de vértices

$$N_p = P_{i,1} = (x_{i,1}, y_{i,1})$$

$$N_q = P_{i,2} = (x_{i,2}, y_{i,2})$$

$$N_r = P_{i,3} = (x_{i,3}, y_{i,3})$$

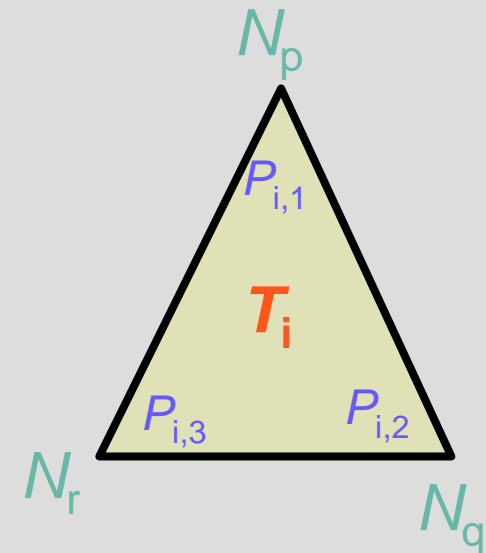
$$\phi_p(x, y) = a_{i,1} + b_{i,1}x + c_{i,1}y$$

$$\phi_q(x, y) = a_{i,2} + b_{i,2}x + c_{i,2}y$$

$$\phi_r(x, y) = a_{i,3} + b_{i,3}x + c_{i,3}y$$

$$\phi_\alpha(N_\beta) = \delta_{\alpha\beta}$$

$$\begin{bmatrix} 1 & x_{i,1} & y_{i,1} \\ 1 & x_{i,2} & y_{i,2} \\ 1 & x_{i,3} & y_{i,3} \end{bmatrix} \cdot \begin{bmatrix} a_{i,1} & a_{i,2} & a_{i,3} \\ b_{i,1} & b_{i,2} & b_{i,3} \\ c_{i,1} & c_{i,2} & c_{i,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

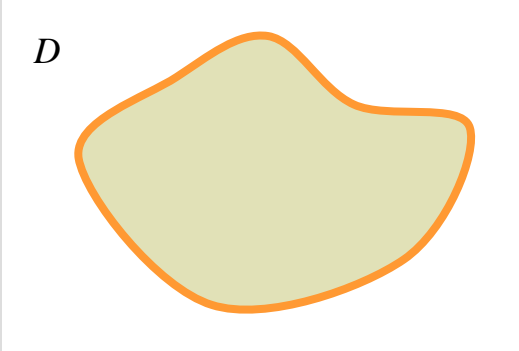


Expressões das funções base no triângulo  $T_i$

# Problema de minimização equivalente

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \quad \text{no interior de } D$$
$$V(x, y) = g(x, y) \quad \text{em } \partial D$$

Equação diferencial  
com condição fronteira



$$\text{minimizar } I = \iint_D \left[ \frac{1}{2} \left( \frac{\partial V}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial V}{\partial y} \right)^2 - \frac{\rho}{\varepsilon} V \right] dx dy$$

sujeito a  $V(x, y) = g(x, y)$  em  $\partial D$

Problema de otimização  
com restrições

Solução aproximada  $V(x, y) = \sum_{j=1}^m \gamma_j \phi_j(x, y)$

Parâmetros a determinar de forma a resolver o problema de

minimizar  $I = \iint_D \left[ \frac{1}{2} \left( \frac{\partial V}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial V}{\partial y} \right)^2 - \frac{\rho}{\epsilon} V \right] dx dy$   
 sujeito a  $V(x, y) = g(x, y)$  em  $\partial D$

$\gamma_{n+1}, \gamma_{n+2}, \dots, \gamma_m$  utilizados para satisfazer condições fronteira  $\rightarrow$  pontos da fronteira

$\gamma_1, \gamma_2, \dots, \gamma_n$  utilizados para minimizar  $I \rightarrow$  pontos interiores  $\rightarrow \frac{\partial I}{\partial \gamma_j} = 0$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$\alpha_{ij} = \iint_D \left( \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right) dx dy \longrightarrow \beta_i = \iint_D \frac{\rho}{\varepsilon} \phi_i dx dy - \sum_{j=n+1}^m \alpha_{ij} \gamma_j$$

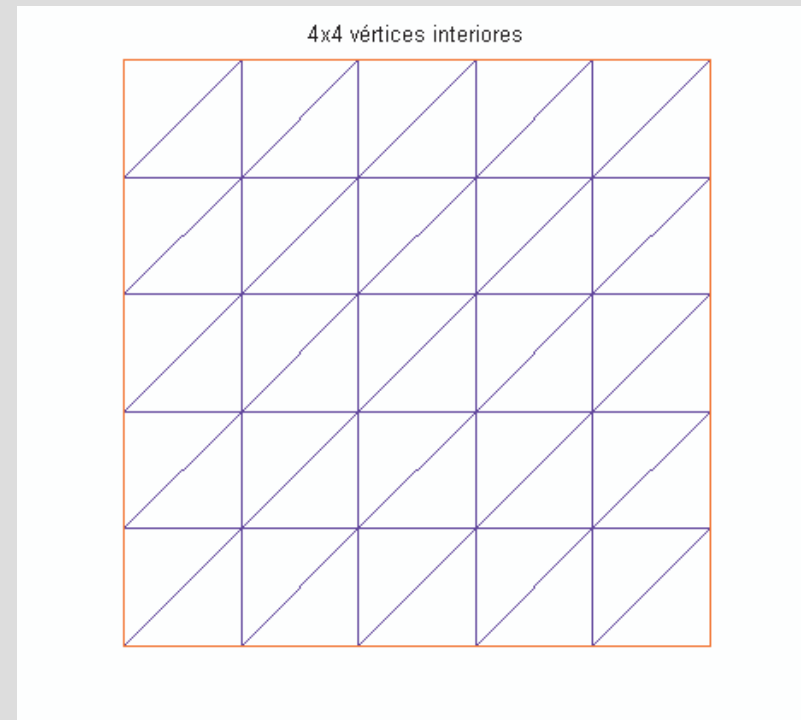
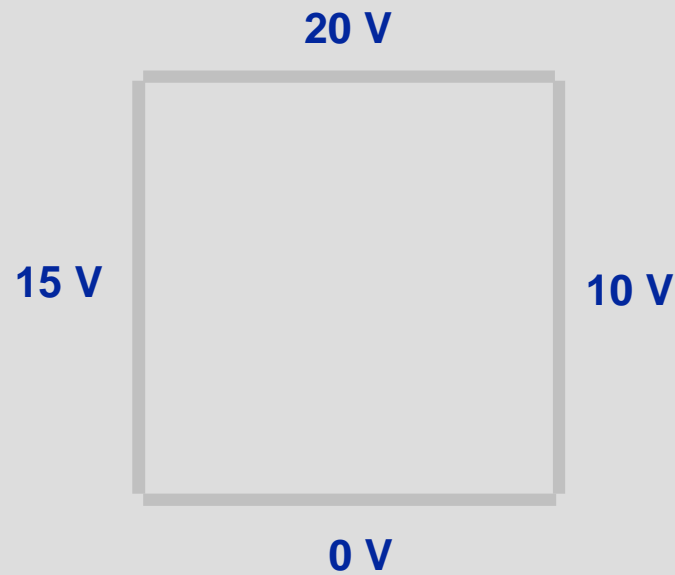
Integração nos triângulos onde  $\phi_i$  e  $\phi_j$  são simultaneamente não nulas  $\Leftrightarrow$  triângulos a que  $N_i$  e  $N_j$  pertencem em simultâneo

$\partial/\partial x$  e  $\partial/\partial y$  são constantes em cada triângulo!

$\alpha_{ij}=0$  se  $N_i$  e  $N_j$  não pertencem a nenhum triângulo simultaneamente  $\Rightarrow$  matriz esparsa!

1. **Divisão da região em elementos**
2. **Numeração de nós**
3. **Determinação das funções base**
4. **Determinação de  $\gamma_{n+1}, \dots, \gamma_m$  a partir das cond. fronteira**
5. **Obtenção do sistema de equações ( $\alpha_{ij}$  e  $\beta_i$ )**
6. **Resolução do sistema de equações para obter  $\gamma_1, \dots, \gamma_n$** 
  - Método directo ou
  - Método iterativo

- Considere a região da figura onde  $\rho = 0$ .

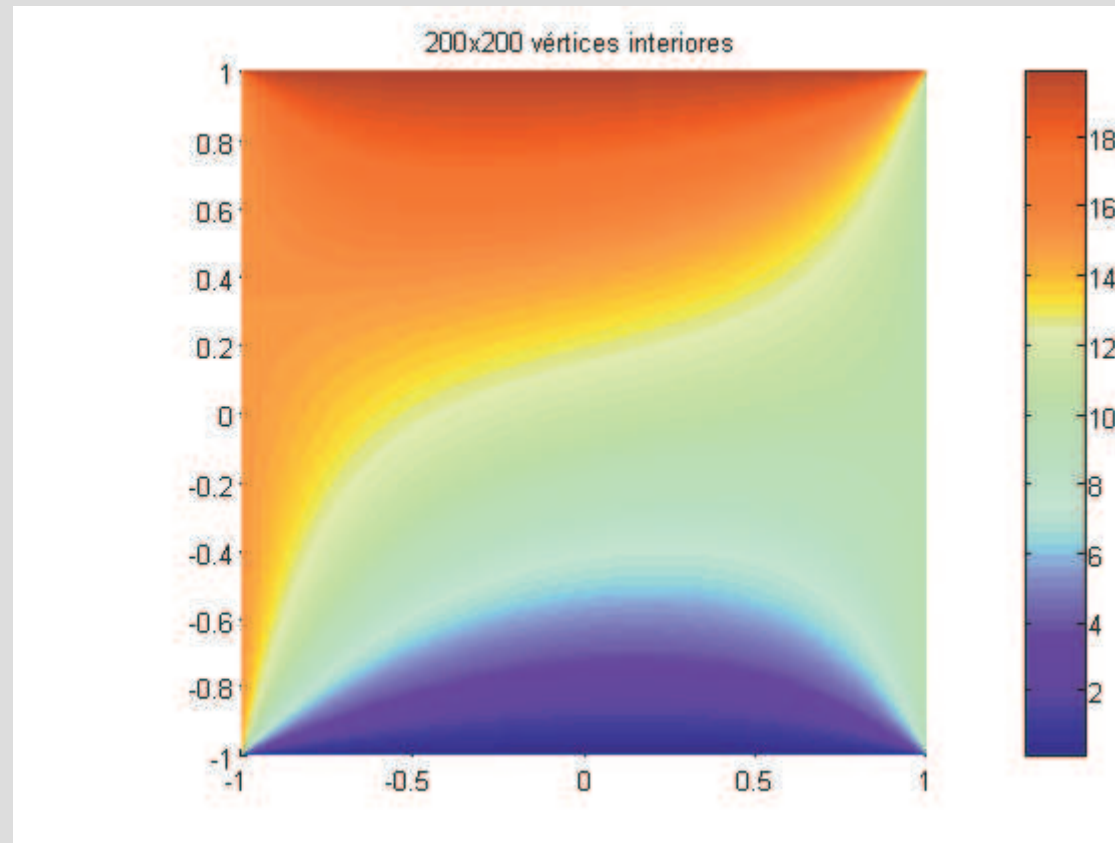


# Elementos finitos – Exemplo 2

$$\begin{bmatrix}
 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4
 \end{bmatrix}
 \begin{bmatrix}
 \gamma_1 \\
 \gamma_2 \\
 \gamma_3 \\
 \gamma_4 \\
 \gamma_5 \\
 \gamma_6 \\
 \gamma_7 \\
 \gamma_8 \\
 \gamma_9 \\
 \gamma_{10} \\
 \gamma_{11} \\
 \gamma_{12} \\
 \gamma_{13} \\
 \gamma_{14} \\
 \gamma_{15} \\
 \gamma_{16}
 \end{bmatrix}
 =
 \begin{bmatrix}
 15 \\
 0 \\
 0 \\
 10 \\
 15 \\
 0 \\
 0 \\
 10 \\
 15 \\
 0 \\
 0 \\
 10 \\
 35 \\
 20 \\
 20 \\
 30
 \end{bmatrix}$$

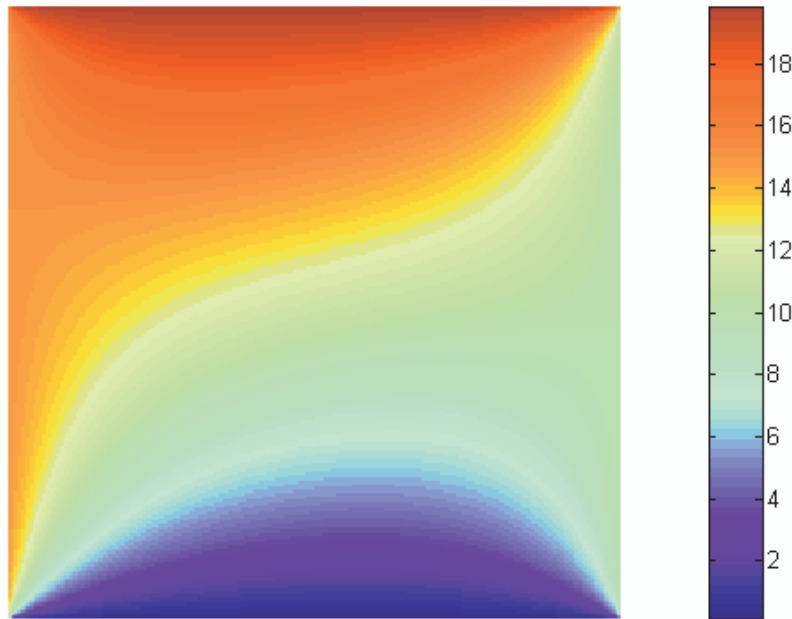


## Elementos finitos – Exemplo 2



### Diferenças finitas

200 x 200



### Elementos finitos

200x200 vértices interiores

