6.1 - Internal stability

\[ \dot{x} = F(x), \quad x(t) \in \mathbb{R}^n \]

\( x^* \) is an equilibrium point of \( \dot{x} = F(x) \) if \( x(t) \equiv x^* \) is a solution of this equation.

Lemma \( x^* \) is an equilibrium point of \( \dot{x} = F(x) \) \iff \( F(x) = 0 \)
An equilibrium point \( x^* \in \mathbb{R} \) of \( \dot{x} = F(x) \) is

- **stable** if \( \forall U \) neighborhood of \( x^* \) \( \exists V \) a neighborhood of \( x^* \) such that \( x(0) \in V \Rightarrow \phi(t, 0, x(0)) \in U \).

- **asymptotically stable** if it is stable and \( \exists W \) a neighborhood of \( x^* \) such that \( x(0) \in W \Rightarrow \lim_{t \to \infty} \phi(t, 0, x(0)) = x^* \).

- **unstable** if it is not stable.

\((A, B, C, D)\) is (internally) **stable** if \( x^* = 0 \in \mathbb{R}^n \) is an asymptotically stable point of \( \dot{x} = Ax \). In this case the matrix \( A \) is also said to be **stable**.
Theorem

The matrix $A$ is stable $\iff \sigma(A) \subset \mathbb{C}^- := \{ \lambda \in \mathbb{C} : \Re(\lambda) < 0 \}$

Proof

- The solutions of $\dot{x} = Ax$ are of the form $\phi(t, 0, x(0)) = e^{At}x(0)$.
- The system is stable if and only if sse all the entries of the matrix $e^{At}$ converge to zero.
- Let $\lambda_l = a_l + bi_l$, $l = 1, \ldots, r$ be the distinct eigenvalues of $A$.
- Each entry $ij$ of $e^{At}$ is of the form $(e^{At})_{ij} = \sum_{l=1}^{r} \alpha_{ij}^l(t)$ where

$$\alpha_{ij}^l(t) = e^{a_l t} (p_{ij}^l(t) \cos b_l t + q_{ij}^l(t) \sin b_l t)$$

and $p_{ij}^l(t)$ and $q_{ij}^l(t)$ are polynomial functions of $t$.

- $a_l < 0$, $l = 1, \ldots, r \Rightarrow (e^{At})_{ij} \to 0$, $i, j = 1 \ldots, n$

- For each eigenvalue $\lambda_l$ there is a least one $\alpha_{i^*j^*}^l(t)$ which is nonzero.

- $(e^{At})_{i^*j^*} \to 0 \Rightarrow a_l < 0$.

- The entries of the matrix $e^{At}$ converge to zero $\iff \Re(\lambda) < 0$, $\forall \lambda \in \sigma(A)$.
6. Stability

6.2 - External stability

**Bounded Input Bounded Output - BIBO stability**

\((A, B, C, D)\) is **BIBO stable** if

\[ \forall M > 0 \ \exists N > 0 : x(0) = 0, \|u(t)\| < M, t \geq 0 \Rightarrow \|y(t)\| < N, t \geq 0 \]

**Example**

\[
A = \begin{bmatrix}
2 & 0 \\
0 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad C = [1 \ 1], \quad D = 0
\]

\[y(t) = e^{2t}x_1(0) + e^{-t}x_2(0) + \int_0^t e^{-(t-\tau)}u(\tau)d\tau\]

The system is **BIBO stable**: for zero initial conditions, \(u\) bounded implies \(y\) bounded.

However: for \(x_1(0) \neq 0\), \(u\) bounded does not imply \(y\) bounded.
Theorem The following conditions are equivalent:

1) \((A, B, C, D)\) is BIBO stable
2) All the poles of the system transfer function belong to \(\mathbb{C}^-\)
3) \(\int_0^\infty \|Ce^{At}B\|dt < \infty\).

Remark If \(T(s) = \left[\frac{n_{ij}(s)}{d_{ij}(s)}\right]_{ij},\ i = 1, \ldots, p,\ j = 1, \ldots, m\), where \(n_{ij}(s)\) e \(d_{ij}(s)\) are coprime polynomials (i.e., are polynomial without common factors), the poles of each entry \(ij\) of \(T(s)\) are the roots of the polynomial \(d_{ij}(s)\). The poles of \(T(s)\) are the poles of its entries.

Proof - Exercise! Suggestion:

- Show that the poles of \(C(sI - A)^{-1}B + D\) coincide with the poles of \(C(sI - A)^{-1}B\)
- Show that for each \(\lambda = a + bi\) which is a pole of the \(ij\) entry of \(C(sI - A)^{-1}B\), the entry \(ij\) of \(Ce^{At}B\) contains a summand of the form \(e^{at}(p(t) \cos t + q(t) \sin t)\) where the polynomials \(p(t)\) and \(q(t)\) are not simultaneously null.
Proposition $(A, B, C, D)$ internally stable $\Rightarrow (A, B, C, D)$ BIBO stable.

Proof

- The eigenvalues of $A$ are the poles of $(sI - A)^{-1}$

- The $\mathcal{P}$ of poles of $C(sI - A)^{-1}B + D$ is contained in the set $\mathcal{A}$ of poles of $(sI - A)^{-1}$

- If $A$ is stable: $\mathcal{P} \subset \mathcal{A} = \sigma(A) \subset \mathbb{C}^-$

Remark The reciprocal implication does not hold (see previous example).
Proposition If \((A, B, C, D)\) is a minimal realization then: \((A, B, C, D)\) internally stable \(\iff\) \((A, B, C, D)\) BIBO stable.

Proof: exercise (simple, taking into account the relationship between poles and eigenvalues for minimal realizations.)