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Lyapunov stability of 2D finite-dimensional behaviours

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1. Introduction

The stability of two-dimensional (2D) systems has been the subject of extensive investigation in the past decades; among these research efforts, some have also been focused on the computation of Lyapunov functions. Past research has predominantly been concerned with systems whose set of trajectories is infinite-dimensional, and almost exclusively has concerned specific class of models, for example Fornasini–Marchesini or Roesser models (Fornasini and Marchesini 1980; Lu and Lee 1985). Moreover, in those investigations a specific (usually nonnegative quarter-plane) notion of causality has been assumed.

In this article we follow the behavioural approach: we study the stability of 2D systems described by higher order difference equations without reference to special representations; the central object of interest in our investigation is the set of all admissible trajectories of the system, the *behaviour*, rather than any of its specific representations. Following the pioneering approach of Valcher (2000), stability is defined accordingly at the level of trajectories, although we will be using a different but ultimately equivalent definition to that proposed in Valcher (2000) for a finite-dimensional case. We also adopt the eminently reasonable position proposed in Valcher (2000) to let the system dynamics themselves dictate what notion of causality is most appropriate for the case at hand.

A Lyapunov analysis of stability of infinite-dimensional square 2D behaviours has been presented in Kojima, Rapisarda, and Takaba (2010); in this article we concentrate our attention on the case of finite-dimensional 2D discrete behaviours, i.e. finite-dimensional subspaces of the set of trajectories from $\mathbb{Z}^2$ to $\mathbb{R}^m$. We define a finite-dimensional 2D system stable if all trajectories of the behaviour go to zero along every ‘discrete line’ in a cone, which without loss of generality is taken to be the first orthant of the lattice $\mathbb{Z}^2$; this is a different but equivalent definition from that adopted in Section 3, Definition 3.1 of Valcher (2000), and follows the approach of Napp Avelli and Rocha (2010a, b), Pillai and Shankar (1998), Rocha (2008). The main result of this article is a characterisation of stability for finite-dimensional 2D behaviours in term of the existence of a *Lyapunov function*, defined as a quadratic function of the system variables and their 2D shifts which is positive along each discrete line in the first orthant, and whose increment along each such line is negative. In this article we also give necessary and sufficient conditions for a quadratic function of the system variables and their 2D shifts to be a Lyapunov function; and we illustrate an algorithm to compute a Lyapunov function for a given finite-dimensional 2D behaviour.

In the following we make extensive use of the concepts and calculus of 2D quadratic difference forms.
(Kojima and Takaba 2006), and their association with four-variable polynomial matrices. We will also use extensively the concepts and terminology of the behavioural approach to 2D systems. In order to make this article self-contained, we have included some background material in Section 2; the reader interested in a more thorough introduction to 2D behavioural system theory is referred to Pillai and Willems (2002), Rocha (1990), Valcher (2000). The main result of this article is illustrated in Section 3. In Section 4 we outline an algorithm for the construction of Lyapunov functions.

2. Preliminaries
We consider sets \( B \) of trajectories defined over \( \mathbb{Z}^2 \) that can be described by a set of linear constant coefficient partial difference equations, i.e.

\[
B = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) \subseteq (\mathbb{R}^n)^{\mathbb{Z}^2},
\]

where \( \sigma_i \)'s are the 2D shift operators defined by

\[
\sigma_i(w(k_1, k_2)) = w((k_1, k_2) + e_i),
\]

for \((k_1, k_2) \in \mathbb{Z}^2 \) and \( e_i \) the \( i \)th element of the canonical basis of \( \mathbb{R}^2 \), \( i = 1, 2 \); and \( R(s_1, s_1^{-1}, s_2, s_2^{-1}) \) is a 2D \((p \times w)\)-dimensional Laurent-polynomial matrix. We call (1) a kernel representation of the behaviour \( B \).

We now introduce finite-dimensional behaviours, and briefly discuss their representation by means of state equations.

Given a full column rank Laurent-polynomial matrix \( R \in \mathbb{R}^{p \times n}[s_1, s_1^{-1}, s_2, s_2^{-1}] \), we define its Laurent variety (or simply variety) as

\[
V(R) := \{(\lambda_1, \lambda_2) \in (\mathbb{C} \setminus 0)^2 \mid \text{rank}(R(\lambda_1, \lambda_2)) < \text{rank}(R) = w\},
\]

where rank \( (R(\lambda_1, \lambda_2)) \) is the rank of the complex matrix \( R(\lambda_1, \lambda_2) \), while rank \( (R) \) is the rank of the Laurent-polynomial matrix \( R \). It can be shown (Zerz 1996) that any two different representations of \( B \) share the same Laurent variety; consequently, in the following we refer to the variety of \( B \), denoted by \( V(B) \), as the Laurent variety of any of its kernel representations. It is well known (Rocha 1990; Bisio and Valcher 2005) that a behaviour \( B \) is finite-dimensional (when considered as a subspace of the vector space over \( \mathbb{R} \) consisting of all trajectories from \( \mathbb{Z}^2 \) to \( \mathbb{R}^n \)) if and only if \( V(B) \) consists of a finite number of points, or equivalently if \( B \) admits a right factor prime representation (see Fornasini and Valcher (1997) for a definition).

It was shown in Fornasini, Rocha, and Zampieri (1993) that for every finite-dimensional behaviour \( B \) there exists a hybrid representation of first-order, i.e. there exist matrices \( A_1 \in \mathbb{R}^{p \times n}, A_2 \in \mathbb{R}^{p \times n} \) and \( C \in \mathbb{R}^{w \times n} \) such that \( B \) consists of all trajectories \( w \) for which there exists a trajectory \( x : \mathbb{Z}^2 \to \mathbb{R}^n \) such that

\[
\begin{align*}
\sigma_1 x &= A_1 x \\
\sigma_2 x &= A_2 x \\
w &= Cx,
\end{align*}
\]

holds, where also the matrices \( A_1 \) and \( A_2 \) commute: \( A_1 A_2 = A_2 A_1 \). In particular, \( A_1, A_2 \) and \( C \) can be chosen so that the state variable \( x \) is observable from \( w \), i.e.

\[
[(w, x) \text{ satisfy (2)} \text{ and } w = 0] \implies [x = 0].
\]

There are several different characterisation of observability in terms of the algebraic properties of the representation; the first one is

\[
\ker \begin{bmatrix} \sigma_1 I - A_1 \\ \sigma_2 I - A_2 \\ C \end{bmatrix} = \{0\}.
\]

This condition is equivalent with the extended observability matrix, defined as the column block matrix

\[
O(A_1, A_2, C) := 
\begin{bmatrix}
C \\
C A_1 \\
C A_2 \\
C A_1 A_2 \\
\vdots \\
C A_2^{s_2-1}
\end{bmatrix},
\]

having rank equal to \( n \). This implies that there exists a matrix \( E \) such that \( E \cdot O(A_1, A_2, C) = I_n \), where \( I_n \) is the \( n \times n \) identity matrix. Thus we obtain that

\[
x = X(s_1, s_1^{-1}, s_2, s_2^{-1})w
\]

where

\[
X(s_1, s_1^{-1}, s_2, s_2^{-1}) := E \in \mathbb{R}^{[s_1, s_1^{-1}, s_2, s_2^{-1}] \times w}.
\]

In this case the state map \( X(s_1, s_1^{-1}, s_2, s_2^{-1}) \) is minimal, i.e. the dimension of the state variable \( x \) is minimal among all possible representations (2). Note that here \( X \) is a polynomial matrix, i.e. its entries contain only nonnegative powers of \( s_1 \) and \( s_2 \).
In the following, discrete lines in the lattice $\mathbb{Z}^2$ will play an important role; we now introduce the basic notation and discuss behaviours restricted to lines.

The set of lines in the first orthant of $\mathbb{Z}^2$ is defined by

$$L := \{ \ell \subset \mathbb{N}^2 \mid \ell = (a(b), b) \in \mathbb{N}^2 | a \in \mathbb{N}, a, b \in \mathbb{N} \text{ are coprime} \}.$$

The lines in the first orthant corresponding to the vertical and horizontal axes will be denoted in the following by $\ell_1$ and $\ell_2$, respectively. Given a 2D behaviour $B$ and a line $\ell \in L$, we define the restriction of $B$ to $\ell$ as

$$B \mid _\ell := \{w_\ell : \mathbb{Z} \to \mathbb{R}^v \mid \text{there exists } w \in B \text{ such that } w|_\ell = w_\ell \},$$

where $w|_\ell$ denotes the restriction of the trajectory $w$ to the domain $\ell$. Note that $w_\ell$ is a 1D trajectory, while $w|_\ell$ is a trajectory depending on two indices. It has been shown in Napp Avelli and Rocha (2010b; Theorems 6 and 7) that a 2D behaviour restricted to a line is a 1D behaviour; that is, $B$ is the kernel of some polynomial operator in the 1D shift defined by a Laurent-polynomial matrix $R(s) \in \mathbb{R}[s, s^{-1}]$. It is easy to see that if $B$ is described by (2) and if $B_1$ and $B_2$ denote the restrictions of $B$ to the axes, then $B_1$ is described in the state-space form as

$$\sigma_i x = A_i x,$$
$$w = C x,$$

$i = 1, 2$. Note that these state representations may be non-minimal even if (2) is minimal.

In many modelling and control problems it is necessary to study quadratic functionals of the system variables and their derivatives; for example, in linear quadratic optimal control, $H_\infty$-control, dissipativity theory, etc. Following the seminal work of Willems and Trentelman (1998), successively extended to the 2D case in Kojima, Rapisarda, and Takaba (2007), Kojima et al. (2010) and Pillai and Willems (2002), we will use polynomial matrices in four variables as a tool to express quadratic functionals of functions of two independent variables and their shifts. We next review the definitions regarding these functionals which are most relevant to the problems treated in this article.

In the following, we use the multi-indices $k = (k_1, k_2)$ and $l = (l_1, l_2)$, and multi-indeterminates denoted by $\zeta = (\zeta_1, \zeta_2)$ and $\eta = (\eta_1, \eta_2)$. We also denote $\delta^k = \delta_{k_1} \delta_{k_2}$ and $\eta^k = \eta_{l_1} \eta_{l_2}$. We denote by $\mathbb{R}^{w \times n}[\zeta, \eta]$ the set of real polynomial $w \times n$ matrices in the four indeterminates $\zeta$ and $\eta$; that is, an element of $\mathbb{R}^{w \times n}[\zeta, \eta]$ is of the form

$$\Phi(\zeta, \eta) = \sum_{k,l} \Phi_{k,l} \delta^k \eta^l,$$

where $\Phi_{k,l} \in \mathbb{R}^{w \times n}$; the sum ranges over a finite set of multi-indices $k, l \in \mathbb{N}^2$. The four-variable polynomial matrix $\Phi(\zeta, \eta)$ is called symmetric if $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^T$, equivalently if $\Phi_{k,l} = \Phi_{l,k}$ for all $l$ and $k$. In this article we restrict our attention to the symmetric elements in $\mathbb{R}^{w \times n}[\zeta, \eta]$, and denote this subset by $\mathbb{R}_s^{w \times n}[\zeta, \eta]$. Any symmetric $\Phi$ induces a quadratic functional

$$Q(\Phi) : (\mathbb{R}^w)^2 \times (\mathbb{R}^w)^2 \to (\mathbb{R}^w)^2$$

$$Q(\Phi)(w) = \sum_{k,l} (\delta^k w)^T \Phi_{k,l} \delta^l (w)$$

where the $k$-th shift operator $\delta^k$ is defined as $\delta^k = \delta^{k_1} \delta^{k_2}$ (similarly for $\delta^l$). We call $Q(\Phi)$ the quadratic difference form (in the following abbreviated with QDF) associated with $\Phi$. Given two QDFs $Q_{\Phi_1}, Q_{\Phi_2}$, we say that $Q_{\Phi_1}$ is equivalent to $Q_{\Phi_2}$ on $B$, denoted by $Q_{\Phi_1} \equiv Q_{\Phi_2}$, if

$$Q_{\Phi_1}(w) = Q_{\Phi_2}(w) \text{ for all } w \in B.$$

We call a QDF $Q(\Phi)$ non-negative along $B$, denoted by $Q(\Phi) \geq 0$, if $Q(\Phi)(w) \geq 0$ for all $w \in B$. We call $Q(\Phi)$ positive along $B$, denoted by $Q(\Phi) > 0$, if $Q(\Phi) \geq 0$, and moreover $\forall w \in B(Q(\Phi)(w) = 0) \Rightarrow \{w = 0\}$.

In the following we will also be using operations on QDFs. Given a QDF $Q(\Phi)$ and a line $\ell = \{a(b), b) | a \in \mathbb{N}\} \in L$, we define the increment of $Q(\Phi)$ along the line $\ell$, denoted by $\nabla_\ell(Q(\Phi))$, as

$$\nabla_\ell Q(\Phi)(w)(a(a, b)) := Q(\Phi)(w)(a + 1)(a, b) - Q(\Phi)(w)(a(a, b))$$

The increment along the vertical and horizontal, line will be denoted by $\nabla_{\ell_1}$ and $\nabla_{\ell_2}$, respectively.

3. 2D stability and Lyapunov functions

In this article we will consider stability as defined in Rocha (2008), i.e. with respect to a cone $S$. A set $S \subset \mathbb{R} \times \mathbb{R}$ is called a cone if $a S \subset S$ for all $a \geq 0$. A cone $S$ is solid if it contains an open ball in $\mathbb{R} \times \mathbb{R}$, and pointed if $S \cap (-S) = \{(0, 0)\}$. A cone is proper if it is closed, pointed, solid and convex. Since an appropriate change of independent variables transforms any proper cone $S$ into the first orthant, in the following we assume, without loss of generality, that $S$ is the first orthant in $\mathbb{Z}^2$. For the sake of brevity, in the following we will use the expression ‘stable’ instead of ‘stable with respect to the first orthant’.

The definition of asymptotic stability that we shall use in the rest of this article is the following: note that it is the discrete counterpart of that considered in the
Definition 3.1: Let \( B \) be a 2D behaviour. \( B \) is asymptotically stable if

\[
[w \in B] \Rightarrow \forall (a, b) \in \mathbb{R}^2 \lim_{a \to \infty} w(a(a, b)) = 0.
\]

It is straightforward to see that \( B \) is asymptotically stable if and only if \( \forall \ell \in L \) it holds that \( w_\ell \), the 1D trajectory associated with the restriction \( w_1 \), goes to zero as the independent variable goes to infinity. This definition of stability is equivalent to the definition considered in Rocha (2008) for (finite-dimensional) 2D behaviours. It was shown in Rocha (2008) that in the discrete case all stable behaviours according to Definition 3.1 are finite-dimensional; note that in the continuous case this is not necessarily true.

Having defined stability as in Definition 3.1, we now define Lyapunov functions as follows.

Definition 3.2: A functional \( F: (\mathbb{R}^n)^2 \to (\mathbb{R}^n)^2 \) is a Lyapunov function for a 2D behaviour \( B \) if for all \( \ell \in L \) it holds that for all \( w \in B \),

\[
F(w)|_\ell > 0 \quad \text{and} \quad \forall \ell F(w) < 0.
\]

If \( F \) is a quadratic functional of \( w \in B \) and its shifts, we call it a quadratic Lyapunov function (QLF) for \( B \).

In order to state the main result of this section, a characterisation of asymptotic stability in terms of Lyapunov functions, we need some preliminary concepts and results. The first one is the notion of quadratic functional of the state. Let \( B \) be a finite-dimensional 2D behaviour, and let (2) be a minimal state representation of \( B \). We say that a quadratic functional \( Q_\Phi(w) \) of \( w \in B \) and its shifts is a quadratic function of the state of \( B \) if there exists a symmetric constant matrix \( P \) such that for all trajectories \( (w, x) \) satisfying (2) it holds that \( Q_\Phi(w) = x^T P x \). As the following result shows, in the finite-dimensional case any quadratic functional of the system variables and their shifts is a quadratic functional of the state.

Proposition 3.3: Let \( B \) be a finite-dimensional 2D behaviour, and let (2) be a state representation of \( B \) with state \( x \). Let \( Q_\Phi: (\mathbb{R}^n)^2 \to (\mathbb{R}^n)^2 \) be a QDF. Then there exists a symmetric matrix \( P \in \mathbb{R}^{n \times n} \) such that for all \( w \in B \) with associated state trajectory \( x \), the following holds:

\[
Q_\Phi(w) = x^T P x.
\]

Proof: Write

\[
Q_\Phi(w) = \sum_{i,j,k_1,k_2=0}^N (\sigma_{i,j,k_1,k_2}^w) \Phi_{i,j,k_1,k_2}(\sigma_{i,j,k_1,k_2}^w) = (w^T \sigma_1 w^T \sigma_2 w^T \ldots) \Phi_{0000} \Phi_{0010} \Phi_{0001} \ldots \left( \begin{array}{c} w^T \sigma_1 w^T \sigma_2 w^T \ldots \end{array} \right) \Phi_{1000} \Phi_{1010} \Phi_{0011} \ldots \left( \begin{array}{c} Cx \end{array} \right).
\]

Now observe that \( \sigma_{i,j,k_1,k_2}^w = C A_{i,j,k_1,k_2} x \); consequently, the last expression can be rewritten as

\[
(Cx)^T (CA_1 x)^T (CA_2 x)^T \ldots \Phi_{0000} \Phi_{0010} \Phi_{0001} \ldots \left( \begin{array}{c} Cx \end{array} \right).
\]

Now define

\[
P := O(A_1, A_2, C)^T \left( \begin{array}{cccc} \Phi_{0000} & \Phi_{0010} & \Phi_{0001} & \ldots \\Phi_{1000} & \Phi_{1010} & \Phi_{0011} & \ldots \\Phi_{0100} & \Phi_{0110} & \Phi_{0101} & \ldots \ldots & \ldots & \ldots \end{array} \right).
\]

the claim follows.

It follows from the proof of Proposition 3.3 that every QDF \( Q_\Phi \) can be written as \( Q_\Phi(w) = (X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})^T P X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})^T w) = x^T P x \) for \( w \in B \), where \( X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) \) is a polynomial operator in the shift that induces the state variable \( x \) when acting on the trajectory \( w \); note that if the expression for \( X \) is derived from the extended observability matrix (3), then \( X \) is a polynomial matrix, and therefore \( \Phi \) is a polynomial matrix in four indeterminates. Consequently, every QDF \( Q_\Phi \) is equivalent on \( B \) to a QDF \( Q_{\Phi'} \) induced by a four-variable polynomial matrix of the form \( \Phi'(\xi_1, \xi_2, \eta_1, \eta_2) = X(\xi_1, \xi_2)^T P X(\eta_1, \eta_2) \), with \( P \) symmetric and \( X \) inducing a state variable. We call such a QDF \( Q_{\Phi'} \) a canonical representative of \( Q_\Phi \). Note that if \( \hat{x} \) is another minimal state variable for \( B \), then it is easy to see that there exists a nonsingular matrix \( T \) such that \( \hat{x} = Tx \). Consequently, \( Q_\Phi(w) = \hat{x}^T P \hat{x} \) where \( P = (T^{-1})^T P T^{-1} \). If a well-ordering (Becker and Weispfenning 1993) has been fixed in the space of polynomials \( \mathbb{R}^{n \times n}[\xi_1, \xi_2, \eta_1, \eta_2] \), then a unique canonical representative can be defined (Kojima et al. 2007).
We now give an example of the computation of a canonical representative of a QDF.

Example 3.4: Let $\mathcal{B} = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ be a 2D behaviour where

$$R(s_1, s_1^{-1}, s_2, s_2^{-1}) = \begin{pmatrix}
(s_2 - \frac{1}{2})(s_1 - \frac{1}{4}) & 0 \\
(s_1 - \frac{1}{2})(s_2 - \frac{1}{4}) & 0 \\
0 & s_1 - \frac{1}{3} \\
0 & s_2 - \frac{1}{5}
\end{pmatrix},$$

and $\Phi(\eta_1, \eta_2, \zeta_1, \zeta_2)$ a four-variable polynomial matrix given by

$$\Phi(\eta_1, \eta_2, \zeta_1, \zeta_2) = \begin{pmatrix}
1 + \eta_1 \zeta_1 & \zeta_1 \\
\eta_1 & \eta_2 \zeta_2
\end{pmatrix}.$$

It is a matter of straightforward verification to check that the variety of $\mathcal{B}$ consists of the points

$$\mathcal{V}(\mathcal{B}) = \left\{ \begin{pmatrix} 1 \ 1 \\
\frac{1}{2} \ \frac{1}{2} \\
\frac{1}{4} \ \frac{1}{4} \\
\frac{1}{3} \ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} 1 \ 1 \\
\frac{1}{2} \ 1 \\
\frac{1}{4} \ \frac{1}{4} \\
\frac{1}{3} \ \frac{1}{3} \end{pmatrix} \right\};$$

since $\mathcal{V}(\mathcal{B})$ is finite, it follows that $\mathcal{B}$ is finite-dimensional.

Following the procedure illustrated in Fornasini et al. (1993), a minimal state-space realisation of $\mathcal{B}$ as in (2) is given by the matrices

$$A_1 = \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{3} & 0 \\
0 & 0 & \frac{1}{4}
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{5} & 0 \\
0 & 0 & \frac{1}{4}
\end{pmatrix}, \quad C = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.$$

It is easy to compute that $x = X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w$ is the state variable corresponding to the matrices $A_1, A_2$ and $C$, where

$$X(s_1, s_1^{-1}, s_2, s_2^{-1}) = \begin{pmatrix}
-4(s_1 - \frac{1}{2}) & 0 \\
0 & 1 \\
4(s_1 - \frac{1}{4}) & 0
\end{pmatrix}.$$

We now compute a canonical representative of $Q_\Phi$. It is easy to check that $Q_\Phi(w) = x^TPx$, where

$$P = O(A_1, A_2, C)^T \begin{pmatrix}
\Phi_{0000} & \Phi_{0010} & \Phi_{0011} & \ldots \\
\Phi_{1000} & \Phi_{1010} & \Phi_{1011} & \ldots \\
\Phi_{1100} & \Phi_{1110} & \Phi_{1111} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \times O(A_1, A_2, C),$$

$$= \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} \\
0 & \frac{1}{5} & 0 \\
1 & 0 & \frac{1}{3} \\
\frac{1}{4} & 0 & \frac{1}{6} \\
0 & \frac{1}{25} & 0 \\
0 & \frac{1}{4} & 0 \\
1 & 0 & \frac{1}{16} \\
0 & \frac{1}{4} & 0 \\
1 & 0 & \frac{1}{16} \\
0 & \frac{1}{15} & 0
\end{pmatrix} \times \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} \\
0 & \frac{1}{5} & 0 \\
1 & 0 & \frac{1}{3} \\
\frac{1}{4} & 0 & \frac{1}{6} \\
0 & \frac{1}{25} & 0 \\
0 & \frac{1}{4} & 0 \\
1 & 0 & \frac{1}{16} \\
0 & \frac{1}{4} & 0 \\
1 & 0 & \frac{1}{16} \\
0 & \frac{1}{15} & 0
\end{pmatrix} = \begin{pmatrix}
5 & 1 & 9 \\
\frac{4}{3} & \frac{3}{2} & \frac{3}{8} \\
\frac{1}{2} & 1 & 1 \\
\frac{3}{2} & \frac{25}{3} & \frac{3}{8} \\
\frac{9}{8} & \frac{1}{9} & \frac{3}{8}
\end{pmatrix}.$$

Hence, a canonical representative for $Q_\Phi$ is

$$\Phi'(\zeta_1, \zeta_2, \eta_1, \eta_2) = X(\zeta_1, \zeta_2)^T PX(\eta_1, \eta_2).$$

The notion of canonical representative of a QDF is important for the proof of the following theorem, that constitutes the main result of this section. It relates the 2D stability of the 2D behaviour $\mathcal{B}$ with the 1D stability of the 1D behaviours resulting from the restriction of $\mathcal{B}$ to the axes and with the existence of a quadratic Lyapunov functional.
Theorem 3.5: Let $B$ be a finite-dimensional 2D behaviour and denote by $B_1$ and $B_2$ the restrictions of $B$ to the vertical and horizontal axes, respectively. The following statements are equivalent:

1. $B$ is stable;
2. $B_1$ and $B_2$ are stable 1D behaviours;
3. There exist three variable polynomial matrices $\Phi$, $\Delta_1$, and $\Delta_2$ such that

$$Q_\Phi > 0, \quad Q_\Delta > 0 \quad \text{and} \quad \nabla_t Q_\Phi = -Q_\Delta, \quad i = 1, 2.$$

4. There exists a QLF for $B$.

Proof: We begin with some general considerations about finite-dimensional behaviours which will make the proof of the result easier. Since $B$ is finite-dimensional, $\mathcal{V}(B)$ is finite and every trajectory of $B$ is a linear combination of polynomial exponentials of the form

$$w_{\lambda_1, \lambda_2}(k_1, k_2) = p_{\lambda_1, \lambda_2}(k_1, k_2)\lambda_1^{k_1}\lambda_2^{k_2}$$

for some suitable nonzero $w$-vector polynomial function $p_{\lambda_1, \lambda_2}$, i.e.

$$p_{\lambda_1, \lambda_2}(k_1, k_2) = \sum_{(i, j) \in I} \alpha_{ij}k_1^i k_2^j,$$

where $I \subseteq \mathbb{N}^2$ is a finite bi-index set and $\alpha_{ij} \in \mathbb{R}^2$. This implies that the trajectories of $B_1$ are linear combinations of trajectories of the form

$$p_{\lambda_1, \lambda_2}(k; \xi)\lambda_1^{k_1}, \quad i = 1, 2.$$  

Furthermore, it follows from (Rocha 2008, Theorem 8) that $B$ is stable if and only if $\mathcal{V}(B)$ is finite and for all $(\lambda_1, \lambda_2) \in \mathcal{V}(B)$, it holds that $|\lambda_i| < 1$, $i = 1, 2$.

$(1) \Rightarrow (2)$: Let $\tilde{\lambda}_1 \in \mathcal{V}(B_1)$. By assumption, $B$ is finite-dimensional and the trajectories of $w \in B$ are of the form described in (7). Also, the trajectories of $B_1$ are of the form described in (8) with $i = 1$. This implies that there exists $\tilde{\lambda}_2$ such that $p_{\lambda_1, \lambda_2}(k_1, k_2)\lambda_1^{k_1}\lambda_2^{k_2} \in B$. Since $B$ is stable, it follows that $|\tilde{\lambda}_1^{k_1}\tilde{\lambda}_2^{k_2}| < 1$ for all $(\alpha_1, \alpha_2) \in \mathbb{N}^2$ which is equivalent to saying that $|\lambda_i| < 1$ for $i = 1, 2$, i.e. $B_1$ is stable for $i = 1, 2$.

$(2) \Rightarrow (1)$: Let $(\lambda_1, \lambda_2) \in \mathcal{V}(B)$. Since $B_1$ is stable, $i = 1, 2$ and $\lambda_1 \in \mathcal{V}(B_1)$ and $\lambda_2 \in \mathcal{V}(B_2)$ we have that $|\lambda_1^{k_1}| < 1$ and $|\lambda_2^{k_2}| < 1$ for all $(\alpha_1, \alpha_2) \in \mathbb{N}^2$. Therefore $|\lambda_i| < 1$ for all $i = 1, 2$, i.e. $B$ is stable.

$(2) \Rightarrow (3)$: Consider a representation of $B$ as in (2). Then $B_1$ is described by (5); by assumption $A_1, A_2$ are Schur matrices, i.e. all their eigenvalues have modulus less than one. Since these matrices commute, there exists (Narendra and Balakrishnan 1994) a matrix $P > 0$, $\Delta_1 > 0$ and $\Delta_2 > 0$ of suitable sizes such that

$$A_1^TPA_1 - P = -\Delta_1, \quad \text{for } i = 1, 2.$$  

Define

$$Q_\Phi(w) := (X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w)\Sigma PX(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w = x^\top P x,$$

$$Q_\Delta(w) := (X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w)\Sigma \Delta X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w = x^\top \Delta x,$$

$i = 1, 2$, where $X$ is the polynomial operator in the shift inducing the state variable $x$. Now since $\nabla_\xi Q_\Phi(w) = x^\top (A_1^TPA_1 - P)x$, $\forall w \in B$, $i = 1, 2$, it is easy to verify that $Q_\Phi, Q_\Delta$ and $Q_\Delta$ satisfy the conditions of statement (3).

$(3) \Rightarrow (4)$: We show that $Q_\Phi$ is a QLF for $B$. It follows from the Proposition 3.3 that there exists a symmetric polynomial matrix $P$ such that $Q_\Phi(w) = x^\top P x$ with $x = X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w$. Observe that the 1D dynamics along a line $\ell = (a \in \mathbb{N}) \in L$ are described by $\sigma_k x = A_k A_k x$, $w = C x$, for some $i, j$ fixed but otherwise arbitrary. Thus, it is enough to prove that $P$ satisfies a matrix Lyapunov equation for $A_1 A_2$, i.e. $(A_1 A_2)^TP(A_1 A_2) - P < 0$. First, observe that

$$(A_1 A_2)^TP(A_1 A_2) - P = (A_1^TPA_1 - P)(A_2^TPA_2 - P) < (A_2^TPA_2 - P),$$

where we have used the fact that

$$[A_1^TPA_1 - P < 0] \Rightarrow [(A_1^t)^TP(A_1 - P < 0)],$$

as can be readily proved by induction. From the same argument it follows that $(A_1^t)^TP(A_1^t - P < 0$ and consequently $(A_1 A_2)^TP(A_1 A_2) - P < 0$. Now define $Q_\Phi(w) := x^\top P x$, and conclude that along the line $\ell$ it holds that $\nabla_\xi Q_\Phi < 0$, as was to be proved.

$(4) \Rightarrow (2)$ It is a matter of straightforward verification to check that $x |\xi_i|$ is a state trajectory for the 1D behaviour $B_i$, $i = 1, 2$. Moreover, if $P$ is a matrix corresponding to a canonical representative of the Lyapunov function $Q_\Phi$, then $(x |\xi_i)\Sigma X |\xi_i$ is a Lyapunov function for $B_i$. This implies that for $i = 1, 2$ $w_i$ goes to zero along the line $\xi_i$, $i = 1, 2$, i.e. $B_i$ is stable.

We can now characterise Lyapunov functions in terms of canonical representatives.

Definition 3.6: Let $A_1, A_2 \in \mathbb{R}^{n \times n}$. A matrix $P > 0$ is said to be a common Lyapunov solution (CLS) for $A_1$ and $A_2$ if

$$A_1^TPA_1 - P < 0,$$

$$A_2^TPA_2 - P < 0.$$
Using this definition, we can give yet another equivalent statement to those of Theorem 3.5.

**Proposition 3.7:** Let $B$ be a finite-dimensional 2D behaviour and $\Phi$ be a QDF. $\Phi$ is a QLF for $B$ if and only if given any minimal state-space representation $(A_1, A_2, C)$ of $B$ together with a state map $X \in \mathbb{R}^{2 \times n} [\xi_1, \xi_2]$ inducing the state variable $x$ for the representation, the following two conditions hold:

1. There exists a symmetric matrix $P > 0$ which is a \( A \)-CLS for $A_1$ and $A_2$;
2. the canonical representative of $Q_x$ is equal to $X^T PX$.

**Proof:** The claim follows easily using the same arguments of the proof (4) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) of Theorem 3.5. \( \square \)

An interesting question is the construction of all Lyapunov functions for a given 2D behaviour $B$. The relevance of the computation of Lyapunov functions goes beyond stability analysis. Besides its theoretical interest, the computation of Lyapunov functions is crucial, for instance, for the construction of positive storage functions in the context of non-controllable 1D dissipative systems (Çamlıbel, Willems, and Belur 2003; Pal and Belur 2008). We believe that this will also hold true in the 2D case. We address this issue in the next section.

### 4. Construction of Lyapunov functions

The result of Proposition 3.7 shows that the problem of finding 2D Lyapunov functions can be reduced to that of finding common solutions to a pair of 1D Lyapunov equations. Based on this fact, in this section we provide an algorithm for the construction of such functions. We remark that it is crucial in our approach to assume that both $A_1$ and $A_2$ are diagonalisable. Note that, although not general, this is the generic case.

We present some preliminary results which will be instrumental to this end. The first one introduces two useful maps. In the following, we denote by $\mathbb{R}^{n \times n}$ the set of $n \times n$ symmetric matrices.

**Definition 4.1:** Let $A_1, A_2 \in \mathbb{R}^{n \times n}$. The Lyapunov map associated with $A_i$, $i = 1, 2$, is defined as

\[
L_i : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}, \quad P \mapsto A_i^T P A_i - P.
\]

The name Lyapunov map is adopted from Peeters and Rapisarda (2001).

**Lemma 4.2:** If the matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$ in (10) commute then the Lyapunov maps $L_1$ and $L_2$ associated with $A_1, A_2$ commute.

**Proof:**

\[
(L_1 L_2)(P) = A_1^T [A_2^T PA_2 - P] A_1 - [A_2^T PA_2 - P] = A_1^T [A_2^T PA_2 - A_1^T PA_1 - A_2^T PA_2 + P] = A_1^T [A_2^T PA_2 - A_2^T PA_2 + A_1^T PA_1 + P] = A_1^T [A_2^T PA_2 - P] A_2 - [A_1^T PA_1 - P] = (L_2 L_1)(P).
\]

The next result states a necessary and sufficient condition for the existence of a common Lyapunov solution for $A_1$ and $A_2$, given in terms of the Lyapunov maps $L_i$.

**Theorem 4.3:** Let the matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$ in (10) be Schur commuting matrices. Then the associated Lyapunov maps $L_i$ are invertible. Moreover, a matrix $P > 0$ is a $A_i$-CLS for $A_1$ and $A_2$ if and only if there exists a matrix $S > 0$ such that

\[
P = (L_2^{-1} L_1^{-1})(S).
\]

**Proof:** The fact that the maps $L_i$ are invertible follows from standard knowledge regarding the stability of 1D systems. We prove the second part of the claim.

$\Rightarrow$: This part of the claim can be proved using the same argument of the proof of statement (ii) of Theorem 1 of Narendra and Balakrishnan (1994). By assumption we have that $L_2(P) = L_1^{-1}(S) < 0$ because of the linearity of maps $L_i$, as $L_i^{-1}(-S) > 0$. Thus, $P$ is a Lyapunov solution for $A_2$. In the same way we prove that $P$ is a Lyapunov solution for $A_1$. Thus $P$ is a $A_i$-CLS for $A_1$ and $A_2$.

$\Leftarrow$: Let $P$ be a $A_i$-CLS for $A_1$ and $A_2$, and define $Q_i = L_i(P)$, $i = 1, 2$. Note that $Q_i < 0$, $i = 1, 2$. Define $S = L_1(Q_2) = L_1(L_2(P)) = L_1(L_1(P)) = L_2(Q_1)$. Then, $(L_2^{-1} L_1^{-1})(S) = L_2^{-1}(Q_2) = P$. Note that $S > 0$. \( \square \)

The result of Theorem 4.3 characterises the common Lyapunov solutions for a pair of Schur commuting matrices; it also constitutes a generalisation of Theorem 1 of Narendra and Balakrishnan (1994), since it shows that the condition (11) is not only sufficient, but also necessary for the existence of a common Lyapunov function. Moreover, Theorem 4.3 also suggests an algorithm to compute a $A_i$-CLS by inversion of the maps $L_i$.

We now proceed to investigate further the properties of these maps; a similar approach has been adopted for one polynomial Lyapunov equation in Peeters and Rapisarda (2001).

**Definition 4.4:** Let $A_1, A_2 \in \mathbb{R}^{n \times n}$. Then, $0 \neq v \in \mathbb{C}^{n}$ is an eigenvector of the matrix pair $(A_1, A_2)$ if there exists $(\lambda, \mu) \in \mathbb{C}^{2}$ such that

\[
v^{\ast} [\lambda I - A_1 - \mu I - A_2] = 0.
\]
In this case we say that \((\lambda, \mu)\) is an eigenvalue (pair) of \((A_1, A_2)\). Equivalently, we say that \(0 \neq \tilde{v} \in \mathbb{C}^{n \times n}\) is an eigenvector of a pair \((L_1, L_2)\) of linear transformations from \(\mathbb{C}^{n \times n}\) to \(\mathbb{C}^{n \times n}\) such that there exist \((\lambda, \mu) \in \mathbb{C}^2\) such that 
\[L_1(\tilde{v}) = (\lambda \tilde{v}), \quad L_2(\tilde{v}) = (\mu \tilde{v})\]. In this case we say that \((\lambda, \mu)\) is an eigenvalue (pair) of \((L_1, L_2)\).

**Lemma 4.5:** Let \(A_1\) and \(A_2\) be two \(n \times n\) diagonalisable commuting matrices and \(L_1, L_2\) the Lyapunov maps associated with \(A_1, A_2\), respectively. Then, the pair \((A_1, A_2)\) has \(n\) linearly independent eigenvectors. Moreover, given \(V = \{v_1, \ldots, v_n\}\) a basis of \(\mathbb{C}^n\) formed by eigenvectors of \((A_1, A_2)\) with corresponding eigenvalues \((\lambda_i, \mu_i)\), we have that

\[
\tilde{V} = \{\tilde{v}_i = v_i^\lambda + v_i^\mu | 1 \leq i \leq j \leq n\}
\]

is a set of \(\frac{(n+1)n}{2}\) linearly independent eigenvectors of \((L_1, L_2)\), each associated with the pair of eigenvalues \((\lambda_i, \lambda_j - 1, \mu_i, \mu_j - 1)\), for \(1 \leq i \leq j \leq n\).

**Proof:** That the pair \((A_1, A_2)\) has \(n\) linearly independent eigenvectors follows from the well-known fact that commuting matrices are diagonalisable if and only if they have a basis of common eigenvectors. In order to prove that the matrices \(\tilde{v}_i\) form a basis for the set of \(n \times n\) Hermitian matrices, assume by contradiction that there is a non-zero linear combination of the \(\tilde{v}_i\) that is zero; then there exist complex numbers \(k_{ij}, i = 1, \ldots, n\) such that 
\[
\sum_{i=1}^n \sum_{j=1}^n k_{ij} v_i^\lambda v_j^\mu = 0.
\]

This equality is equivalent with 
\[
\sum_{i=1}^n \sum_{j=1}^n k_{ij} v_i^\lambda v_j^\mu = 0.
\]

Since the \(v_j^\mu\) are linearly independent, \(j = 1, \ldots, n\); it follows that \(\sum_{i=1}^n k_{ij} v_i^\lambda = 0\) for all \(j = 1, \ldots, n\). Now apply the linear independence of the \(v_i^\lambda\), \(i = 1, \ldots, n\), to conclude that all coefficients \(k_{ij}\) are zero. The claim that the matrices \(\tilde{v}_i\) are eigenvectors of \(L_i\) follows easily after verifying that 
\[L_1(\tilde{v}_i) = (\lambda_i \tilde{v}_i)\] \[L_2(\tilde{v}_i) = (\mu_i \tilde{v}_i)\]. Note that these formulas also show what the eigenvalues associated with each eigenvector are. 

The result of Lemma 4.5 shows that if the matrices \(A_1, i = 1, 2\) are diagonalisable, a basis of common eigenvectors of \(L_i, i = 1, 2\), can be computed in a straightforward way from a basis of common eigenvectors of \(A_1\) and \(A_2\). Consequently, the inversion necessary to compute the matrix \(P\) as in Theorem 4.3 is a straightforward matter. These considerations lead us to state the following algorithm to compute Lyapunov functions for a given stable behaviour \(B\).

**Algorithm:**

**Input:** A stable, finite-dimensional behaviour \(B\).

**Output:** \(\Phi \in \mathbb{R}^{I \times I} [\xi_1, \xi_2, \eta_1, \eta_2] \) inducing a Lyapunov function for \(B\).

**Step 1:** Compute a representation of \(B\) as in (2), together with a state map \(X \in \mathbb{R}^{I \times I} [\xi_1, \xi_2]\) inducing the state variable \(x\) for the representation.

**Step 2:** Using the matrices \(A_1\) and \(A_2\) from Step 1 construct \(V\) and \(\tilde{V}\) as described in Lemma 4.5.

**Step 3:** Select \(a_{ij}, 1 \leq i \leq j \leq n\) such that 
\[S = \sum_{1 \leq i \leq j \leq n} a_{ij} \tilde{v}_{ij} > 0.\]

**Step 4:** Output

\[\Phi(\xi_1, \xi_2, \eta_1, \eta_2) := (X(\xi_1, \xi_2))^T \sum_{1 \leq i \leq j \leq n} a_{ij} \tilde{v}_{ij} (\lambda_i \lambda_j - 1)(\mu_i \mu_j - 1) \tilde{v}_{ij} X(\eta_1, \eta_2).\]

Some remarks are in order.

**Remark 1:** Note that since \(\{\tilde{v}_{ij}\}_{1 \leq i \leq j \leq n}\) forms a basis for the space of \(n \times n\) Hermitian matrices, the construction of the matrix \(S\) in the step 3 generates all positive definite matrices; consequently, the algorithm produces all possible Lyapunov functions for a given stable 2D behaviour. In particular, an \(S\) can, for instance, be chosen as \(S = \sum_{1 \leq i \leq j \leq n} v_{ij}^\lambda v_{ij}^\mu\).

**Remark 2:** The increments \(v_i, i = 1, 2\) of the Lyapunov function \(Q_\Phi\) computed in Step 4 are easily seen to be equal, respectively, to 
\[\Delta_1(\xi_1, \xi_2, \eta_1, \eta_2) = X(\xi_1, \xi_2)^T \left( \sum_{1 \leq i \leq j \leq n} \frac{1}{\lambda_i \lambda_j - 1} a_{ij} \tilde{v}_{ij} \right) X(\eta_1, \eta_2),\]

\[\Delta_2(\xi_1, \xi_2, \eta_1, \eta_2) = X(\xi_1, \xi_2)^T \left( \sum_{1 \leq i \leq j \leq n} \frac{1}{\mu_i \mu_j - 1} a_{ij} \tilde{v}_{ij} \right) X(\eta_1, \eta_2).\]

**Remark 3:** It follows from Theorem 9.1.1 of Gohberg, Lancaster, and Rodman (2006) that there exist non-diagonalisable matrices which do not have a basis of common generalised eigenvectors. The problem of finding efficient algorithms to compute a common Lyapunov solution in this non-generic case is a matter of further investigation.

**Example 4.6:** Consider \((A_1, A_2, X)\) as in Example 3.4. In step 3, take, for instance \(S = I_3\), the \(3 \times 3\) identity matrix. Step 4 produces

\[
\Phi(\xi_1, \xi_2, \eta_1, \eta_2) = \begin{pmatrix} \left(9(\xi_1 - \frac{1}{2}) - \frac{1}{4}\right)(\eta_1 - \frac{1}{4}) + \frac{225}{16} \left(\xi_1 - \frac{1}{4}\right)(\eta_1 - \frac{1}{4}) & \frac{64}{75} \end{pmatrix}
\]

5. Conclusions

In this article we have illustrated a Lyapunov approach to the stability of finite-dimensional 2D systems. We have adopted as definition of stability, the one given in Definition 3.1, namely the asymptotic stability along all lines in the first orthant. The main results are Theorem 3.5, which characterises stability in terms of
the existence of a Lyapunov function, defined as a quadratic functional of the system variables which is positive along all lines, and whose increments are negative along all lines; and the algorithm given in Section 4 for the computation of Lyapunov functions for stable 2D behaviours $B$, which is valid in the case $B$ has a state space representation as in (2) with diagonalisable matrices $A_i$, $i=1,2$. The development of constructive algorithms for the general case is still a problem under current investigation.

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