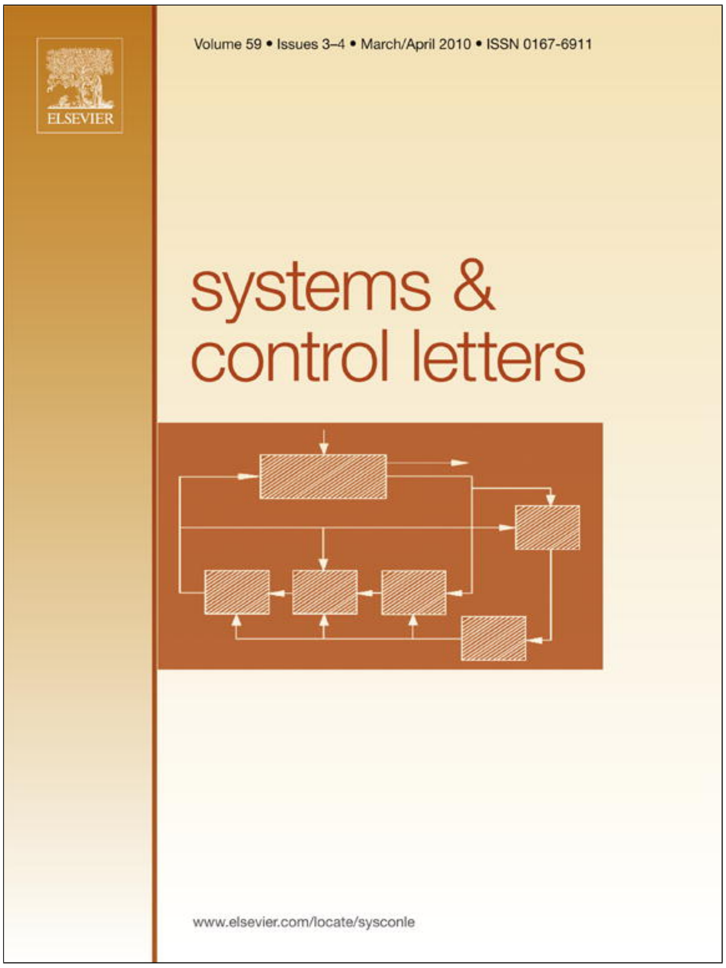
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Autonomous multidimensional systems and their implementation by behavioral control

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ABSTRACT

We consider autonomous n-dimensional systems over $\mathbb{Z}^n$ with different degrees of autonomy, which are characterized by the freeness of the system projections on certain sub-lattices of $\mathbb{Z}^n$. The main purpose of this contribution is to investigate the implementation of such systems in the context of behavioral control. Taking into account that stable nD behaviors are autonomous, we apply our results in order to revisit the problem of stabilization of multidimensional behaviors and complete the results presented in previous contributions.

1. Introduction

In several control applications, both in one- as in multidimensional systems, such as for instance pole-placement and stabilization, the control objective to be implemented by the controller is required to be autonomous, i.e., to have no free variables.

Whereas in the 1D case the property of autonomy is equivalent to the finite dimensionality of a system (meaning that each trajectory is generated from a finite number of initial conditions), nD autonomous systems are generally infinite dimensional. But even in this case the amount of information (initial conditions) necessary to generate the trajectories of an autonomous nD system may vary. This has led to the definition of autonomy degree for behavioral systems proposed in [1], where also a relation with the primeness degree of the correspondent system representation matrix was established for behaviors over $\mathbb{N}^n$. Here we slightly reformulate that definition in terms of the freeness of the projections of the behavior on certain sub-lattices of $\mathbb{Z}^n$. Moreover, we show that such projections are again behaviors and use this fact to extend the results of [1, Section 7] on the autonomy of multidimensional systems to behaviors over $\mathbb{Z}^n$.

The possibility of obtaining (or implementing) a given control objective by means of regular interconnection is a central question in behavioral control [2–6]. Roughly speaking, regular implementation corresponds to the possibility of interconnecting a given behavior with a suitable "non-redundant" controller in order to obtain the desired controlled behavior. This is a crucial issue in the context of feedback control [7,6].

In the 1D case, the requirement that a finite-dimensional control objective is regularly implementable from a given system does not impose by itself any restrictions on the original (to be controlled) behavior. In this paper, we show that the same does not apply to the multidimensional case. More concretely, we consider the problem of regular implementation of autonomous nD behaviors with different autonomous degrees and give conditions in terms of the original behavior for the solvability of this problem. The obtained results are then applied to the stabilization of nD behaviors, allowing to complete the analysis carried out in [8,9].

A preliminary version of this contribution has been partially presented in [10]. This paper is organized as follows: we begin by introducing some necessary background from the field of nD discrete behaviors over $\mathbb{Z}^n$, centering around concepts such as controllability, autonomy, orthogonal module, etc. Section 3 is devoted to the study of the projection of a multidimensional behavior on a sub-lattice of $\mathbb{Z}^n$. Section 4 presents the notion of autonomous degree of a system and its relation with the primeness degree of the corresponding system representation matrix. In Section 5 we investigate the regular implementation of autonomous behaviors. Finally in Section 6 we apply the results of Section 5 to characterize all stabilizable behaviors.

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2. Preliminaries: discrete multidimensional behaviors over $\mathbb{Z}^n$

In order to state more precisely the questions to be considered, we introduce some preliminary notions and results. Denote $\underline{s} = (s_1, \ldots, s_n)$, $\underline{s}^{-1} = (s_1^{-1}, \ldots, s_n^{-1})$, $\mathbb{C}[\underline{s}, \underline{s}^{-1}]$ the Laurent-polynomial ring and $\mathbb{C}(\underline{s}, \underline{s}^{-1})$ its quotient field. We consider nD behaviors $\mathfrak{B}$ defined over $\mathbb{Z}^n$ that can be described by a set of linear partial difference equations, i.e.,

$$\mathfrak{B} = \ker R(\underline{\sigma}, \underline{\sigma}^{-1}) := \{w \in \mathcal{U} \mid R(\underline{\sigma}, \underline{\sigma}^{-1})w \equiv 0\}, \tag{1}$$

where $\mathcal{U}$ is the trajectory universe, here taken to be $(\mathbb{C}^q)^{\mathbb{Z}^n}$, $q \in \mathbb{N}$, $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n)$, $\underline{\sigma}^{-1} = (\sigma_1^{-1}, \ldots, \sigma_n^{-1})$, the σ_i's are the elementary nD shift operators (defined by $\sigma_i w(\underline{k}) = w(\underline{k} + e_i)$, for $\underline{k} \in \mathbb{Z}^n$, where e_i is the ith element of the canonical basis of $\mathbb{C}^n$) and $R(\underline{s}, \underline{s}^{-1}) \in \mathbb{C}^{p\times q}[\underline{s}, \underline{s}^{-1}]$ is an nD Laurent-polynomial matrix known as *representation* of $\mathfrak{B}$. These behaviors are known as *kernel behaviors*; however, throughout this paper we simply refer to them as *behaviors* or *nD behaviors*. If no confusion arises, given an nD Laurent-polynomial matrix $A(\underline{\sigma}, \underline{\sigma}^{-1})$, we sometimes write A instead of $A(\underline{\sigma}, \underline{\sigma}^{-1})$ and $A(\underline{s}, \underline{s}^{-1})$.

Instead of characterizing $\mathfrak{B}$ by means of a representation matrix R, it is also possible to characterize it by means of its *orthogonal module* Mod $(\mathfrak{B})$, which consists of all the nD Laurent-polynomial rows $r(\underline{s}, \underline{s}^{-1}) \in \mathbb{C}^q[\underline{s}, \underline{s}^{-1}]$ such that $\mathfrak{B} \subset \ker r(\underline{\sigma}, \underline{\sigma}^{-1})$, and can be shown to coincide with the $\mathbb{C}[\underline{s}, \underline{s}^{-1}]$-module $RM(R)$ generated by the rows of R, i.e., Mod $(\mathfrak{B}) = \mathrm{RM}(R(\underline{s}, \underline{s}^{-1}))$ [11].

The notions of controllability and autonomy play an important role in the sequel.

Definition 1. A behavior $\mathfrak{B} \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ there exists $\delta > 0$ such that for all subsets $U_1, U_2 \subset \mathbb{Z}^n$ with $d(U_1, U_2) > \delta$, there exists a $w \in \mathfrak{B}$ such that $w\,|_{U_1} = w_1\,|_{U_1}$ and $w\,|_{U_2} = w_2\,|_{U_2}$.

In the above definition, $d(\cdot, \cdot)$ denotes the Euclidean metric on $\mathbb{Z}^n$ and $w\,|_U$, for some $U \subset \mathbb{Z}^n$, denotes the trajectory w restricted to the domain U. It was shown that this definition of controllability is equivalent to say that $\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/\mathrm{Mod}\ (\mathfrak{B})$ is a torsion free $\mathbb{C}[\underline{s}, \underline{s}^{-1}]$-module (see [12, Cor. 2 and Th. 5]).

In contrast with the one-dimensional case, multidimensional behaviors admit a stronger notion of controllability called rectifiability. The following theorem shows several characterizations of rectifiable behaviors that have appeared in several papers.

Theorem 2 (*See [3, Lemma 2.12], [13, Prop. 2.1] and [14, Th. 9 and Th. 10, page 819]*). *Let $\mathfrak{B} = \ker R$ be a behavior. Then the following are equivalent:*

1. *$\mathfrak{B}$ is rectifiable;*
2. *there exists an invertible operator U, where U is an nD Laurent-polynomial matrix, such that*
$$U(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}) := \{Uw \mid w \in \mathfrak{B}\} = \{v \mid RU^{-1}v = 0\}$$
is equal to $\ker [I_\ell\ 0]$, *where I_ℓ is the $\ell \times \ell$ identity matrix, for some $\ell \in \{1, \ldots, q\}$;*
3. *$\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/\mathrm{Mod}\ (\mathfrak{B})$ is a free $\mathbb{C}[\underline{s}, \underline{s}^{-1}]$-module.*

On the other hand, we shall say that a behavior $\mathfrak{B} = \ker R$ is autonomous if it has no free variables, i.e., if the projection mapping $\pi_i : \mathfrak{B} \to (\mathbb{C})^{\mathbb{Z}^n}$, $w = (w_1, \ldots, w_q) \mapsto w_i$ is not surjective for any i. This is equivalent to the condition that $R(\underline{s}, \underline{s}^{-1})$ has full column rank (over $\mathbb{C}[\underline{s}, \underline{s}^{-1}]$) (see [15, Lemma 5]) or also to the condition that $ann(\mathfrak{B}) := \{d \in \mathbb{C}[\underline{s}, \underline{s}^{-1}] \mid d(\underline{\sigma}, \underline{\sigma}^{-1})w = 0 \text{ for all } w \in \mathfrak{B}\} \neq 0$.

It was also shown in [15] that every nD behavior $\mathfrak{B}$ can be decomposed into a sum

$$\mathfrak{B} = \mathfrak{B}^c + \mathfrak{B}^a,$$

where $\mathfrak{B}^c$ is the *controllable part* of $\mathfrak{B}$ (defined as the largest controllable sub-behavior of $\mathfrak{B}$) and $\mathfrak{B}^a$ is a (non-unique) autonomous sub-behavior said to be an *autonomous part* of $\mathfrak{B}$. In general, this cannot be made a direct sum when $n > 1$.

If the controllable-autonomous decomposition happens to be a direct sum decomposition, i.e., if $\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a$, we say that the autonomous part of $\mathfrak{B}^a$ is an *autonomous direct summand* of $\mathfrak{B}$.

Remark 3. It is important to remark that if $\mathfrak{B} = \ker R$ and $\mathfrak{B}^c = \ker R^c$, then the rank of R and the rank of R^c must coincide. Indeed, by [3, Cor. 2.10], $\mathfrak{B}$ and $\mathfrak{B}^c$ must have the same number of inputs (free variables) and the same number of outputs. Moreover, the number of outputs in a behavior coincides with the rank of its representation matrix [16].

When the controllable part $\mathfrak{B}^c$ is rectifiable, it is possible to take advantage of the simplified form of the rectified behaviors in order to derive various results. In particular, it is not difficult to obtain the next proposition that characterizes the autonomous direct summands of a behavior.

Proposition 4. *Let $\mathfrak{B} = \ker R(\underline{\sigma}, \underline{\sigma}^{-1}) \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ be an nD behavior with rectifiable controllable part $\mathfrak{B}^c$ and $U(\underline{\sigma}, \underline{\sigma}^{-1})$ be a corresponding rectifying operator such that $U(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}^c) =$* $\ker [I_\ell\ 0]$. *Then the following are equivalent:*

1. $\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a$,
2. $\mathfrak{B}^a = \ker \left(\begin{bmatrix} P & 0 \\ X & I_{q-\ell} \end{bmatrix} U \right)$, *with $P(\underline{s}, \underline{s}^{-1}) \in \mathbb{C}^{r\times \ell}[\underline{s}, \underline{s}^{-1}]$, p the number of rows of R, such that $RU^{-1} = [P\ 0]$ and $X(\underline{s}, \underline{s}^{-1})$ a Laurent-polynomial matrix of suitable size.*

Note that the behaviors $\mathfrak{B}^a$ of Proposition 4 always exist and are autonomous. Thus, this result states that every behavior $\mathfrak{B}$ with rectifiable controllable part has an autonomous part which is a direct summand of $\mathfrak{B}$; moreover, it gives a parametrization for all such summands.

3. Projection of $\mathfrak{B}$ on a sub-lattice of $\mathbb{Z}^n$

In this section we introduce the notion of sub-lattice of $\mathbb{Z}^n$ (denoted by $\mathfrak{L}$) and the restriction of a behavior $\mathfrak{B}$ to such a sub-lattice (denoted by $\mathfrak{B}_{\mathfrak{L}}$). Moreover, we show that $\mathfrak{B}_{\mathfrak{L}}$ has itself the structure of a (kernel) behavior, i.e., it can be represented by the kernel of linear shift-invariant operator as in (1). This result turns out to be very useful, for instance, to reformulate the definition of degree of autonomy that will be presented in the next section.

Definition 5. We define an ℓ-dimensional sub-lattice of $\mathbb{Z}^n$ as $\mathfrak{L} := \{(k_1, \ldots, k_n) \in \mathbb{Z}^n \mid k_{j_{\ell+1}} = \cdots = k_{j_n} = 0,\ j_{\ell+1}, \ldots, j_n \in \{1, \ldots, n\}\}$. The restriction of a behavior $\mathfrak{B}$ to $\mathfrak{L}$ is defined as follows:

$$\begin{aligned} \mathfrak{B}_{\mathfrak{L}} := \{&w_{\mathfrak{L}} : \mathfrak{L} \mapsto \mathbb{C}^q \mid \exists\, w \in \mathfrak{B} \text{ such that } w_{\mathfrak{L}}(k_1, \ldots, k_n) \\ &= w(k_1, \ldots, k_n) \text{ for all } (k_1, \ldots, k_n) \in \mathfrak{L}\} \\ = \{&w_{\mathfrak{L}} : \mathfrak{L} \mapsto \mathbb{C}^q \mid \exists\, w \in \mathfrak{B} \text{ such that } w_{\mathfrak{L}} = w\,|_{\mathfrak{L}}\}. \end{aligned}$$

Given a sub-lattice $\mathfrak{L} := \{(k_1, \ldots, k_n) \in \mathbb{Z}^n \mid k_{j_{\ell+1}} = \cdots = k_{j_n} = 0,\ j_{\ell+1}, \ldots, j_n \in \{1, \ldots, n\}\}$, define $\underline{s}_{\,\mathfrak{L}} := (s_{i_1}, \ldots, s_{i_\ell})$ as the indeterminate vector corresponding to the non-zero coordinates in $\mathfrak{L}$. We define $\underline{s}_{\,\mathfrak{L}}^{-1} := (s_{i_1}^{-1}, \ldots, s_{i_\ell}^{-1})$, $\underline{\sigma}_{\,\mathfrak{L}}$ and $\underline{\sigma}_{\,\mathfrak{L}}^{-1}$ in a similar way.

Theorem 6. *Let $\mathfrak{B} \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ be a behavior and $\mathfrak{L}$ an ℓ-dimensional sub-lattice of $\mathbb{Z}^n$. Then $\mathfrak{B}_{\mathfrak{L}}$ is a behavior.*

Proof. It was shown that behaviors, as defined previously in (1), are precisely the closed shift-invariant subspaces of $(\mathbb{C}^q)^{\mathbb{Z}^n}$, $q \in \mathbb{N}$, for the topology of pointwise convergence (t.p.c), see [17–19,11]. Hence, it is enough to show that $\mathfrak{B}_{\mathfrak{L}}$ is a closed shift-invariant subspace of $(\mathbb{C}^q)^{\mathbb{Z}^n}$ under the t.p.c.

Since $\mathbb{C}^q$ is a finite-dimensional vector space, it is a linearly compact space when endowed with the discrete topology. Furthermore, the topological product of a countable number of linearly compact spaces is still linearly compact (see [20, Section 10.9 (7)]). Consequently, $(\mathbb{C}^q)^{\mathbb{Z}^n}$ equipped with the product topology, i.e. with the t.p.c., is linearly compact.

We define the continuous linear projection map $\pi_{\mathfrak{L}}$ as

$$\pi_{\mathfrak{L}} : (\mathbb{C}^q)^{\mathbb{Z}^n} \to (\mathbb{C}^q)^{\mathfrak{L}}; \qquad w \mapsto w\,|_{\mathfrak{L}}\,.$$

Finally, we use the fact that every continuous linear operator from a linearly compact space into another maps closed subspaces into closed subspaces ([20, Section 10.9 (1)]). Hence we have that $\pi_{\mathfrak{L}}(\mathfrak{B}) = \mathfrak{B}_{\mathfrak{L}}$ is a closed subspace of $(\mathbb{C}^q)^{\mathbb{Z}^n}$ and since it is obviously shift-invariant with respect to the shifts in $\mathbb{C}[\underline{\sigma}_{\,\mathfrak{L}}, \underline{\sigma}_{\,\mathfrak{L}}^{-1}]$, we conclude that $\mathfrak{B}_{\mathfrak{L}}$ is a (kernel) behavior. $\square$

It follows from the proof of Theorem 6 that $\mathfrak{B}_{\mathfrak{L}}$ can clearly be identified with a behavior over $\mathbb{Z}^\ell$, i.e., there exists a Laurent-polynomial matrix $\overline{R}$ with entries in $\mathbb{C}[\underline{s}_{\,\mathfrak{L}}, \underline{s}_{\,\mathfrak{L}}^{-1}]$ such that $\mathfrak{B}_{\mathfrak{L}} = \ker \overline{R}$. A row r of $\overline{R}$ can be thought as an equation that is satisfied by all trajectories of $\mathfrak{B}_{\mathfrak{L}}$. The next result characterizes the set of all equations that are satisfied by the elements of $\mathfrak{B}_{\mathfrak{L}}$.

Theorem 7. *Let $\mathfrak{B}$ be a behavior and $\mathfrak{L}$ an ℓ-dimensional sub-lattice of $\mathbb{Z}^n$. Then* $\mathrm{Mod}\,(\mathfrak{B}_{\mathfrak{L}}) = \mathrm{Mod}\,(\mathfrak{B}) \cap \mathbb{C}^q[\underline{s}_{\,\mathfrak{L}}, \underline{s}_{\,\mathfrak{L}}^{-1}]$.

Proof. ($\supset$) Consider $r(\underline{\sigma}, \underline{\sigma}^{-1}) \in \mathrm{Mod}\,(\mathfrak{B}) \cap \mathbb{C}^q[\underline{s}_{\,\mathfrak{L}}, \underline{s}_{\,\mathfrak{L}}^{-1}]$ and $w_{\mathfrak{L}} \in \mathfrak{B}_{\mathfrak{L}}$. Since $\mathfrak{L} + \underline{j} \subset \mathfrak{L}$ for all $\underline{j} \in \mathfrak{L}$ and $r(\underline{\sigma}, \underline{\sigma}^{-1}) = r(\underline{s}_{\,\mathfrak{L}}, \underline{s}_{\,\mathfrak{L}}^{-1})$, the action of $r(\underline{\sigma}, \underline{\sigma}^{-1})$ on $w_{\mathfrak{L}}$ is well defined. Moreover, letting $w \in \mathfrak{B}$ be such that $w\,|_{\mathfrak{L}} = w_{\mathfrak{L}}$,

$$(r(\underline{\sigma}, \underline{\sigma}^{-1})w_{\mathfrak{L}})(\ell) = (r(\underline{\sigma}, \underline{\sigma}^{-1})w)(\ell) = 0 \quad \text{for all } \ell \in \mathfrak{L}$$

since $r \in \mathrm{Mod}\,(\mathfrak{B})$. Thus, $r(\underline{\sigma}, \underline{\sigma}^{-1}) \in \mathrm{Mod}\,(\mathfrak{B}_{\mathfrak{L}})$.

($\subset$) Let $r \in \mathrm{Mod}\,(\mathfrak{B}_{\mathfrak{L}})$, i.e., $rw_{\mathfrak{L}} = 0$ for all $w_{\mathfrak{L}} \in \mathfrak{B}_{\mathfrak{L}}$. Since $\mathfrak{B}_{\mathfrak{L}} = \ker \overline{R}$ for some Laurent-polynomial matrix $\overline{R}$ with entries in $\mathbb{C}[\underline{s}_{\,\mathfrak{L}}, \underline{s}_{\,\mathfrak{L}}^{-1}]$ and $\mathrm{Mod}\,(\mathfrak{B}_{\mathfrak{L}}) = RM(\overline{R})$, we have that $r \in \mathbb{C}^q[\underline{s}_{\,\mathfrak{L}}, \underline{s}_{\,\mathfrak{L}}^{-1}]$. Hence, it is enough to prove that $r \in \mathrm{Mod}\,(\mathfrak{B})$. Suppose that $r \notin \mathrm{Mod}\,(\mathfrak{B})$. Then, there exists $\widetilde{w} \in \mathfrak{B}$ such that $r(\underline{\sigma}_{\,\mathfrak{L}}, \underline{\sigma}_{\,\mathfrak{L}}^{-1})\widetilde{w} \neq 0$. In particular, we have that there exists $\underline{t} = (t_1, \ldots, t_n) \in \mathbb{Z}^n$ such that $[r(\underline{\sigma}_{\,\mathfrak{L}}, \underline{\sigma}_{\,\mathfrak{L}}^{-1})\widetilde{w}](\underline{t}) \neq 0$. Define $w := \underline{\sigma}^{\underline{t}}\, \widetilde{w}$, where $\underline{\sigma}^{\underline{t}} := \sigma_1^{t_1} \cdots \sigma_n^{t_n}$. Note that due to shift-invariance $w \in \mathfrak{B}$. Now,

$$\begin{aligned} [r(\underline{\sigma}_{\,\mathfrak{L}}, \underline{\sigma}_{\,\mathfrak{L}}^{-1})w](\underline{0}) &= [r(\underline{\sigma}_{\,\mathfrak{L}}, \underline{\sigma}_{\,\mathfrak{L}}^{-1})\, \underline{\sigma}^{\underline{t}}\widetilde{w}](\underline{0}) \\ &= [r(\underline{\sigma}_{\,\mathfrak{L}}, \underline{\sigma}_{\,\mathfrak{L}}^{-1})\widetilde{w}](\underline{t}) \neq 0. \end{aligned}$$

As $\underline{0} \in \mathfrak{L}$, it follows that $w\,|_{\mathfrak{L}} \notin \ker r$. Finally, since $w\,|_{\mathfrak{L}} \in \mathfrak{B}_{\mathfrak{L}}$ we have that $r \notin \mathrm{Mod}\,(\mathfrak{B}_{\mathfrak{L}})$, which is a contradiction with the assumption. $\square$

The following corollary, which characterizes the projection of a behavior in a sub-lattice, is a straightforward conclusion of the two previous results.

Corollary 8. *Let $\mathfrak{B}$ be a behavior and $\mathfrak{L}$ a sub-lattice of $\mathbb{Z}^n$. Then $\mathfrak{B}_{\mathfrak{L}} = \ker \widetilde{R}$ where $RM(\widetilde{R}) = \mathrm{Mod}\,(\mathfrak{B}) \cap \mathbb{C}^q[\underline{s}_{\,\mathfrak{L}}, \underline{s}_{\,\mathfrak{L}}^{-1}]$.*

We now present a simple example in order to illustrate the previous results.

Example 9. Let $\mathfrak{B} = \ker R \subset (\mathbb{C}^2)^{\mathbb{Z}^3}$ be a behavior with

$$R(\sigma_1, \sigma_2, \sigma_3, \sigma_1^{-1}, \sigma_2^{-1}, \sigma_3^{-1}) = \begin{pmatrix} \sigma_1\sigma_2 - 1 & 0 \\ 1 & \sigma_3^2 - 2 \\ \sigma_1 & \sigma_2 - \sigma_1^{-1} \\ 1 - \sigma_1\sigma_3 & 2 - \sigma_2^{-1} \end{pmatrix}$$

and $\mathfrak{L}_1 = \{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_3 = 0\}$, $\mathfrak{L}_2 = \{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_1 = k_2 = 0\}$ two sub-lattices of $\mathbb{Z}^3$. By Corollary 8, one obtains that

$$\begin{aligned} \mathfrak{B}_{\mathfrak{L}_1} &= \{w_{\mathfrak{L}_1} \in (\mathbb{C}^2)^{\mathfrak{L}_1} \mid \text{there exists } w \in \mathfrak{B} \text{ such that } w\,|_{\mathfrak{L}} = w_{\mathfrak{L}_1}\} \\ &= \ker \begin{pmatrix} \sigma_1\sigma_2 - 1 & 0 \\ \sigma_1 & \sigma_2 - \sigma_1^{-1} \end{pmatrix} \quad \text{and} \\ \mathfrak{B}_{\mathfrak{L}_2} &= \{w_{\mathfrak{L}_2} \in (\mathbb{C}^2)^{\mathfrak{L}_2} \mid \text{there exists } w \in \mathfrak{B} \text{ such that } w\,|_{\mathfrak{L}} = w_{\mathfrak{L}_2}\} \\ &= \ker\,(1\ \sigma_3^2 - 2). \end{aligned}$$

4. Autonomous *n*D behaviors

Given an autonomous behavior, a natural question to ask is how much information is necessary in order to fully determine the system trajectories, i.e., how large is the initial condition set. This question has been analyzed in [1] by introducing the notion of autonomy degrees for behaviors and relating them to the different types of primeness of the corresponding representation matrices. Although the results presented in [1] concern behaviors over $\mathbb{N}^n$, it is possible to extend them to behaviors over $\mathbb{Z}^n$, as we shall do in the sequel using a slightly different formulation.

We first consider some elementary examples.

Example 10. 1. Let $\mathfrak{B}^1 = \ker R_1 \subset \mathbb{C}^{\mathbb{Z}^3}$, where $R_1 = (s_3 - 1)$, be an autonomous 3D behavior. Then the trajectories $w(k_1, k_2, k_3) \in \mathfrak{B}^1$ can be assigned freely on a plane, parallel to the span of e_1 and e_2.

2. Let $\mathfrak{B}^2 = \ker R_2 \subset \mathbb{C}^{\mathbb{Z}^3}$, where $R_2 = \begin{pmatrix} s_3 - 1 \\ s_2 - 1 \end{pmatrix}$, be an autonomous 3D behavior. Then the trajectories $w(k_1, k_2, k_3) \in \mathfrak{B}^2$ can be assigned freely on a line, parallel to the e_1-axis.

3. Let $\mathfrak{B}^3 = \ker R_3 \subset \mathbb{C}^{\mathbb{Z}^3}$, where $R_3 = \begin{pmatrix} s_3 - 1 \\ s_2 - 1 \\ s_1 - 1 \end{pmatrix}$, be an autonomous 3D behavior. Then the trajectories $w(k_1, k_2, k_3) \in \mathfrak{B}^3$ can be assigned freely only on a point, and $\mathfrak{B}^3$ is therefore finite dimensional.

Definition 11. Let $\mathfrak{B} \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ be a nonzero *n*D behavior. We define the *autonomous degree* of $\mathfrak{B}$, denoted by *autodeg*$(\mathfrak{B})$, as $n - \ell$, where ℓ is the largest value for which there exists an ℓ-dimensional sub-lattice $\mathfrak{L}$ of $\mathbb{Z}^n$ such that $\mathfrak{B}_{\mathfrak{L}}$ is not autonomous. The autonomy degree of the zero behavior is defined to be ∞.

Note that the larger the autonomy degree, the smaller is the freedom to assign initial conditions. Indeed, a behavior that is not autonomous has autonomous degree equal to zero.

Example 12. Consider $\mathfrak{B}^1$, $\mathfrak{B}^2$ and $\mathfrak{B}^3$ as in Example 10. We define $\mathfrak{L}_1 := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_3 = 0\}$ and obtain that

$$\mathfrak{B}^1_{\mathfrak{L}_1} := \{w_{\mathfrak{L}_1} : \mathfrak{L}_1 \mapsto \mathbb{C} \mid \exists\, w \in \mathfrak{B}^1 \text{ such that } w(k_1, k_2, 0) = w_{\mathfrak{L}_1}\}$$

can clearly be identified with $(\mathbb{C})^{\mathbb{Z}^2}$ and therefore is not autonomous. Since $dim(\mathfrak{L}_1) = 2$ we have that $autodeg(\mathfrak{B}^1) = 1$. Next, we define $\mathfrak{L}_2 := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_2 = k_3 = 0\}$. It is not difficult to see that

$$\mathfrak{B}^2_{\mathfrak{L}_2} := \{w_{\mathfrak{L}_2} : \mathfrak{L}_2 \mapsto \mathbb{C} \mid \exists\, w \in \mathfrak{B}^2 \text{ such that } w(k_1, 0, 0) = w_{\mathfrak{L}_2}\}$$

is not autonomous and that there does not exist a two-dimensional sub-lattice $\overline{\mathfrak{L}}$ of $\mathbb{Z}^3$ such that $\mathfrak{B}^2_{\overline{\mathfrak{L}}}$ is not autonomous. Hence, $autodeg(\mathfrak{B}^2) = 2$. Finally, we define $\mathfrak{L}_3 := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_1 = k_2 = k_3 = 0\}$. Again it is easy to check that

$$\mathfrak{B}^3_{\mathfrak{L}_3} := \{w_{\mathfrak{L}_3} : \mathfrak{L}_3 \mapsto \mathbb{C} \mid \exists\, w \in \mathfrak{B}^3 \text{ such that } w(0, 0, 0) = w_{\mathfrak{L}_3}\}$$

is not autonomous and that there does not exist a one-dimensional sub-lattice $\overline{\mathfrak{L}}$ of $\mathbb{Z}^n$ such that $\mathfrak{B}^3_{\overline{\mathfrak{L}}}$ is not autonomous. Thus, $autodeg(\mathfrak{B}^3) = 3$.

It is possible to relate the autonomy degree of $\mathfrak{B} = \ker R$ with the right primeness degree of R, which we define as follows.

Definition 13. Let $R \in \mathbb{C}^{p\times q}[\underline{s}, \underline{s}^{-1}]$ be an nD Laurent-polynomial matrix and $p \geq q$. Let $m_1, \ldots, m_s \in \mathbb{C}[\underline{s}, \underline{s}^{-1}]$ be the $q \times q$ order minors of R. The ideal generated by these minors is denoted by $I(R) = \langle m_1, \ldots, m_s\rangle$ and let $Z(I(R))$ denote the set of all points in $(\mathbb{C} \setminus 0)^n$ at which every element of $I(R)$ vanishes. We define $primdeg(R) := n - dimZ(I(R))$ to be the *right primeness degree of* R.

Example 14. Consider the matrices R_1, R_2 and R_3 in Example 10. Then $I(R_1) = \langle s_3 - 1\rangle$, $dimZ(I(R_1)) = 2$ and therefore $primdeg(R_1) = 1$, whereas $I(R_2) = \langle s_2-1, s_3-1\rangle$, $dimZ(I(R_2)) = 1$ and therefore $primdeg(R_2) = 2$. Finally we have that $I(R_3) = \langle s_1 - 1, s_2 - 1, s_3 - 1\rangle$, $dimZ(I(R_3)) = 0$ and therefore $primdeg(R_3) = 3$.

Note that here the primeness of the representation matrices coincides with the autonomy degrees of the associated behaviors. In fact, this also holds in the general case, as stated in the following theorem.

Theorem 15. *Let $\mathfrak{B} = \ker R$ be a behavior. Then, the autonomy degree of $\mathfrak{B}$ is equal to the right primeness degree of R, i.e., $autodeg(\mathfrak{B}) = primdeg(R)$.*

Proof. It was proved in [12, Lemma 4] that $ann(\mathfrak{B}) = ann(\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/RM(R))$. Thus, by [1, Th. 5.2.], one concludes that the right primeness degree of R is $d \in \{1, \ldots, n\}$ if and only if $ann(\mathfrak{B}) \cap \mathbb{C}[\widetilde{\underline{s}}, \widetilde{\underline{s}}^{-1}] \neq 0$ for all $\widetilde{\underline{s}} = (s_{j_1}, \ldots, s_{j_{n-d+1}})$ with $j_i \in \{1, \ldots, n\}$. This is equivalent to say that $ann(\mathfrak{B}) \cap \mathbb{C}[\underline{s}_{\mathfrak{L}}, \underline{s}_{\mathfrak{L}}^{-1}] \neq 0$ for every $(n - d + 1)$-dimensional sub-lattice $\mathfrak{L}$ of $\mathbb{Z}^n$.

Suppose that $autodeg(\mathfrak{B}) = d$, i.e., for every $(n - d + 1)$-dimensional sub-lattice $\mathfrak{L}$, $\mathfrak{B}_{\mathfrak{L}}$ is autonomous or equivalently $ann(\mathfrak{B}_{\mathfrak{L}}) \neq 0$. Since it follows from Theorem 7 that $ann(\mathfrak{B}_{\mathfrak{L}}) = ann(\mathfrak{B}) \cap \mathbb{C}[\underline{s}_{\mathfrak{L}}, \underline{s}_{\mathfrak{L}}^{-1}]$, we conclude that $ann(\mathfrak{B}) \cap \mathbb{C}[\underline{s}_{\mathfrak{L}}, \underline{s}_{\mathfrak{L}}^{-1}] \neq 0$, and hence (because $\mathfrak{L}$ is an arbitrary sub-lattice of dimension $n-d+1$) $primdeg(R) = d$. □

5. Regular implementation of autonomous behaviors

Given two behaviors $\mathfrak{B}^1$ and $\mathfrak{B}^2$ their *interconnection* is defined as the intersection $\mathfrak{B}^1 \cap \mathfrak{B}^2$. This interconnection is said to be *regular* if

$\mathrm{Mod}\ (\mathfrak{B}^1) \cap \mathrm{Mod}\ (\mathfrak{B}^2) = 0.$

The following result can be found in, for instance, [3, Lemma 3, page 115].

Lemma 16. *Given the two behaviors $\mathfrak{B}^1 = \ker R^1$ and $\mathfrak{B}^2 = \ker R^2$. The following are equivalent:*

1. *$\mathfrak{B}^1 \cap \mathfrak{B}^2$ is a regular interconnection,*
2. *$\mathfrak{B}^1 + \mathfrak{B}^2 = (\mathbb{C}^q)^{\mathbb{Z}^n}$,*
3. *$\operatorname{rank} R^1 + \operatorname{rank} R^2 = \operatorname{rank} \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}$.*

Regular interconnections correspond to a lack of overlapping between the laws of the interconnected behaviors and play an important role in behavioral control, [5–7,2].

A sub-behavior $\mathfrak{B}^d \subset \mathfrak{B}$ is said to be regularly implementable from $\mathfrak{B}$ if there exists a controller behavior $\mathfrak{C}$ such that $\mathfrak{B} \cap \mathfrak{C} = \mathfrak{B}^d$ and this interconnection is regular. In this case we denote $\mathfrak{B} \cap_{reg} \mathfrak{C} = \mathfrak{B}^d$.

A relevant question (for instance in the framework of pole-placement) is the regular implementation of autonomous behaviors. The following proposition is a direct consequence of the results in [3], and states that every regularly implementable autonomous sub-behavior of $\mathfrak{B}$ is an autonomous part of $\mathfrak{B}$.

Proposition 17. *Let $\mathfrak{B}^d \subset \mathfrak{B}$ be two behaviors, with $\mathfrak{B}^d$ autonomous. If $\mathfrak{B}^d$ is regularly implementable from $\mathfrak{B}$, then*

$\mathfrak{B} = \mathfrak{B}^c + \mathfrak{B}^d.$

This result can be intuitively explained by the fact that an autonomous part of a behavior may be somehow considered as obstructions to the (regular) control of that behavior, as happens for instance with the non-controllable modes in the context of pole-placement for classical state-space systems.

A more surprising result is the fact that the possibility of implementing autonomous sub-behaviors of $\mathfrak{B}$ by regular interconnection may also impose conditions in the controllable part of $\mathfrak{B}$, depending on the autonomy degree of such sub-behaviors.

Theorem 18. *Let $\mathfrak{B}$ be a behavior. If $\mathfrak{B}^d \subset \mathfrak{B}$ is regularly implementable from $\mathfrak{B}$ and has autonomy degree larger than 1, then $\mathfrak{B}^c$ (the controllable part of $\mathfrak{B}$) is rectifiable.*

Proof. In order to prove the result we will make use of the duality between $\mathfrak{B}$ and Mod $(\mathfrak{B})$. It turns out that $\mathfrak{B} \cap_{reg} \mathfrak{C} = \mathfrak{B}^d$ if and only if $\mathrm{Mod}\ (\mathfrak{B}) \oplus \mathrm{Mod}\ (\mathfrak{C}) = \mathrm{Mod}\ (\mathfrak{B}^d)$, see for instance [6, page 1074]. The assumption that $\mathfrak{B}^d$ has autonomy degree ≥ 2 amounts to say that the height of the annihilator of $\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/(\mathrm{Mod}\ (\mathfrak{B}) \oplus \mathrm{Mod}\ (\mathfrak{C}))$ is ≥ 2, see [1, Lemma 4.7, page 54]. Equivalently, the annihilator of $\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/(\mathrm{Mod}\ (\mathfrak{B}) \oplus \mathrm{Mod}\ (\mathfrak{C}))$ contains at least two coprime elements, see [1, Lemma 3.6].

Further, the interconnection $\mathfrak{B} \cap \mathfrak{C}$ is regular if and only if $\mathfrak{B}^c \cap \mathfrak{C}^c$ is regular, where $\mathfrak{B}^c$ and $\mathfrak{C}^c$ denote the corresponding controllable parts, see [9, Lemma 12]. Obviously $\mathfrak{B}^c \cap \mathfrak{C}^c \subset \mathfrak{B} \cap \mathfrak{C}$ and therefore $autodeg(\mathfrak{B}^c \cap \mathfrak{C}^c) \geq autodeg(\mathfrak{B} \cap \mathfrak{C})$.

Thus we have, by assumption, that the annihilator of $\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/(\mathrm{Mod}\ (\mathfrak{B}^c) \oplus \mathrm{Mod}\ (\mathfrak{C}^c))$ contains at least two coprime elements, say d_1, d_2. Note that since $\mathfrak{B}^c$ and $\mathfrak{C}^c$ are controllable, $\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/\mathrm{Mod}\ (\mathfrak{B}^c)$ and $\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/\mathrm{Mod}\ (\mathfrak{C}^c)$ are torsion free.

Using Theorem 2 we prove that $\mathfrak{B}^c$ is rectifiable by showing that the $\mathbb{C}[\underline{s}, \underline{s}^{-1}]$-module $\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/\mathrm{Mod}\ (\mathfrak{B}^c)$ is free.

Consider an element $\xi \in \mathbb{C}^q[\underline{s}, \underline{s}^{-1}]$. There are coprime elements d_1, d_2 with $d_1\xi = a_1 + b_1$, $d_2\xi = a_2 + b_2$ with $a_1, a_2 \in \mathrm{Mod}\ (\mathfrak{B}^c)$, $b_1, b_2 \in \mathrm{Mod}\ (\mathfrak{C}^c)$. The element $\tau_1 = \frac{a_1}{d_1} = \frac{a_2}{d_2} \in \mathbb{C}^q(\underline{s}, \underline{s}^{-1})$ has the property $d_1\tau_1,\ d_2\tau_1 \in \mathbb{C}^q[\underline{s}, \underline{s}^{-1}]$. Since d_1, d_2 are coprime, this implies that $\tau_1 \in \mathbb{C}^q[\underline{s}, \underline{s}^{-1}]$. Since $\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/\mathrm{Mod}\ (\mathfrak{B}^c)$ has no torsion, one obtains $\tau_1 \in \mathrm{Mod}\ (\mathfrak{B}^c)$.

The same argument shows that $\tau_2 = \frac{b_1}{d_1} = \frac{b_2}{d_2}$ belongs to $\mathrm{Mod}\ (\mathfrak{C}^c)$. Hence $\xi = \tau_1 + \tau_2 \in \mathrm{Mod}\ (\mathfrak{B}^c) \oplus \mathrm{Mod}\ (\mathfrak{C}^c)$ and $\mathbb{C}^q[\underline{s}, \underline{s}^{-1}] = \mathrm{Mod}\ (\mathfrak{B}^c) \oplus \mathrm{Mod}\ (\mathfrak{C}^c)$. Then $\mathrm{Mod}\ (\mathfrak{B}^c)$ and $\mathrm{Mod}\ (\mathbb{C}^c)$ are projective modules and therefore free. Finally, since $\mathrm{Mod}\ (\mathbb{C}^c) \approx \mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/\mathrm{Mod}\ (\mathfrak{B}^c)$ one obtains that $\mathbb{C}^q[\underline{s}, \underline{s}^{-1}]/\mathrm{Mod}\ (\mathfrak{B}^c)$ is free. This concludes the proof. □

Given a behavior $\mathfrak{B}$, the possibility of implementing an autonomous behavior with the autonomous degree equal to one by regular interconnection from $\mathfrak{B}$ does not require that the controllable part of $\mathfrak{B}$ is rectifiable. This is shown in the following example.

Example 19. Let $\mathfrak{B} = \ker\ (\sigma_1 - 1\ \sigma_2 - 1)$ and $\mathfrak{C} = \ker \begin{pmatrix} 0 & \sigma_1 - 1 \\ 0 & \sigma_2 - 1 \end{pmatrix}$ be two behaviors over $\mathbb{Z}^2$. Then it is easy to see that the interconnection of $\mathfrak{B}$ and $\mathfrak{C}$ is regular and that

$$\mathfrak{B} \cap_{reg} \mathfrak{C} = \ker \begin{pmatrix} \sigma_1 - 1 & \sigma_2 - 1 \\ 0 & \sigma_1 - 1 \\ 0 & \sigma_2 - 1 \end{pmatrix}$$

is an autonomous behavior (with autonomous degree equal 1). However, the controllable part of $\mathfrak{B}$, $\mathfrak{B}^c = \ker\ (\sigma_1 - 1\ \sigma_2 - 1)$, is not rectifiable.

Theorem 18 generalizes the one obtained in [9] for the 2D case. However, the proof given here is completely different from the one in [9], which is not adaptable to the nD case.

It is not difficult to conclude that if $\widetilde{\mathfrak{B}^a}$ is a sub-behavior of $\mathfrak{B} = \ker R$ with autonomous degree not less than 2, that is regularly implementable from $\mathfrak{B}$ then $\widetilde{\mathfrak{B}^a}$ is described as

$$\widetilde{\mathfrak{B}^a} = \ker \begin{pmatrix} P & 0 \\ C_1 & C_2 \end{pmatrix} U,$$

where U is a rectifying operator such that $U(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}) = \ker (P\ 0)$, where P has the same number of rows of R, C_2 has full column rank and rank $(C_1 C_2) = \text{rank } C_2$. The fact that $autodeg(\widetilde{\mathfrak{B}^a}) \geq 2$ also implies that $autodeg(\ker P) \geq 2$. As a consequence, by Proposition 4, all the autonomous direct summands of $\mathfrak{B}$ must have autonomous degree larger than or equal to 2. Taking into account that such direct summands are regularly implementable from $\mathfrak{B}$, this allows to conclude the following.

Proposition 20. *Let $\mathfrak{B}$ be a behavior. Then, there exists a sub-behavior of $\mathfrak{B}$ with autonomous degree larger than or equal to 2 that is regularly implementable from $\mathfrak{B}$ if and only if*

$$\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a$$

with $\mathfrak{B}^c$ rectifiable and $autodeg(\mathfrak{B}^a) \geq 2$.

6. Stabilizability

In this section we apply the results obtained in the previous section to the context of stabilization and characterize all stabilizable behaviors.

A discrete 1D behavior $\mathfrak{B} \subset (\mathbb{C}^q)^{\mathbb{Z}}$ is said to be *stable* if all its trajectories tend to the origin as time goes to infinity. In the nD case, we shall define stability with respect to a specified stability region, as in [8], by adapting the ideas in [21] to the discrete case. For this purpose we identify a *direction* in $\mathbb{Z}^n$ with an element $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ whose components are coprime integers, and define a *stability cone* in $\mathbb{Z}^n$ as the set of all positive integer linear combinations of n linearly independent directions.

By a *half-line* associated with a direction $\underline{d} \in \mathbb{Z}^n$ we mean the set of all points of the form $\alpha\underline{d}$ where α is a nonnegative integer; clearly, the half-lines in a stability cone S are the ones associated with the directions $\underline{d} \in S$.

Given a stability cone $S \subset \mathbb{Z}^n$, a trajectory $w \in (\mathbb{C}^q)^{\mathbb{Z}^n}$ is said to be *S-stable* if it tends to zero along every half line in S. A behavior $\mathfrak{B}$ is *S-stable* if all its trajectories are *S-stable*.

It turns out that stable behaviors on $(\mathbb{C}^q)^{\mathbb{Z}^n}$ must be finite dimensional.

Lemma 21 *([8, Lemma 2]). Every nD behavior $\mathfrak{B} \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ which is stable with respect to some stability cone S is a finite-dimensional linear subspace of the trajectory universe, $(\mathbb{C}^q)^{\mathbb{Z}^n}$, i.e., $autodeg(\mathfrak{B}) = n$.*

As for stabilization, our definition of S-stabilizability is similar to the one proposed in [21], but has the extra requirement of regularity.

Definition 22. Given a stability cone $S \subset \mathbb{Z}^n$, we say that a behavior $\mathfrak{B} \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ is *S-stabilizable* if there exists an S-stable sub-behavior $\mathfrak{B}^s \subset \mathfrak{B}$ that is implementable from $\mathfrak{B}$ by regular interconnection.

The following theorem provides a characterization of all stabilizable behaviors.

Theorem 23. *Let $\mathfrak{B} = \ker R(\underline{\sigma}, \underline{\sigma}^{-1}) \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ be a behavior and $S \subset \mathbb{Z}^n$ be a stability cone. Then the following statements are equivalent:*

1. *$\mathfrak{B}$ is S-stabilizable,*
2. *$\mathfrak{B}^c$ is rectifiable and if U is a rectifying operator such that $RU = [P\ 0]$, then $\ker P(\underline{\sigma}, \underline{\sigma}^{-1})$ is S-stable,*
3. *$\mathfrak{B}^c$ is rectifiable and every autonomous direct summand of $\mathfrak{B}$ is stable.*

Proof. $1 \Rightarrow 2$: Assume that $\mathfrak{B}$ is S-stabilizable. Then, by Lemma 21 and Theorem 18, $\mathfrak{B}^c$ is rectifiable. If $\mathfrak{B} = \ker R = \ker PR^c$ with R^c such that $\mathfrak{B}^c = \ker R^c$ and U is a rectifying operator for $\mathfrak{B}^c$ then $PR^c = P(I\ 0)U$, $U(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}) = \ker (P\ 0)$ and $U(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}^c) = \ker (I\ 0)$.

If $\mathcal{K} = \ker (K_1\ K_2)U$ is a controller behavior such that its interconnection with $\mathfrak{B}$ is regular and yields an autonomous behavior then, by Lemma 16,

$$\text{rank} \begin{pmatrix} P & 0 \\ K_1 & K_2 \end{pmatrix} = \text{rank } (P\ 0) + \text{rank } (K_1\ K_2).$$

On the other hand, P must have full column rank (by Remark 3) as well as K_2 (otherwise $U(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B} \cap \mathcal{K}) = \ker \begin{pmatrix} P & 0 \\ K_1 & K_2 \end{pmatrix}$ would not be full columnrank) and therefore we have that

$$\text{rank} \begin{pmatrix} P & 0 \\ K_1 & K_2 \end{pmatrix} = \text{rank } P + \text{rank } K_2.$$

Thus rank $K_2 = \text{rank } (K_1\ K_2)$.

In particular this implies that if $w_1 \in \ker P$, then there exists a trajectory w_2 such that $(w_1, w_2) \in \mathfrak{B} \cap \mathcal{K}$.

In this way, we conclude that if $\mathfrak{B}$ is S-stabilizable, then P must be stable, i.e., $1 \Rightarrow 2$.

$2 \Rightarrow 1$: Take $\mathcal{K}$ such that $U(\underline{\sigma}, \underline{\sigma}^{-1})(\mathcal{K}) = \ker (0\ I)$. Thus $\mathfrak{B} \cap_{reg} \mathcal{K}$ is stable since P is stable.

$2 \Leftrightarrow 3$: Easy from the characterization, obtained in Proposition 4, of all autonomous direct summands of $\mathfrak{B}$. □

7. Conclusions

In this paper, we dealt with behavioral systems described by constant coefficient linear partial difference equations on $\mathbb{Z}^n$. In particular, we have studied in detail autonomous behaviors and their implementation in the context of behavioral control.

By showing that the projection of a behavior on a sub-lattice of $\mathbb{Z}^n$ is again a behavior, we found a natural definition for the autonomous degree of a behavior and proved that the autonomous degree of a behavior over $\mathbb{Z}^n$ coincides with the primeness degree of any of its representation matrices, completing the results in [1].

Next, we investigated the regular implementation of autonomous behaviors from a given behavior $\mathfrak{B}$, proving that a surprisingly strong property on the structure of $\mathfrak{B}$ (namely, the rectifiability of its controllable part) is required in order to regularly implement an autonomous behavior with autonomous degree larger or equal to 2.

Finally, we applied these results to address the problem of stabilization of nD behaviors over $\mathbb{Z}^n$ and provided a full characterization of the stabilizability property, thus completing the results in [8,9].

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