Transfer function realization

Definition
A state space model \((A, B, C, D)\) is a realization of a transfer function \(H(s)\) if its transfer function coincides with \(H(s)\), i.e., if:

\[
C(sI - A)^{-1}B + D = H(s)
\]

A transfer function \(H(s)\) is said to be realizable if there exists a state space model \((A, B, C, D)\) with transfer function \(H(s)\).

Lemma: \(H(s)\) is a realizable transfer function \(\Rightarrow\) \(H(s)\) is rational and proper

Exercise: prove this result!
Scalar case - SISO (Single Input Single Output)

Realization of \( h(s) = \frac{n(s)}{d(s)} \) with  
\[
\begin{align*}
n(s) &= \beta_{n-1}s^{n-1} + \ldots + \beta_1 s + \beta_0 \\
d(s) &= s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0
\end{align*}
\]

Remark:

- In order to realize proper, but not strictly proper transfer functions, one should make the division of the numerator by the denominator so as to obtain \( h(s) = \frac{n(s)}{d(s)} + D \), with \( n(s) \) e \( d(s) \) as above.

- In case the denominator is not a monic polynomial, one should divide both the numerator and the denominator by the coefficient of \( s^n \) (which is supposed to be nonzero).
\[ \hat{y}(s) = h(s)\hat{u}(s) \iff \hat{y}(s) = n(s)[d(s)]^{-1}\hat{u}(s) \]

Defining \( \hat{v}(s) := [d(s)]^{-1}\hat{u}(s) \) we get:

\[ d(s)\hat{v}(s) = \hat{u}(s) \text{ e } \hat{y}(s) = n(s)\hat{v}(s) \]

Defining: \( \hat{x}_1(s) = s^{n-1}\hat{v}(s), \ldots, \hat{x}_n(s) = \hat{v}(s) \), yields:

\[
\begin{align*}
  s\hat{x}_n(s) &= \hat{x}_{n-1}(s) \\
  \vdots \\
  s\hat{x}_2(s) &= \hat{x}_1(s) \\
  s\hat{x}_1(s) &= -\alpha_{n-1}\hat{x}_1(s) - \ldots - \alpha_0\hat{x}_n(s) + \hat{u}(s) \\
  \hat{y}(s) &= \beta_{n-1}\hat{x}_1(s) + \ldots + \beta_0\hat{x}_n(s)
\end{align*}
\]
In the time domain (and using matrix notation), we have:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix}
    -\alpha_{n-1} & \cdots & -\alpha_1 & -\alpha_0 \\
    I_{n-1} & 0 & \ddots & 0 \\
    0 & 0 & \ddots & 0 \\
\end{bmatrix} x(t) + \begin{bmatrix}
    1 \\
    0 \\
    \vdots \\
    0
\end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix}
    \beta_{n-1} & \cdots & \beta_0
\end{bmatrix} C x(t)
\end{align*}
\]

This realization is known as **controllable canonical realization** - Check that it is indeed controllable!
Remarks on the controllable canonical realization

- The matrix $A$ is a companion matrix associated to the polynomial $d(s)$.
- **Lemma:**
  1. $d(\lambda) = 0 \iff \lambda \in \sigma(A)$
  2. The eigenvectors of $A$ associated to $\lambda \in \sigma(A)$ are multiples of $v_\lambda := [\lambda^{n-1} \ldots \lambda 1]^T$
  3. $n(\lambda) = 0 \iff Cv_\lambda = 0$

**Theorem**

The controllable canonical form is observable if and only if $d(s)$ and $n(s)$ are coprime polynomials

**Proofs:** lectures
Matrix case - MIMO (Multi Input Multi Output)

Exercise: Consider a matrix transfer function \( H(s) = \frac{N(s)}{d(s)} \), where \( N(s) \) a matrix with polynomial entries and \( d(s) \) is a polynomial. Propose a method to obtain a state space realization for \( H(s) \) based on the SISO case.
Minimality

Definition - A realization \((A, B, C, D)\) of a transfer function/matrix \(H(s)\) is said to be minimal if no other realization of \(H(s)\) has smaller dimension.

Theorem: The following statements are equivalent:

1. \((A, B, C, D)\) is a minimal realization of \(H(s)\)
2. \((A, B)\) is controllable and \((C, A)\) is observable.

Theorem: Given a transfer function/matrix \(H(s)\), all the minimal realizations of \(H(s)\) are algebraically equivalent.

Proofs: lectures
**Theorem**: Let \((A, B, C, D)\) be a realization of \(H(s)\). The following statements are equivalent:

(1) \((A, B, C, D)\) is minimal.

(2) The poles of \(H(s)\) are the eigenvalues of \(A\).

**Proof**: lectures