# Dynamic modeling of a Stewart platform using the generalized momentum approach 

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#### Abstract

Dynamic modeling of parallel manipulators presents an inherent complexity, mainly due to system closed-loop structure and kinematic constraints.

In this paper, an approach based on the manipulator generalized momentum is explored and applied to the dynamic modeling of a Stewart platform. The generalized momentum is used to compute the kinetic component of the generalized force acting on each manipulator rigid body. Analytic expressions for the rigid bodies inertia and Coriolis and centripetal terms matrices are obtained, which can be added, as they are expressed in the same frame. Gravitational part of the generalized force is obtained using the manipulator potential energy. The computational load of the dynamic model is evaluated, measured by the number of arithmetic operations involved in the computation of the inertia and Coriolis and centripetal terms matrices. It is shown the model obtained using the proposed approach presents a low computational load. This could be an important advantage if fast simulation or model-based real-time control are envisaged.


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## 1. Introduction

The dynamic model of a parallel manipulator operated in free space can be mathematically represented, in the Cartesian space, by a system of nonlinear differential equations that may be written in matrix form as:

$$
\begin{equation*}
\mathbf{I}(\mathbf{x}) \cdot \ddot{\mathbf{x}}+\mathbf{V}(\mathbf{x}, \dot{\mathbf{x}}) \cdot \dot{\mathbf{x}}+\mathbf{G}(\mathbf{x})=\mathbf{f} \tag{1}
\end{equation*}
$$

$\mathbf{I}(\mathbf{x})$ being the inertia matrix, $\mathbf{V}(\mathbf{x}, \dot{\mathbf{x}})$ the Coriolis and centripetal terms matrix, $\mathbf{G}(\mathbf{x})$ a vector of gravitational generalized forces, $\mathbf{x}$ the generalized position of the moving platform (end-effector) and $\mathbf{f}$ the controlled generalized force applied on the end-effector. Thus,

$$
\begin{equation*}
\mathbf{f}=\mathbf{J}^{T}(\mathbf{x}) \cdot \tau \tag{2}
\end{equation*}
$$

where $\tau$ is the generalized force developed by the actuators and $\mathbf{J}(\mathbf{x})$ is a jacobian matrix.
Generally speaking, the dynamic model can play an important role in system simulation and control. In the first case, the manipulator trajectory (position, velocity and acceleration) is to be computed for the given actuating forces or torques (direct dynamics). In the second case, the actuators forces or torques required to generate a given manipulator trajectory should be calculated (inverse dynamics). Mainly in this case, the efficiency of the computation for the manipulator dynamics is of paramount importance, as manipulator real-time control is usually necessary [1].

[^0]
## Nomenclature

## List of symbols

ã a skew-symmetric matrix representing operator [a×]
$\{\mathrm{B}\},\{\mathrm{P}\},\left\{\mathrm{C}_{i}\right\},\left\{\mathrm{S}_{i}\right\}$ right-handed Cartesian frames attached to base, moving platform, cylinder $i$ and piston $i$ (origin located at centre of mass)
$b_{C} \quad$ distance from the cylinder centre of mass to point $B_{i}$
$\mathbf{b}_{i} \quad$ position of point $B_{i}$ relative to frame $\{\mathrm{B}\}$
$B_{i}, P_{i} \quad$ kinematic chains connecting points, to the base and moving platform
$b_{S} \quad$ distance from the piston centre of mass to point $P_{i}$
f generalized force acting on the end-effector
$\mathbf{f}_{c(\text { kin })} \quad$ kinetic component of the generalized force acting on a body
${ }_{C_{i}} \mathbf{f}_{\left.C_{i}(\text { kin })(\text { tra })\right|_{B}}, ~ S_{i} \mathbf{f}_{\left.S_{i}(\text { kin })(\text { tra })\right|_{B}}$ kinetic component of the generalized force applied to the cylinder, or piston, due to its translation, expressed in frame $\{B\}$
${ }^{P} \mathbf{f}_{\left.C_{i}(\text { kin })(\text { tra })\right|_{B}},{ }^{P} \mathbf{f}_{\left.S_{i}(\text { kin })(\text { tra })\right|_{B}}$ kinetic component of the generalized force applied to $\{\mathrm{P}\}$, due to cylinder, or piston, translation, expressed in frame $\{\mathrm{B}\}$
 pressed in frame $\{B\}$
${ }^{P} \mathbf{f}_{\left.C_{i}(\text { kin })(\text { rot })\right|_{\mathrm{B}}},{ }^{P} \mathbf{f}_{S_{i}\left(\text { kin) (rot) }\left.\right|_{\mathrm{B}}\right.}$ kinetic component of the generalized force applied to $\{\mathrm{P}\}$, due to cylinder, or piston, rotation, expressed in frame $\{\mathrm{B}\}$
$\left.{ }^{P} \mathbf{f}_{P(\text { gra })}\right|_{B E E},{ }^{P} \mathbf{f}_{\left.C_{i}(\text { gra })\right|_{B E}},\left.{ }^{P} \mathbf{f}_{S_{i}(\text { gra }) \mid}\right|_{B E}$ gravitational component of the generalized force acting on $\{\mathrm{P}\}$, due to moving platform,

${ }^{P} \mathbf{F}_{\left.P(\text { kin })\right|_{B}}$ force vector acting on moving platform
$\mathbf{G}(\mathbf{x}) \quad$ vector of gravitational generalized forces
$\mathbf{H}_{c} \quad$ angular momentum about the centre of mass of body $c$
$\mathbf{H}_{\left.P\right|_{B}}, \mathbf{H}_{\left.C_{i}\right|_{B}}, \mathbf{H}_{\left.S_{i}\right|_{B}}$ angular momentum of the moving platform, cylinder and piston, about the centre of mass, written in frame $\{\mathrm{B}\}$
$\mathbf{I}(\mathbf{x}) \quad$ inertia matrix
$\mathbf{I}_{c} \quad$ inertia matrix of a rigid body
$\mathbf{I}_{c(\text { rot })} \quad$ rotational inertia matrix
$\mathbf{I}_{c(t r a)} \quad$ translational inertia matrix
$\mathbf{I}_{P_{I_{B}}} \quad$ moving platform inertia matrix, written in frame $\{\mathrm{B}\}$
$\mathbf{I}_{\left.P(\text { rot })\right|_{B}}, \mathbf{I}_{\left.C_{i}(\text { rot })\right|_{B}}, \mathbf{I}_{\left.S_{i}(\text { rot })\right|_{B}}$ inertia matrix of the rotating moving platform, cylinder, or piston, about frames $\{\mathrm{P}\},\left\{\mathrm{C}_{i}\right\},\left\{\mathrm{S}_{i}\right\}$, expressed in frame $\{B\}$
$\mathbf{I}_{P(t r a)}$ translational inertia matrix of the moving platform
$\mathbf{J}(\mathbf{x}) \quad$ jacobian matrix
$\mathbf{J}_{A} \quad$ jacobian matrix relating Euler angles derivatives and angular velocity
$\mathbf{J}_{B_{i}} \quad$ jacobian matrix relating cylinder linear velocity and moving platform generalized velocity
$\mathbf{J}_{C} \quad$ kinematic jacobian matrix relating active joints velocities and moving platform generalized velocity
$\mathbf{J}_{D_{i}} \quad$ jacobian matrix relating leg angular velocity and moving platform generalized velocity
$\mathbf{J}_{E} \quad$ Euler angles jacobian matrix, relating active joints velocities and moving platform generalized velocity (linear velocity and Euler angles derivatives)
$\mathbf{J}_{G_{i}} \quad$ jacobian matrix relating piston linear velocity and moving platform generalized velocity
$\mathbf{l}=\left[\begin{array}{llll}l_{1} & l_{2} & \ldots & l_{6}\end{array}\right]^{T}, \quad \mathbf{i}$ position and velocity of the active joints
$l_{i} \quad$ coordinate (length) of the prismatic joint $i$
$\hat{\mathbf{1}}_{i} \quad$ versor of vector $\mathbf{1}_{i}$
$m_{P}, m_{C}, m_{S}$ mass of the moving platform, cylinder and piston
${ }^{P} \mathbf{M}_{\left.P(\text { kin })\right|_{B}}$ moment about the origin of $\{\mathrm{P}\}$ acting on the moving platform, expressed in frame $\{\mathrm{B}\}$
$P_{c} \quad$ potential energy of a rigid body
${ }^{B}{ }_{P}^{B} \mathbf{p}_{\left.C_{i}\right|_{B}},{ }^{B} \mathbf{p}_{\left.S_{i}\right|_{B}}$ position of the cylinder, or piston, centre of mass, relative to frame $\{\mathrm{B}\}$
${ }^{P} \mathbf{p}_{\left.i\right|_{B}} \quad$ position of point $P_{i}$ relative to frame $\{\mathrm{B}\}$
$\mathbf{q}_{c} \quad$ generalized momentum of a rigid body
$\mathbf{Q}_{c} \quad$ linear momentum
$\mathbf{q}_{\left.P\right|_{B}} \quad$ generalized momentum of the moving platform, expressed in frame $\{\mathrm{B}\}$
$\mathbf{Q}_{\left.P\right|_{B}}, \mathbf{Q}_{\left.C_{i}\right|_{B}}, \mathbf{Q}_{\left.S_{i}\right|_{B}}$ linear momentum of the moving platform, cylinder and piston, written in frame $\{\mathrm{B}\}$
$r_{B}, r_{P} \quad$ base and moving platform radius
${ }^{B} \mathbf{R}_{P},{ }^{B} \mathbf{R}_{C_{i}}$ rotation matrix of frame $\{\mathrm{P}\}$, or frame $\left\{\mathrm{C}_{i}\right\}$, relative to frame $\{\mathrm{B}\}$
$\mathbf{T} \quad$ matrix transformation defined by $\mathbf{T}=\operatorname{diag}\left(\left[\begin{array}{ll}\mathfrak{J} & \mathbf{J}_{A}\end{array}\right]\right)$
$\mathbf{u}_{c} \quad$ generalized velocity (linear and angular) of a rigid body
$\mathbf{V}(\mathbf{x}, \dot{\mathbf{x}}) \quad$ Coriolis and centripetal terms matrix
$\mathbf{v}_{c}, \omega_{c} \quad$ rigid body linear and angular velocities
$\mathbf{x} \quad$ generalized position (position and orientation)
${ }^{B} \mathbf{X}_{\left.P\right|_{\mathrm{BIE}}}=\left[{ }^{B} \mathbf{X}_{\left.P(p o s)\right|_{B}}^{T}{ }^{B} \mathbf{X}_{\left.P(o)\right|_{E}}^{T}\right]^{T}$ generalized position of frame $\{\mathrm{P}\}$ relative to frame $\{\mathrm{B}\}$ (orientation is given by the Euler angles system)
${ }^{B} \mathbf{X}_{\left.P(\text { pos })\right|_{B}}=\left[\begin{array}{lll}x_{P} & y_{P} & z_{P}\end{array}\right]^{T}$ position of the origin of frame $\{\mathrm{P}\}$ relative to frame $\{\mathrm{B}\}$ (expressed in frame $\{\mathrm{B}\}$ )
${ }^{B} \mathbf{X}_{\left.P(0)\right|_{E}}=\left[\begin{array}{lll}\psi_{P} & \theta_{P} & \varphi_{P}\end{array}\right]^{T} \quad$ Euler angles system representing orientation of frame $\{\mathrm{P}\}$ relative to $\{\mathrm{B}\}$
${ }^{B} \dot{\mathbf{x}}_{\left.P\right|_{B E E}}=\left[\begin{array}{ll}{ }^{B} \dot{\mathbf{x}}_{P(\text { pos }) \mid \mathrm{IB}}^{T} & { }^{B} \dot{\mathbf{x}}_{P(o) \mid E E}^{T}\end{array}\right]^{T}$ generalized velocity of the moving platform, expressed in the Euler angles system (linear
velocity and Euler angles derivatives)
${ }^{B}{ }_{B}^{B} \dot{\mathbf{x}}_{P_{\mid B}}=\left[{ }^{B} \dot{\mathbf{x}}_{\left.P(p o s)\right|_{B}}^{T}{ }^{B}{ }^{B} \omega_{\left.P\right|_{B}}^{T}\right]^{T}$ generalized velocity of the moving platform, expressed in frame $\{\mathrm{B}\}$ (linear and angular velocities)
${ }^{B} \dot{\mathbf{x}}_{\left.P(\text { pos })\right|_{B}} \equiv{ }^{B} \mathbf{v}_{\left.P\right|_{B}}$ linear velocity of moving platform relative to frame $\{\mathrm{B}\}$
${ }^{B} \dot{\mathbf{x}}_{\left.P(o)\right|_{E}} \quad$ Euler angles time derivatives
$\phi_{B}, \phi_{P}$ separation half-angles between kinematic chains connecting points, at base and moving platform
$\tau_{P(k i n)}$ actuating forces corresponding to
$\tau \quad$ forces developed by the actuators
${ }^{B} \omega_{\left.P\right|_{B}},{ }^{B} \omega_{\left.l_{i}\right|_{B}}$ angular velocity of the moving platform, or leg (cylinder or piston), relative to frame $\{B\}$

Dynamic modeling of parallel manipulators presents an inherent complexity, mainly due to system closed-loop structure and kinematic constraints. Several approaches have been applied to the dynamic analysis of parallel manipulators, the New-ton-Euler and the Lagrange methods being the most popular ones. The Newton-Euler approach uses the free body diagrams of the rigid bodies. Do and Yang [2] and Reboulet and Berthomieu [3] use this method on the dynamic modeling of a Stewart platform. Ji [4] presents a study on the influence of leg inertia on the dynamic model of a Stewart platform. Dasgupta and Mruthyunjaya [5] used the Newton-Euler approach to develop a closed-form dynamic model of the Stewart platform. This method was also used by Khalil and Ibrahim [6], Riebe and Ulbrich [7], and Guo and Li [8], among others.

The Lagrange method describes the dynamics of a mechanical system from the concepts of work and energy. Nguyen and Pooran [9] use this method to model a Stewart platform, modeling the legs as point masses. Lebret et al. [10] follow an approach similar to the one used by Nguyen and Pooran [9]. Lagrange's method was also used by Gregório and Parenti-Castelli [11] and Caccavale et al. [12], for example.

These methods use classical mechanics principles, as is the case for all the approaches found in the literature, namely the ones based on the principle of virtual work [13,14], screw theory [15], recursive matrix method [16], and Hamilton's principle [17]. Thus, all approaches are equivalent as they are describing the same physical system [18]. All methods lead to equivalent dynamic equations, although these equations present different levels of complexity and associated computational loads [1]. Minimize the number of operations involved in the computation of the manipulator dynamic model has been the main goal of recent proposed techniques [1,16,19-21].

In this paper, the author presents a new approach to the dynamic modeling of a six degrees-of-freedom (dof) Stewart platform: the use of the generalized momentum concept [22]. This method is used to compute the kinetic component of the generalized force acting on each manipulator rigid body. Analytic expressions for the rigid bodies inertia and Coriolis


Fig. 1. Stewart platform kinematic structure.
and centripetal terms matrices are obtained, which can be added, as they can be expressed in the same frame. Gravitational part of the generalized force is obtained using the manipulator potential energy. The computational load of the dynamic model is evaluated, measured by the number of arithmetic operations involved in the computation of the inertia and Coriolis and centripetal terms matrices. It is shown the model obtained using the proposed approach presents a low computational load. This could be an important advantage if fast simulation or model-based real-time control are envisaged.

The paper is organized as follows. Section 2 introduces the Stewart platform parallel manipulator. Section 3 presents the manipulator dynamic model using the generalized momentum approach. In Section 4, the computational effort of the dynamic model is evaluated. A numerical simulation of the manipulator inverse dynamics is presented in Section 5. Conclusions are drawn in Section 6.

## 2. Stewart platform kinematic structure

A Stewart platform comprises a fixed platform (base) and a moving (payload) platform, linked together by six independent, identical, open kinematic chains (Fig. 1). Each chain (leg) comprises a cylinder and a piston (or spindle) that are connected together by a prismatic joint, $l_{i}$. The upper end of each leg is connected to the moving platform by a spherical joint whereas the lower end is connected to the fixed base by a universal joint. Points $B_{i}$ and $P_{i}$ are the connecting points to the base and moving platforms, respectively. They are located at the vertices of two semi-regular hexagons inscribed in circumferences of radius $r_{B}$ and $r_{P}$. The separation angles between points $B_{1}$ and $B_{6}, B_{2}$ and $B_{3}$, and $B_{4}$ and $B_{5}$ are denoted by $2 \phi_{B}$. In a similar way, the separation angles between points $P_{1}$ and $P_{2}, P_{3}$ and $P_{4}$, and $P_{5}$ and $P_{6}$ are denoted by $2 \phi_{P}$.

For kinematic modeling purposes, two frames, $\{P\}$ and $\{B\}$, are attached to the moving and base platforms, respectively. Its origins are the platforms centres of mass. The generalized position of frame $\{P\}$ relative to frame $\{B\}$ may be represented by the vector:

$$
{ }^{B} \mathbf{x}_{P|B| E}=\left[\begin{array}{llllll}
x_{P} & y_{P} & z_{P} & \psi_{P} & \theta_{P} & \varphi_{P}
\end{array}\right]^{T}=\left[\begin{array}{ll}
{ }^{B} \mathbf{X}_{P(p o s)| |_{B}}^{T} & { }^{B} \mathbf{X}_{P(0) \mid E}^{T} \tag{3}
\end{array}\right]^{T}
$$

where ${ }^{B} \mathbf{X}_{\left.P(p o s)\right|_{B}}=\left[\begin{array}{lll}x_{P} & y_{P} & z_{P}\end{array}\right]^{T}$ is the position of the origin of frame $\{\mathrm{P}\}$ relative to frame $\{\mathrm{B}\}$, and ${ }^{B} \mathbf{X}_{\left.P(o)\right|_{E}}=\left[\begin{array}{lll}\psi_{P} & \theta_{P} & \varphi_{P}\end{array}\right]^{T}$ defines an Euler angles system representing orientation of frame $\{\mathrm{P}\}$ relative to $\{\mathrm{B}\}$. The used Euler angles system corresponds to the basic rotations [23]: $\psi_{P}$ about $\mathbf{z}_{P} ; \theta_{P}$ about the rotated axis $\mathbf{y}_{P^{\prime}}$; and $\varphi_{P}$ about the rotated axis $\mathbf{x}_{P^{\prime \prime}}$. The rotation matrix is given by:

$$
{ }^{B} \mathbf{R}_{P}=\left[\begin{array}{ccc}
C \psi_{P} C \theta_{P} & C \psi_{P} S \theta_{P} S \varphi_{P}-S \psi_{P} C \varphi_{P} & C \psi_{P} S \theta_{P} C \varphi_{P}+S \psi_{P} S \varphi_{P}  \tag{4}\\
S \psi_{P} C \theta_{P} & S \psi_{P} S \theta_{P} S \varphi_{P}+C \psi_{P} C \varphi_{P} & S \psi_{P} S \theta_{P} C \varphi_{P}-C \psi_{P} S \varphi_{P} \\
-S \theta_{P} & C \theta_{P} S \varphi_{P} & C \theta_{P} C \varphi_{P}
\end{array}\right]
$$

$S(\cdot)$ and $C(\cdot)$ correspond to the sine and cosine functions, respectively.
The manipulator position and velocity kinematic models are known, being obtainable from the geometrical analysis of the kinematics chains. The velocity kinematics is represented by the Euler angles jacobian matrix, $\mathbf{J}_{E}$, or the kinematic jacobian, $\mathbf{J}_{c}$. These jacobians relate the velocities of the active joints (actuators) to the generalized velocity of the moving platform:

$$
\begin{align*}
& \dot{\mathbf{i}}=\mathbf{J}_{E} \cdot{ }^{B} \dot{\mathbf{X}}_{P_{|B| E}}=\mathbf{J}_{E} \cdot\left[\begin{array}{c}
{ }^{B} \dot{\mathbf{x}}_{\left.P(p o s)\right|_{B}} \\
{ }^{B} \dot{\mathbf{X}}_{\left.P(o)\right|_{E}}
\end{array}\right]  \tag{5}\\
& \dot{\mathbf{i}}=\mathbf{J}_{C} \cdot{ }^{B} \dot{\mathbf{X}}_{P_{\left.\right|_{B}}}=\mathbf{J}_{C} \cdot\left[\begin{array}{c}
{ }^{B} \dot{\mathbf{x}}_{P(\text { pos }) \mid B} \\
{ }^{B} \omega_{\left.P\right|_{B}}
\end{array}\right] \tag{6}
\end{align*}
$$

with

$$
\begin{align*}
& \dot{\mathbf{i}}=\left[\begin{array}{llll}
\dot{l}_{1} & \dot{l}_{2} & \ldots & \dot{l}_{6}
\end{array}\right]^{T}  \tag{7}\\
& { }^{B} \boldsymbol{\omega}_{\left.P\right|_{B}}=\mathbf{J}_{A} \cdot{ }^{B} \dot{\mathbf{x}}_{\left.P(o)\right|_{E}} \tag{8}
\end{align*}
$$

and [23]

$$
\mathbf{J}_{A}=\left[\begin{array}{ccc}
0 & -S \psi_{P} & C \theta_{P} C \psi_{P}  \tag{9}\\
0 & C \psi_{P} & C \theta_{P} S \psi_{P} \\
1 & 0 & -S \theta_{P}
\end{array}\right]
$$

Vectors ${ }^{B} \dot{\mathbf{X}}_{P(\text { pos })_{B}} \equiv{ }^{B} \mathbf{v}_{\left.P\right|_{B}}$ and ${ }^{B} \omega_{P_{\left.\right|_{B}}}$ represent the linear and angular velocity of the moving platform relative to $\{\mathrm{B}\}$, and ${ }^{B} \dot{\mathbf{x}}_{\left.P(o)\right|_{E}}$ represents the Euler angles time derivative.

## 3. Dynamic modeling using the generalized momentum approach

The generalized momentum of a rigid body, $\mathbf{q}_{c}$, may be obtained using the following general expression:

$$
\begin{equation*}
\mathbf{q}_{c}=\mathbf{I}_{c} \cdot \mathbf{u}_{c} \tag{10}
\end{equation*}
$$

Vector $\mathbf{u}_{c}$ represents the generalized velocity (linear and angular) of the body and $\mathbf{I}_{c}$ is its inertia matrix. Vectors $\mathbf{q}_{c}$ and $\mathbf{u}_{c}$, and inertia matrix $\mathbf{I}_{c}$ must be expressed in the same referential.

Eq. (10) may also be written as:

$$
\mathbf{q}_{c}=\left[\begin{array}{l}
\mathbf{Q}_{c}  \tag{11}\\
\mathbf{H}_{c}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{I}_{c(t r a)} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_{c(r o t)}
\end{array}\right] \cdot\left[\begin{array}{l}
\mathbf{v}_{c} \\
\boldsymbol{\omega}_{c}
\end{array}\right]
$$

where $\mathbf{Q}_{c}$ is the linear momentum vector due to rigid body translation, and $\mathbf{H}_{c}$ is the angular momentum vector due to body rotation. $\mathbf{I}_{c(t r a)}$ is the translational inertia matrix, and $\mathbf{I}_{c(\text { rot })}$ the rotational inertia matrix. $\mathbf{v}_{c}$ and $\omega_{c}$ are the body linear and angular velocities. The kinetic component of the generalized force acting on the body can be computed from the time derivative of Eq. (10):

$$
\begin{equation*}
\mathbf{f}_{c(k i n)}=\dot{\mathbf{q}}_{c}=\dot{\mathbf{I}}_{c} \cdot \mathbf{u}_{c}+\mathbf{I}_{c} \cdot \dot{\mathbf{u}}_{c} \tag{12}
\end{equation*}
$$

with force and momentum expressed in the same frame.

### 3.1. Moving platform modeling

The linear momentum of the moving platform, written in frame $\{B\}$, may be obtained from the following expression:

$$
\begin{equation*}
\mathbf{Q}_{\left.P\right|_{B}}=m_{P} \cdot{ }^{B} \mathbf{v}_{\left.P\right|_{B}}=\mathbf{I}_{P(\text { tra })} \cdot{ }^{B} \mathbf{v}_{\left.P\right|_{B}} \tag{13}
\end{equation*}
$$

$\mathbf{I}_{P(t r a)}$ is the translational inertia matrix of the moving platform,

$$
\mathbf{I}_{P(t r a)}=\operatorname{diag}\left(\left[\begin{array}{lll}
m_{P} & m_{P} & m_{P} \tag{14}
\end{array}\right]\right)
$$

$m_{P}$ being its mass.
The angular momentum about the mobile platform centre of mass, written in frame $\{B\}$, is:

$$
\begin{equation*}
\mathbf{H}_{\left.P\right|_{B}}=\mathbf{I}_{\left.P(\text { rot })\right|_{B}} \cdot{ }^{B} \omega_{\left.P\right|_{B}} \tag{15}
\end{equation*}
$$

$\mathbf{I}_{\left.P(\text { rot })\right|_{B}}$ represents the rotational inertia matrix of the moving platform, expressed in the base frame $\{\mathrm{B}\}$.
The inertia matrix of a rigid body is constant when expressed in a frame that is fixed relative to that body. Furthermore if the frame axes coincide with the principal directions of inertia of the body, then all inertia products are zero and the inertia matrix is diagonal. Therefore, the rotational inertia matrix of the moving platform, when expressed in frame $\{\mathrm{P}\}$, may be written as:

$$
\mathbf{I}_{\left.P(\text { rot })\right|_{p}}=\operatorname{diag}\left(\left[\begin{array}{lll}
I_{P_{x x}} & I_{P_{y y}} & I_{P_{z z}} \tag{16}
\end{array}\right]\right)
$$

This inertia matrix can be written in frame $\{\mathrm{B}\}$ using the following transformation [24]:

$$
\begin{equation*}
\mathbf{I}_{\left.P(\text { rot })\right|_{B}}={ }^{B} \mathbf{R}_{P} \cdot \mathbf{I}_{\left.P(\text { rot })\right|_{P}} \cdot{ }^{B} \mathbf{R}_{P}^{T} \tag{17}
\end{equation*}
$$

The generalized momentum of the moving platform about its centre of mass, expressed in frame $\{B\}$, can be obtained from the simultaneous use of Eqs. (13) and (15):

$$
\mathbf{q}_{\left.P\right|_{B}}=\left[\begin{array}{cc}
\mathbf{I}_{P(\text { tra })} & \mathbf{0}  \tag{18}\\
\mathbf{0} & \mathbf{I}_{\left.P(\text { rot })\right|_{B}}
\end{array}\right] \cdot\left[\begin{array}{l}
{ }^{B} \mathbf{v}_{P_{\mid B}} \\
{ }^{B} \boldsymbol{\omega}_{\left.P\right|_{B}}
\end{array}\right]
$$

where

$$
\mathbf{I}_{P_{\left.\right|_{B}}}=\left[\begin{array}{cc}
\mathbf{I}_{P(\text { tra })} & \mathbf{0}  \tag{19}\\
\mathbf{0} & \mathbf{I}_{\left.P(\text { rot })\right|_{B}}
\end{array}\right]
$$

is the moving platform inertia matrix written in the base frame $\{B\}$.


Fig. 2. Position of the centre of mass of the cylinder $i$.

The combination of Eqs. (8) and (15) results into:

$$
\begin{equation*}
\mathbf{H}_{\left.P\right|_{B}}=\mathbf{I}_{\left.P(\text { rot })\right|_{B}} \cdot \mathbf{J}_{A} \cdot{ }^{B} \dot{\mathbf{x}}_{\left.P(o)\right|_{E}} \tag{20}
\end{equation*}
$$

Accordingly, Eq. (18) may be rewritten as:

$$
\begin{align*}
& \mathbf{q}_{P_{\left.\right|_{B}}}=\left[\begin{array}{cc}
\mathbf{I}_{P(\text { tra })} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_{\left.P(\text { rot })\right|_{B}}
\end{array}\right] \cdot\left[\begin{array}{cc}
\mathfrak{J} & 0 \\
0 & \mathbf{J}_{A}
\end{array}\right] \cdot\left[\begin{array}{c}
{ }^{B} \mathbf{v}_{\left.P\right|_{B}} \\
{ }^{{ }^{\dot{\mathbf{X}}}}{ }_{\left.P(o)\right|_{E}}
\end{array}\right]  \tag{21}\\
& \mathbf{q}_{\left.P\right|_{B}}=\mathbf{I}_{P_{\left.\right|_{B}}} \cdot \mathbf{T} \cdot{ }^{B} \dot{\mathbf{X}}_{\left.P\right|_{B \mid E}} \tag{22}
\end{align*}
$$

T being a matrix transformation defined by:

$$
\mathbf{T}=\left[\begin{array}{ll}
\mathfrak{J} & \mathbf{0}  \tag{23}\\
\mathbf{0} & \mathbf{J}_{A}
\end{array}\right]
$$

The time derivative of Eq. (22) results into:

$$
\begin{equation*}
{ }^{P} \mathbf{f}_{\left.P(k i n)\right|_{B}}=\dot{\mathbf{q}}_{\left.P\right|_{B}}=\frac{d}{d t}\left(\mathbf{I}_{\left.\right|_{\mid B}} \cdot \mathbf{T}\right) \cdot{ }^{B} \dot{\mathbf{x}}_{\left.\right|_{|B| E}}+\mathbf{I}_{\left.P\right|_{B}} \cdot \mathbf{T} \cdot{ }^{B} \ddot{\mathbf{X}}_{\left.P\right|_{B \mid E}} \tag{24}
\end{equation*}
$$

${ }^{P} \mathbf{f}_{P(\text { kin })_{B}}$ is the kinetic component of the generalized force acting on $\{\mathrm{P}\}$ due to the moving platform motion, expressed in frame $\{\mathrm{B}\}$. The corresponding actuating forces, $\tau_{P(\text { kin })}$, may be computed from the following relation:

$$
\begin{equation*}
\tau_{P(k i n)}=\mathbf{J}_{C}^{-T} \cdot{ }^{P} \mathbf{f}_{\left.P(k i n)\right|_{B}} \tag{25}
\end{equation*}
$$

where

$$
{ }^{P} \mathbf{f}_{\left.P(k i n)\right|_{B}}=\left[\begin{array}{ll}
{ }^{P} \mathbf{F}_{\left.P(k i n)\right|_{B}}^{T} & { }^{P} \mathbf{M}_{\left.P(k i n)\right|_{B}}^{T} \tag{26}
\end{array}\right]^{T}
$$

Vector ${ }^{P} \mathbf{F}_{P\left(\text { kin }\left.\right|_{B}\right.}$ represents the force vector acting on the centre of mass of the moving platform, and ${ }^{P} \mathbf{M}_{P(\text { kin }) \mid B}$ represents the moment vector acting on the moving platform, expressed in the base frame, $\{\mathrm{B}\}$.

From Eq. (24) it can be concluded that two matrices playing the roles of the inertia matrix and the Coriolis and centripetal terms matrix are:

$$
\begin{align*}
& \mathbf{I}_{\left.P\right|_{B}} \cdot \mathbf{T}  \tag{27}\\
& \frac{d}{d t}\left(\mathbf{I}_{\left.P\right|_{B}} \cdot \mathbf{T}\right) \tag{28}
\end{align*}
$$

It must be emphasized that these matrices do not have the properties of inertia or Coriolis and centripetal terms matrices and therefore should not, strictly, be named as such. Nevertheless, throughout the paper the names "inertia matrix" and "Coriolis and centripetal terms matrix" may be used if there is no risk of misunderstanding.

### 3.2. Cylinder modeling

If the centre of mass of each cylinder is located at a constant distance, $b_{C}$, from the cylinder to base platform connecting point, $B_{i}$, (Fig. 2), then its position relative to frame $\{\mathrm{B}\}$ is:

$$
\begin{equation*}
{ }^{B} \mathbf{p}_{\left.c_{i}\right|_{B}}=b_{C} \cdot \hat{\mathbf{l}}_{i}+\mathbf{b}_{i} \tag{29}
\end{equation*}
$$

where,

$$
\begin{align*}
& \hat{\mathbf{l}}_{i}=\frac{\mathbf{l}_{i}}{\left\|\mathbf{l}_{i}\right\|}=\frac{\mathbf{l}_{i}}{l_{i}}  \tag{30}\\
& \mathbf{l}_{i}={ }^{B} \mathbf{x}_{\left.P(\text { pos })\right|_{B}}+{ }^{P} \mathbf{p}_{\left.i\right|_{B}}-\mathbf{b}_{i} \tag{31}
\end{align*}
$$

The linear velocity of the cylinder centre of mass, ${ }^{B} \dot{\mathbf{p}}_{\mathrm{C}_{i} \mid \mathrm{B}}$, relative to $\{\mathrm{B}\}$ and expressed in the same frame, may be computed as:

$$
\begin{equation*}
{ }^{B} \dot{\mathbf{p}}_{\left.c_{i}\right|_{B}}={ }^{B} \omega_{\left.l_{i}\right|_{B}} \times b_{C} \cdot \hat{\mathbf{l}}_{i} \tag{32}
\end{equation*}
$$

where ${ }^{B} \omega_{\left.l_{i}\right|_{B}}$ represents the leg angular velocity, which can be found from:

$$
\begin{equation*}
{ }^{B} \boldsymbol{\omega}_{\left.l_{i}\right|_{B}} \times \mathbf{l}_{i}={ }^{B} \mathbf{v}_{\left.P\right|_{B}}+{ }^{B} \boldsymbol{\omega}_{\left.P\right|_{B}} \times{ }^{P} \mathbf{p}_{\left.i\right|_{B}} \tag{33}
\end{equation*}
$$

As the leg (both the cylinder and piston) cannot rotate along its own axis, the angular velocity along $\hat{\mathbf{l}}_{i}$ is always zero, and vectors $\mathbf{I}_{i}$ and ${ }^{B} \omega_{\left.l_{i}\right|_{B}}$ are always perpendicular. This enables Eq. (33) to be rewritten as:

$$
\begin{equation*}
{ }^{B} \boldsymbol{\omega}_{l_{i \mid B}}=\frac{1}{\mathbf{1}_{i}^{T} \cdot \mathbf{1}_{i}} \cdot\left[\mathbf{l}_{i} \times\left({ }^{B} \mathbf{v}_{P_{\left.\right|_{B}}}+{ }^{B} \boldsymbol{\omega}_{\left.P\right|_{B}} \times{ }^{P} \mathbf{p}_{i_{\mid B}}\right)\right] \tag{34}
\end{equation*}
$$

or,

$$
{ }^{B} \boldsymbol{\omega}_{l_{i \mid B}}=\mathbf{J}_{D_{i}} \cdot\left[\begin{array}{c}
{ }^{B} \mathbf{v}_{P_{l_{B}}}  \tag{35}\\
{ }^{B} \omega_{P_{l_{B}}}
\end{array}\right]
$$

where jacobian $\mathbf{J}_{D_{i}}$ is given by:

$$
\begin{align*}
& \mathbf{J}_{D_{i}}=\left[\tilde{\overline{\mathbf{l}}}_{i} \tilde{\overline{\mathbf{I}}} \cdot_{i} \cdot \tilde{\mathbf{p}}_{i \mid \mathrm{l}}^{T}\right]  \tag{36}\\
& \overline{\mathbf{1}}_{i}=\frac{\mathbf{l}_{i}}{\mathbf{1}_{i}^{T} \cdot \mathbf{l}_{i}} \tag{37}
\end{align*}
$$

and, for a given vector $\mathbf{a}=\left[\begin{array}{lll}a_{x} & a_{y} & a_{z}\end{array}\right]^{T}$,

$$
\tilde{\mathbf{a}}=\left[\begin{array}{ccc}
0 & -a_{z} & a_{y}  \tag{38}\\
a_{z} & 0 & -a_{x} \\
-a_{y} & a_{x} & 0
\end{array}\right]
$$

On the other hand, Eq. (32) can be rewritten as:

$$
{ }^{B} \dot{\mathbf{p}}_{\mathrm{C}_{i} \mid B}=\mathbf{J}_{B_{i}} \cdot\left[\begin{array}{c}
{ }^{B} \mathbf{v}_{P_{\mid B}}  \tag{39}\\
{ }^{B} \omega_{P_{l_{B}}}
\end{array}\right]
$$

where the jacobian $\mathbf{J}_{B_{i}}$ is given by:

$$
\mathbf{J}_{B_{i}}=\left[\begin{array}{lll}
b_{c} \cdot \tilde{\mathbf{i}}_{i}^{T} \cdot \tilde{\overline{\mathbf{I}}}_{i} & b_{c} \cdot \tilde{\mathbf{i}}_{i}^{T} \cdot \tilde{\overline{\mathbf{i}}}_{i} \cdot{ }^{P} \tilde{\mathbf{p}}_{i_{1}}^{T} \tag{40}
\end{array}\right]
$$

The linear momentum of each cylinder, $\mathbf{Q}_{c_{i \mid B}}$, can be represented in frame $\{B\}$ as:

$$
\begin{equation*}
\mathbf{Q}_{C_{i \mid B}}=m_{C} \cdot{ }^{B} \dot{\mathbf{p}}_{C_{i l B}} \tag{41}
\end{equation*}
$$

where $m_{C}$ is the cylinder mass.
Introducing jacobian $\mathbf{J}_{B_{i}}$ and matrix transformation $\mathbf{T}$ in the previous equation results into:

$$
\begin{equation*}
\mathbf{Q}_{C_{i} \mid B}=m_{C} \cdot \mathbf{J}_{B_{i}} \cdot \mathbf{T} \cdot{ }^{B} \dot{\mathbf{x}}_{P_{\mid B E E}} \tag{42}
\end{equation*}
$$

The kinetic component of the force applied to the cylinder due to its translation and expressed in $\{\mathrm{B}\}$ can be obtained from the time derivative of Eq. (42):

$$
\begin{equation*}
c_{i_{\mathbf{f}_{C_{i}(\text { kin })(\text { tra })_{B}}}=\dot{\mathbf{Q}}_{\left.C_{i}\right|_{B}}=m_{C} \cdot \frac{d}{d t}\left(\mathbf{J}_{B_{i}} \cdot \mathbf{T}\right) \cdot{ }^{B} \dot{\mathbf{x}}_{P_{P_{B E E}}}+m_{C} \cdot \mathbf{J}_{B_{i}} \cdot \mathbf{T} \cdot{ }^{B} \ddot{\mathbf{x}}_{P_{\text {lEE }}}} \tag{43}
\end{equation*}
$$

When Eq. (43) is multiplied by $\mathbf{J}_{B_{i}}^{T}$, the kinetic component of the force applied to \{P\} due to each cylinder translation is obtained in frame $\{\mathrm{B}\}$ :

$$
\begin{align*}
&{ }^{P} \mathbf{f}_{\left.C_{i}(\text { kin })(t r a)\right|_{B}}=\mathbf{J}_{B_{i}}^{T} \cdot c_{\mathbf{c}_{\mathbf{f}_{C_{i}(\text { kin })(t r a) \mid B}}} \\
&=m_{C} \cdot \mathbf{J}_{B_{i}}^{T} \cdot \frac{d}{d t}\left(\mathbf{J}_{B_{i}} \cdot \mathbf{T}\right) \cdot{ }^{B} \dot{\mathbf{x}}_{P_{\mid B E}}+m_{C} \cdot \mathbf{J}_{B_{i}}^{T} \cdot \mathbf{J}_{B_{i}} \cdot \mathbf{T} \cdot{ }^{B} \ddot{\mathbf{x}}_{P_{\mid B E}} \tag{44}
\end{align*}
$$



Fig. 3. Position of the centre of mass of the piston $i$.

The inertia matrix and the Coriolis and centripetal terms matrix of the translating cylinder being:

$$
\begin{align*}
& m_{C} \cdot \mathbf{J}_{B_{i}}^{T} \cdot \mathbf{J}_{B_{i}} \cdot \mathbf{T}  \tag{45}\\
& m_{C} \cdot \mathbf{J}_{B_{i}}^{T} \cdot \frac{d}{d t}\left(\mathbf{J}_{B_{i}} \cdot \mathbf{T}\right) \tag{46}
\end{align*}
$$

These matrices represent the inertia matrix and the Coriolis and centripetal terms matrix of a virtual moving platform that is equivalent to each translating cylinder.

On the other hand, the angular momentum of each cylinder about its centre of mass can be represented in frame \{B\} by:

$$
\begin{equation*}
\mathbf{H}_{\left.C_{i}\right|_{B}}=\mathbf{I}_{\left.C_{i}(r o t)\right|_{B}} \cdot{ }^{B} \omega_{\left.l_{i}\right|_{B}} \tag{47}
\end{equation*}
$$

It is convenient to express the inertia matrix of the rotating cylinder in a frame fixed to the cylinder itself, $\left\{\mathrm{C}_{i}\right\} \equiv\left\{\mathbf{x}_{C_{i}}, \mathbf{y}_{C_{i}}, \mathbf{z}_{C_{i}}\right\}$. So,

$$
\begin{equation*}
\mathbf{I}_{\left.C_{i}(\text { rot })\right|_{B}}={ }^{B} \mathbf{R}_{C_{i}} \cdot \mathbf{I}_{C_{i}(\text { rot }) \mid c_{i}} \cdot{ }^{B} \mathbf{R}_{C_{i}}^{T} \tag{48}
\end{equation*}
$$

where ${ }^{B} \mathbf{R}_{C_{i}}$ is the orientation matrix of each cylinder frame, $\left\{C_{i}\right\}$, relative to base frame, $\{B\}$.
Cylinder frames were chosen in the following way: axis $\mathbf{x}_{C_{i}}$ coincides with the leg axis and points towards $P_{i}$, axis $\mathbf{y}_{C_{i}}$ is perpendicular to $\mathbf{x}_{c_{i}}$ and always parallel to the base plane, this condition being possible given the existence of a universal joint at $B_{i}$, that negates any rotation along its own axis; axis $\mathbf{z}_{C_{i}}$ completes the referential following the right hand rule, and its projection along axis $\mathbf{z}_{B}$ is always positive. Frame origin is located at the cylinder centre of mass. Thus, matrix ${ }^{B} \mathbf{R}_{C_{i}}$ becomes:

$$
{ }^{B} \mathbf{R}_{C_{i}}=\left[\begin{array}{lll}
\mathbf{x}_{C_{i}} & \mathbf{y}_{c_{i}} & \mathbf{z}_{C_{i}} \tag{49}
\end{array}\right]
$$

where

$$
\begin{align*}
& \mathbf{x}_{C_{i}}=\hat{\mathbf{l}}_{i}  \tag{50}\\
& \mathbf{y}_{C_{i}}=\left[\begin{array}{lll}
-\frac{l_{i y}}{\sqrt{l_{i x}^{2}+l_{i y}^{2}}} & \frac{l_{i x}}{\sqrt{l_{i x}^{2}+l_{i y}^{2}}} & 0
\end{array}\right]^{T}  \tag{51}\\
& \mathbf{z}_{C_{i}}=\mathbf{x}_{C_{i}} \times \mathbf{y}_{C_{i}} \tag{52}
\end{align*}
$$

So, the inertia matrices of the cylinders can be written as:

$$
\mathbf{I}_{C_{i}(r o t) \mid c_{c_{i}}}=\operatorname{diag}\left(\left[\begin{array}{lll}
I_{C_{x x}} & I_{C_{y y}} & I_{C_{z z}} \tag{53}
\end{array}\right]\right)
$$

where $I_{C_{x x}}, I_{C_{y y}}$ and $I_{C_{z z}}$ are the cylinders moments of inertia expressed in its own frame.
Introducing jacobian $\mathbf{J}_{D_{i}}$ and matrix transformation $\mathbf{T}$ in Eq. (47) results into:

$$
\begin{equation*}
\mathbf{H}_{\left.C_{i}\right|_{B}}=\mathbf{I}_{\left.C_{i}(r o t)\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T} \cdot{ }^{B} \dot{\mathbf{X}}_{P_{|B| E}} \tag{54}
\end{equation*}
$$

The kinetic component of the generalized force applied to the cylinder, due to its rotation and expressed in $\{\mathrm{B}\}$ can be obtained from the time derivative of Eq. (54):

$$
\begin{align*}
{ }^{c_{i}} \mathbf{f}_{\left.C_{i}(\text { kin })(\text { rot })\right|_{B}}= & \dot{\mathbf{H}}_{\left.C_{i}\right|_{B}} \\
& =\frac{d}{d t}\left(\mathbf{I}_{\left.C_{i}(\text { rot })\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T}\right) \cdot{ }^{B} \dot{\mathbf{X}}_{\left.\right|_{\mid B E E}}+\mathbf{I}_{C_{i}(\text { rot })_{\mid B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T} \cdot{ }^{B} \ddot{\mathbf{X}}_{P_{\mid B E E}} \tag{55}
\end{align*}
$$

When Eq. (55) is pre-multiplied by $\mathbf{J}_{D_{i}}^{T}$ the kinetic component of the generalized force applied to $\{\mathrm{P}\}$ due to each cylinder rotation is obtained in frame $\{B\}$ :

$$
\begin{align*}
& { }^{P} \mathbf{f}_{C_{i}\left(\text { kin) (rot) }\left.\right|_{B}\right.}=\mathbf{J}_{D_{i}}^{T} \cdot{ }^{C_{i}} \mathbf{f}_{C_{i}\left(\text { kin)(rot) }\left.\right|_{B}\right.} \\
& =\mathbf{J}_{D_{i}}^{T} \cdot \frac{d}{d t}\left(\mathbf{I}_{\left.C_{i}(\text { rot })\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T}\right) \cdot{ }^{B} \dot{\mathbf{X}}_{P_{|B| E}}  \tag{56}\\
& +\mathbf{J}_{D_{i}}^{T} \cdot \mathbf{I}_{\left.C_{i}(r o t)\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T} \cdot{ }^{B} \ddot{\mathbf{X}}_{P_{|B| E}}
\end{align*}
$$

The inertia matrix and the Coriolis and centripetal terms matrix of the rotating cylinder may be written as:

$$
\begin{align*}
& \mathbf{J}_{D_{i}}^{T} \cdot \mathbf{I}_{\left.C_{i}(r o t)\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T}  \tag{57}\\
& \mathbf{J}_{D_{i}}^{T} \cdot \frac{d}{d t}\left(\mathbf{I}_{\left.C_{i}(r o t)\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T}\right) \tag{58}
\end{align*}
$$

These matrices represent the inertia matrix and the Coriolis and centripetal terms matrix of a virtual moving platform that is equivalent to each rotating cylinder.

Table 1
Number of arithmetic operations involved in the computation of the Coriolis and centripetal terms matrices.

|  | Lagrange |  |  | Lagrange-Koditschek representation |  |  | Generalized momentum |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Add. | Mul. | Div. | Add. | Mul. | Div. | Add. | Mul. | Div. |
| Coriolis and centripetal terms matrix of the mobile platform | 485 | 719 | - | 474 | 735 | - | 182 | 326 | - |
| Coriolis and centripetal terms matrix of each translating cylinder | 2982 | 5483 | 7 | 2389 | 4283 | 7 | 238 | 533 | 4 |
| Coriolis and centripetal terms matrix of each rotating cylinder | 5433 | 9215 | 8 | 3908 | 6765 | 8 | 471 | 942 | 8 |
| Coriolis and centripetal terms matrix of each translating piston | 2985 | 5483 | 7 | 2392 | 4283 | 7 | 241 | 533 | 4 |
| Coriolis and centripetal terms matrix of each rotating piston | 5433 | 9215 | 8 | 3908 | 6765 | 8 | 471 | 942 | 8 |
| Total operations | 101,483 | 177,095 | 80 | 76,056 | 133,311 | 180 | 8708 | 18,026 | 144 |

Table 2
Manipulator parameters.

| Para. | Value | Para. | Value $\left(\mathrm{kg} \mathrm{m}^{2}\right)$ | Para. |
| :--- | :--- | :--- | :--- | :--- |
| $r_{B}$ | 1.500 m | $I_{P x x}$ | 0.2 | $I_{S x x}$ |
| $r_{P}$ | 0.750 m | $I_{P y y}$ | 0.2 | $I_{S y y}$ |
| $\phi_{B}$ | $15^{\circ}$ | $I_{P z z}$ | 0.4 | $I_{S z z}$ |
| $\phi_{P}$ | $0^{\circ}$ | $I_{C x x}$ | 0.0 | $b_{C}$ |
| $m_{P}$ | 1.430 kg | $I_{C y y}$ | 0.1 | $b_{S}$ |
| $m_{C}$ | 0.39 kg | $I_{C z z}$ | 0.1 | $0.1 \mathrm{~kg} \mathrm{~m}^{2}$ |
| $m_{S}$ | 0.39 kg |  |  | $0.5 \mathrm{~kg} \mathrm{~m}^{2}$ |

### 3.3. Piston modeling

If the centre of mass of each piston is located at a constant distance, $b_{S}$, from the piston to moving platform connecting point, $P_{i}$, (Fig. 3), then its position relative to frame $\{\mathrm{B}\}$ is:

$$
\begin{equation*}
{ }^{B} \mathbf{p}_{\left.S_{i}\right|_{B}}=-b_{S} \cdot \hat{\mathbf{1}}_{i}+{ }^{B} \mathbf{p}_{\left.i\right|_{B}}+{ }^{B} \mathbf{x}_{\left.P(p o s)\right|_{B}} \tag{59}
\end{equation*}
$$

The linear velocity of the piston centre of mass, ${ }^{B} \dot{\mathbf{p}}_{S_{i} \mid B}$, relative to $\{\mathrm{B}\}$ and expressed in the same frame, may be computed as:

$$
\begin{align*}
{ }^{B} \dot{\mathbf{p}}_{\left.S_{i}\right|_{B}} & =\dot{\mathbf{l}}_{i}+{ }^{B} \boldsymbol{\omega}_{\left.l_{i}\right|_{B}} \times\left(-b_{S} \cdot \hat{\mathbf{l}}_{i}\right)  \tag{60}\\
{ }^{B} \dot{\mathbf{p}}_{\left.S_{i}\right|_{B}} & =\mathbf{J}_{G_{i}} \cdot\left[\begin{array}{l}
{ }^{B} \mathbf{v}_{\left.P\right|_{B}} \\
{ }^{B} \omega_{\left.P\right|_{B}}
\end{array}\right] \tag{61}
\end{align*}
$$

where the jacobian $\mathbf{J}_{G_{i}}$ is given by:

$$
\mathbf{J}_{G_{i}}=\left[\begin{array}{ll}
\mathbf{I}-b_{S} \cdot \tilde{\mathbf{l}}_{i}^{T} \cdot \tilde{\overline{\mathbf{l}}}_{i} & \left(\mathbf{I}-b_{S} \cdot \tilde{\mathbf{l}}_{i}^{T} \cdot \tilde{\overline{\mathbf{l}}}_{i}\right) \cdot{ }^{P} \tilde{\mathbf{p}}_{\left.i\right|_{B}}^{T} \tag{62}
\end{array}\right]
$$

The linear momentum of each piston, $\mathbf{Q}_{S_{i \mid B}}$, can be represented in frame $\{\mathrm{B}\}$ as:

$$
\begin{equation*}
\mathbf{Q}_{\left.S_{i}\right|_{B}}=m_{S} \cdot{ }^{B} \dot{\mathbf{p}}_{\left.S_{i}\right|_{B}} \tag{63}
\end{equation*}
$$

where $m_{S}$ is the piston mass.
Introducing jacobian $\mathbf{J}_{G_{i}}$ and matrix transformation $\mathbf{T}$ in the previous equation results into:

$$
\begin{equation*}
\mathbf{Q}_{\left.S_{i}\right|_{B}}=m_{S} \cdot \mathbf{J}_{G_{i}} \cdot \mathbf{T} \cdot{ }^{B} \dot{\mathbf{X}}_{P_{|B| E}} \tag{64}
\end{equation*}
$$

The kinetic component of the force applied to the piston due to its translation and expressed in $\{B\}$ can be obtained from the time derivative of Eq. (64):

$$
\begin{equation*}
s_{S_{i}}^{\mathbf{S}_{\left.S_{i}(\text { kin })(\text { tra })\right|_{B}}}=_{\mathbf{Q}_{\left.S_{i}\right|_{B}}}=m_{S} \cdot \frac{d}{d t}\left(\mathbf{J}_{B_{i}} \cdot \mathbf{T}\right) \cdot{ }^{B} \dot{\mathbf{x}}_{\left.P\right|_{B E}}+m_{S} \cdot \mathbf{J}_{B_{i}} \cdot \mathbf{T} \cdot{ }^{B} \ddot{\mathbf{X}}_{\left.\right|_{\mid B E}} \tag{65}
\end{equation*}
$$

When Eq. (65) is multiplied by $\mathbf{J}_{G_{i}}^{T}$, the kinetic component of the force applied to $\{\mathrm{P}\}$ due to each piston translation is obtained in frame $\{\mathrm{B}\}$ :

$$
\begin{align*}
& { }^{P} \mathbf{f}_{\left.S_{i}(\text { kin })(\text { tra })\right|_{\mathrm{B}}}=\mathbf{J}_{G_{i}}^{T} \cdot{ }^{S_{i}} \mathbf{f}_{S_{i}\left(\text { kin) }\left.(\text { tra })\right|_{\mathrm{B}}\right.} \\
& =m_{S} \cdot \mathbf{J}_{G_{i}}^{T} \cdot \frac{d}{d t}\left(\mathbf{J}_{G_{i}} \cdot \mathbf{T}\right) \cdot{ }^{B} \dot{\mathbf{x}}_{\left.P\right|_{B \mid E}}+m_{S} \cdot \mathbf{J}_{G_{i}}^{T} \cdot \mathbf{J}_{G_{i}} \cdot \mathbf{T} \cdot{ }^{B} \ddot{\mathbf{x}}_{P \mid \text { BEE }} \tag{66}
\end{align*}
$$

The inertia matrix and the Coriolis and centripetal terms matrix of the translating piston are:

$$
\begin{align*}
& m_{S} \cdot \mathbf{J}_{G_{i}}^{T} \cdot \mathbf{J}_{G_{i}} \cdot \mathbf{T}  \tag{67}\\
& m_{S} \cdot \mathbf{J}_{G_{i}}^{T} \cdot \frac{d}{d t}\left(\mathbf{J}_{G_{i}} \cdot \mathbf{T}\right) \tag{68}
\end{align*}
$$



Fig. 4. Actuators trajectories: (a) position; (b) velocity; (c) acceleration.
which represent the inertia matrix and the Coriolis and centripetal terms matrix of a virtual moving platform that is equivalent to each translating piston.

On the other hand, the angular momentum of each piston about its centre of mass can be represented in frame $\{B\}$ by:

$$
\begin{align*}
& \mathbf{H}_{\left.S_{i}\right|_{B}}=\mathbf{I}_{\left.S_{i}(\text { rot })\right|_{B}} \cdot{ }^{B} \boldsymbol{\omega}_{l_{i}| |_{B}}  \tag{69}\\
& \mathbf{I}_{\left.S_{i}(\text { rot })\right|_{B}}={ }^{B} \mathbf{R}_{S_{i}} \cdot \mathbf{I}_{S_{i}(\text { rot }) \mid S_{S_{i}}} \cdot{ }^{B} \mathbf{R}_{S_{i}}^{T} \tag{70}
\end{align*}
$$

where ${ }^{B} \mathbf{R}_{S_{i}}$ is the orientation matrix of each piston frame, $\left\{\mathrm{S}_{i}\right\}$, relative to the base frame, $\{\mathrm{B}\}$.
As the relative motion between cylinder and piston is a pure translation, $\left\{\mathrm{S}_{i}\right\}$ can be chosen parallel to $\left\{\mathrm{C}_{i}\right\}$, and its origin located at the piston centre of mass. Therefore, ${ }^{B} \mathbf{R}_{S_{i}}={ }^{B} \mathbf{R}_{C_{i}}$.

So, the inertia matrices of the pistons can be written as:

$$
\mathbf{I}_{\left.S_{i}(\text { rot })\right|_{s_{i}}}=\operatorname{diag}\left(\left[\begin{array}{lll}
I_{x x} & I_{S_{y y}} & I_{S_{z z}} \tag{71}
\end{array}\right]\right)
$$

where $I_{S_{x x}}, I_{S_{y y}}$ and $I_{S_{z z}}$ are the pistons moments of inertia expressed in its own frame.
Introducing jacobian $\mathbf{J}_{D_{i}}$ and matrix transformation $\mathbf{T}$ in Eq. (69) results into:

$$
\begin{equation*}
\mathbf{H}_{\left.S_{i}\right|_{B}}=\mathbf{I}_{\left.S_{i}(\text { rot })\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T} \cdot{ }^{B} \dot{\mathbf{x}}_{\left.P\right|_{B \mid E}} \tag{72}
\end{equation*}
$$

The kinetic component of the generalized force applied to the piston, due to its rotation and expressed in $\{B\}$ can be obtained from the time derivative of Eq. (72):

$$
\begin{align*}
s_{i} \mathbf{f}_{S_{i}\left(\text { kin)(rot) }\left.\right|_{B}\right.}= & \dot{\mathbf{H}}_{\left.S_{i}\right|_{B}} \\
& =\frac{d}{d t}\left(\mathbf{I}_{\left.S_{i}(r o t)\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T}\right) \cdot{ }^{B} \dot{\mathbf{X}}_{\left.P\right|_{B E E}}+\mathbf{I}_{\left.S_{i}(r o t)\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T} \cdot{ }^{B} \ddot{\mathbf{X}}_{P_{\mid B E E}} \tag{73}
\end{align*}
$$

Pre-multiplied by $\mathbf{J}_{D_{i}}^{T}$, the kinetic component of the generalized force applied to $\{\mathrm{P}\}$ due to each piston rotation is obtained in frame $\{B\}$ :

$$
\begin{align*}
{ }^{P} \mathbf{S}_{\left.S_{i}(\text { kin })(\text { rot })\right|_{B}}= & \mathbf{J}_{D_{i}}^{T} \cdot S_{i} \mathbf{S}_{S_{i}\left(\text { kin) }\left.(\text { rot })\right|_{B}\right.} \\
= & \mathbf{J}_{D_{i}}^{T} \cdot \frac{d}{d t}\left(\mathbf{I}_{\left.S_{i}(\text { rot })\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T}\right) \cdot{ }^{B} \dot{\mathbf{X}}_{\left.P\right|_{B \mid E}}  \tag{74}\\
& +\mathbf{J}_{D_{i}}^{T} \cdot \mathbf{I}_{\left.S_{i}(\text { rot })\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T} \cdot{ }^{B} \ddot{\mathbf{x}}_{\left.P\right|_{B E E}}
\end{align*}
$$

The inertia matrix and the Coriolis and centripetal terms matrix of the rotating piston will be:

$$
\begin{align*}
& \mathbf{J}_{D_{i}}^{T} \cdot \mathbf{I}_{\left.S_{i}(r o t)\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T}  \tag{75}\\
& \mathbf{J}_{D_{i}}^{T} \cdot \frac{d}{d t}\left(\mathbf{I}_{\left.S_{i}(\text { rot })\right|_{B}} \cdot \mathbf{J}_{D_{i}} \cdot \mathbf{T}\right) \tag{76}
\end{align*}
$$

These matrices represent the inertia matrix and the Coriolis and centripetal terms matrix of a virtual moving platform that is equivalent to each rotating piston.


Fig. 5. Developed actuators forces.

It should be noted that rigid bodies inertia and Coriolis and centripetal terms matrices can be added, as they are expressed in the same frame.

### 3.4. Dynamic model gravitational components

Given a general frame $\{x, y, z\}$, with $\mathbf{z} \equiv-\hat{\mathbf{g}}$, the potential energy of a rigid body is given by:

$$
\begin{equation*}
P_{c}=m_{c} \cdot g \cdot z_{c} \tag{77}
\end{equation*}
$$

where $m_{c}$ is the body mass, $g$ is the modulus of the gravitational acceleration and $z_{c}$ the distance, along $\mathbf{z}$, from the frame origin to the body centre of mass.

The gravitational components of the generalized forces acting on $\{\mathrm{P}\}$ can be easily obtained from the potential energy of the different bodies that compose the system:

$$
\begin{align*}
& { }^{P} \mathbf{f}_{\left.P(\text { gra })\right|_{\text {BEE }}}=\frac{\partial P_{P}\left({ }^{B} \mathbf{x}_{\left.P\right|_{B \mid E}}\right)}{\partial^{B} \mathbf{x}_{P_{|B| E}}}  \tag{78}\\
& { }^{P} \mathbf{f}_{\left.C_{i}(\text { gra })\right|_{\text {BEE }}}=\frac{\partial P_{C_{i}}\left({ }^{B} \mathbf{X}_{P_{\mid \mathrm{BEE}}}\right)}{\partial^{B} \mathbf{X}_{P_{\mid \mathrm{BEE}}}}  \tag{79}\\
& { }^{P} \mathbf{f}_{\left.S_{i}(\text { gra })\right|_{\mathrm{B} \mid E}}=\frac{\partial P_{S_{i}}\left({ }^{B} \mathbf{X}_{\left.P\right|_{|B| E}}\right)}{\partial^{B} \mathbf{X}_{\left.P\right|_{\mid B E E}}} \tag{80}
\end{align*}
$$

The three vectors ${ }^{P} \mathbf{f}_{P(\text { gra })| | B E},{ }^{P} \mathbf{f}_{\left.C_{i}(\text { gra })\right|_{B E}}$ and ${ }^{P} \mathbf{f}_{S_{i}(\text { gra })|B| E}$ represent the gravitational components of the generalized forces acting on $\{\mathrm{P}\}$, expressed using the Euler angles system, due to the moving platform, each cylinder and each piston. Therefore, to be added to the kinetic force components, these vectors must be transformed, to be expressed in frame $\{B\}$. This may be done pre-multiplying the gravitational components force vectors by the matrix transformation $\mathbf{T}^{-1}$.

### 3.5. Manipulator dynamic equations

In previous sections, analytic expressions for the rigid bodies inertia and Coriolis and centripetal terms matrices were obtained. These matrices can be added, as they are expressed in the same frame, resulting in the system inertia matrix, $\mathbf{I}_{\left.\right|_{E}}$, and system Coriolis and centripetal terms matrix, $\mathbf{V}_{l_{E}}$. Therefore, the total kinetic component of the generalized force acting on the moving platform, ${ }^{P} \mathbf{f}_{\left(\text {kin }\left.\right|_{B}\right.}$, may be easily computed using the following equation:

$$
\begin{equation*}
\mathbf{I}_{\mid E} \cdot{ }^{B} \ddot{\mathbf{X}}_{P_{\mid B E E}}+\mathbf{V}_{\mid E} \cdot{ }^{B} \dot{\mathbf{x}}_{P_{|B| E}}={ }^{P} \mathbf{f}_{\left.(k i n)\right|_{B}} \tag{81}
\end{equation*}
$$

In a similar way, ${ }^{P} \mathbf{f}_{(\text {kin) }}$, could also be obtained using Eqs. (24), (44), (56), (66) and (74):

$$
\begin{equation*}
{ }^{P} \mathbf{f}_{\left.P(\text { kin })\right|_{B}}+{ }^{P} \mathbf{f}_{\left.C_{i}(\text { kin })(\text { tra })\right|_{B}}+{ }^{P} \mathbf{f}_{\left.C_{i}(\text { kin })(\text { rot })\right|_{B}}+{ }^{P} \mathbf{f}_{S_{i}\left(\text { kin) }\left.(\text { tra })\right|_{B}\right.}+{ }^{P} \mathbf{f}_{\left.S_{i}(\text { kin })(\text { rot })\right|_{B}}={ }^{P} \mathbf{f}_{\left.(\text {kin })\right|_{B}} \tag{82}
\end{equation*}
$$

This total kinetic component should be added to the total gravitational part of the generalized force acting on the moving platform, $\left.{ }^{P} \mathbf{f}_{(g r a)}\right|_{B}$, which can be obtained using the manipulator potential energy, as in Eqs. (78)-(80):

$$
\begin{equation*}
\mathbf{T}^{-1} \cdot\left({ }^{P} \mathbf{f}_{P(\text { gra })| |_{B E E}}+{ }^{P} \mathbf{f}_{C_{i}\left(\text { gra }\left.\right|_{B \mid E}\right.}+{ }^{P} \mathbf{f}_{\left.S_{i}(\text { gra })\right|_{\mathrm{BEE}}}\right)={ }^{P} \mathbf{f}_{\left.(g r a)\right|_{B}} \tag{83}
\end{equation*}
$$

The total generalized force acting on the moving platform, and the corresponding actuating forces, will be:

$$
\begin{align*}
& { }^{P} \mathbf{f}_{\left.\right|_{B}}={ }^{P} \mathbf{f}_{\text {(kin) }\left.\right|_{B}}+{ }^{P} \mathbf{f}_{\text {gra }\left.\right|_{B}}  \tag{84}\\
& \tau_{P}=\mathbf{J}_{C}^{-T} \cdot{ }^{P} \mathbf{f}_{\left.\right|_{B}} \tag{85}
\end{align*}
$$

## 4. Computational load of the proposed dynamic model

As a means of evaluating efficiency, the number of scalar operations (additions, multiplications and divisions) required for the computation of the inertia and Coriolis and centripetal terms matrices is tabulated. Specification of the model computational load in this manner makes comparison with other models both easy and convenient [25]. Thus, the computational efficiency of the proposed dynamic model is compared with the one resulting from applying the Lagrange method, both directly from Lagrangian equation and using the Koditschek representation [10].

As the largest difference between the methods rests on how the Coriolis and centripetal terms matrices are calculated, the models are evaluated by the number of arithmetic operations involved in the computation of these matrices. The results were obtained using the symbolic computational software Maple ${ }^{\circledR}$ and are presented in Table 1.

The dynamic model obtained using the generalized momentum approach is computationally much more efficient and its superiority manifests precisely in the computation of the matrices requiring the largest relative computational effort: the Coriolis and centripetal terms matrices.

It should be noted, the proposed approach was used in the dynamic modeling of a 6-dof Stewart platform. Nevertheless, it can be applied to any mechanism.

## 5. Numerical simulation

A 6-dof parallel manipulator presenting the kinematic and dynamic parameters shown in Table 2 was considered.
A trajectory was specified in task space. The moving platform initial position is $P_{1}=[0,0,2000,0,0,0]\left(\mathrm{mm}\right.$; ${ }^{\circ}$ ). The moving platform is then displaced to point $P_{2}=[-100,-200,2500,15,-15,15]\left(\mathrm{mm} ;{ }^{\circ}\right)$, and finally it returns to point $P_{1}$.

Third order trigonometric splines were interpolated between the specified points, in order to obtain continuous and smooth trajectories. Fig. 4 shows the corresponding actuators trajectories.

Fig. 5 shows the developed actuators forces, necessary to follow the specified trajectories. Fig. 6 represents the contribution of the moving platform, the six cylinders, and the six pistons to the total forces developed by the actuators.

## 6. Conclusion

Dynamic modeling of parallel manipulators presents an inherent complexity. Despite the intensive study in this topic of robotics, mostly conducted in the last two decades, additional research still has to be done in this area.

In this paper, an approach based on the manipulator generalized momentum was explored and applied to the dynamic modeling of a Stewart platform. The generalized momentum is used to compute the kinetic component of the generalized force acting on the moving platform. Analytic expressions for the rigid bodies inertia and Coriolis and centripetal terms matrices are obtained, which can be added, as they are expressed in the same frame. Having these matrices, the kinetic component of the generalized force acting on the moving platform may be easily computed. This component can be added to the gravitational part of the generalized force, which is obtained through the manipulator potential energy.

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