4. Observability

Observability

Definition

\((A, B, C, D)\) is an observable state space model if the following condition holds:

Given two solutions \((u^i, x^i, y^i), i = 1, 2, \) if \(\exists t^* > 0\) such that \(u^1 \equiv u^2,\)
\(y^1 \equiv y^2\) in \([0, t^*]\) then \(x^1(0) = x^2(0)\).

Interpretation: A system is observable if the initial state can be obtained ("observed") from the knowledge of the input and the output.

Remark: Taking the linearity of the system into account, the condition in the definition may be replaced by:

If \(\exists t^* > 0\) such that \(u \equiv 0, y \equiv 0\) in \([0, t^*]\) then \(x(0) = 0\).
Characterization of observability

Define the **observability matrix** as \( \mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \)

**Theorem** \((A, B, C, D)\) is an observable system ⇔ \( \text{rank} \mathcal{O}(C, A) = n \)

**Proof**

\((A, B, C, D)\) is an observable system ⇔
\{ \( Ce^{At}x(0) = 0, \ t \in [0, t^*] \Rightarrow x(0) = 0 \) \} ⇔
\{ \( CA^kx(0) = 0, \ k = 0, 1, \ldots \Rightarrow x(0) = 0 \) \} ⇔
\{ \( CA^kx(0) = 0, \ k = 0, 1, \ldots, n - 1 \Rightarrow x(0) = 0 \) \} ⇔ \( \ker \mathcal{O}(C, A) = \{0\} \)
⇔ \( \text{rank} \mathcal{O}(C, A) = n \)
4. Observability

Duality between observability and controllability

Dual system of \((A, B, C, D) \rightarrow (A^T, C^T, B^T, D^T)\)

\[
(O(C, A))^T = C(A^T, C^T)
\]

Thus \((C, A)\) is observable \(\iff\) \((A^T, C^T)\) is controllable
Consequences of this duality:

**Theorem:** The following statements are equivalent:

1. $(C, A)$ is **observable**.
2. $W_o(t) := \int_0^t e^{A^T \tau} C^T C e^{A \tau} d\tau$ is **invertible** for all $t > 0$.
3. rank $O(C, A) = n$.
4. rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all eigenvalue $\lambda$ of $A$.

**Theorem:**

Let $(A, B, C, D)$ and $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be two algebraically equivalent systems. Then $(C, A)$ is observable if and only if $(\bar{C}, \bar{A})$ is observable.
Other consequences:

- **Kalman observability decomposition (form)**

Let \((A, B, C, D)\) be a state space system for which \((C, A)\) is non-observable.

Then there exists an invertible matrix \(P\) such that the system \((\bar{A}, \bar{B}, \bar{C}, D) = (PAP^{-1}, PB, CP^{-1}, D)\) has the following structure.

\[
\bar{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \bar{C} = [C_1 \ 0],
\]

- the pair \((C_1, A_{11})\) is observable

This form is called **Kalman observability (staircase) form.**
4. Observability

- Observable realizations for a transfer function

  - Suppose that the system \( \bar{A}, \bar{B}, \bar{C}, D \) is in Kalman observability form.

  - Show that the system transfer function is given by:
    \[
    T(s) = \bar{C}(sI_n - \bar{A})^{-1}\bar{B} + D = C_1(sI_r - A_{11})^{-1}B_1 + D
    \]

  - Conclude that \((A_{11}, B_1, C_1, D)\) is an observable realization for the system transfer function.

- Observable and controllable realizations for a transfer function

  - Check that the application of an observability decomposition followed by the application of a controllability decomposition allows to obtain an observable and controllable realization for a given transfer function starting from an arbitrary realization.