Observability

Definition

(A, B, C, D) is an observable state space model if the following condition holds:

Given two solutions $(u^i, x^i, y^i), i = 1, 2$, if $\exists t^* > 0$ such that $u^1 \equiv u^2$, $y^1 \equiv y^2$ in $[0, t^*]$ then $x^1(0) = x^2(0)$.

Interpretation: A system is observable if the initial state can be obtained ("observed") from the knowledge of the input and the output.

Remark: Taking the linearity of the system into account, the condition in the definition may be replaced by:

If $\exists t^* > 0$ such that $u \equiv 0$, $y \equiv 0$ in $[0, t^*]$ then $x(0) = 0$.

Characterization of observability

Define the observability matrix as $\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Theorem (A,B,C,D) is an observable system $\Leftrightarrow$ $\boxed{\operatorname{rank}\mathcal{O}(C,A) = n}$

Proof

(A,B,C,D) is an observable system $\Leftrightarrow$

$\{Ce^{At}x(0) = 0, t \in [0,t^*] \Rightarrow x(0) = 0\} \Leftrightarrow$

$\{CA^k x(0) = 0, k = 0,1,\ldots \Rightarrow x(0) = 0\} \Leftrightarrow$

$\{CA^k x(0) = 0, k = 0,1,\ldots,n-1 \Rightarrow x(0) = 0\} \Leftrightarrow \ker\mathcal{O}(C,A) = \{0\}$

$\Leftrightarrow \operatorname{rank}\mathcal{O}(C,A) = n$

Duality between observability and controllability

Dual system of $(A, B, C, D) \rightarrow (A^T, C^T, B^T, D^T)$

$$(\mathcal{O}(C, A))^T = \mathcal{C}(A^T, C^T)$$

Thus $\boxed{(C, A) \text{ is observable} \Leftrightarrow (A^T, C^T) \text{ is controllable}}$

Consequences of this duality:

Theorem: **The following statements are equivalent:**

1. (C, A) **is observable.**
2. $W_o(t) := \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$ **is invertible for all** $t > 0$.
3. $\operatorname{rank} \mathcal{O}(C, A) = n$.
4. $\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ **for all eigenvalue** λ **of** A.

Theorem:

Let (A, B, C, D) **and** $(\bar{A}, \bar{B}, \bar{C}, D)$ **be two algebraically equivalent systems. Then** (C, A) **is observable if and only if** $\Leftrightarrow$ $(\bar{C}, \bar{A})$ **is observable.**

Other consequences:

- **Kalman observability decomposition (form)**

Let (A, B, C, D) be a state space system for which (C, A) is non-observable .

Then there exists an invertible matrix P such that the system $(\bar{A}, \bar{B}, \bar{C}, D) = (PAP^{-1}, PB, CP^{-1}, D)$ has the following structure.

$$\bar{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \bar{C} = [C_1 \quad 0], \text{ where}$$

- **the pair (C_1, A_{11}) is observable**

This form is called Kalman observability (staircase) form.

• Observable realizations for a transfer function

- Suppose that the system $\bar{A}, \bar{B}, \bar{C}, D$ is in Kalman observability form.

- Show that the system transfer function is given by:

$$T(s) = \bar{C}(sI_n - \bar{A})^{-1}\bar{B} + D = C_1(sI_r - A_{11})^{-1}B_1 + D$$

- Conclude that (A_{11}, B_1, C_1, D) is an observable realization for the system transfer function.

• Observable and controllable realizations for a transfer function

- Check that the application of an observability decomposition followed by the application of a controllability decomposition allows to obtain an observable and controllable realization for a given transfer function starting from an arbitrary realization.